Inequalities between Arrival Averages and Time Averages in Stochastic Processes Arising from Queueing Theory

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(Received May 1982; accepted December 1983)

A common scenario in stochastic models, especially in queueing theory, is that an arrival counting process both observes and interacts with another continuous time stochastic process. In the case of Poisson arrivals, Wolff [1982] recently proved that the proportion of arrivals finding the process in some state is equal to the proportion of time it spends there under a lack of anticipation assumption. Inspired by Wolff's approach, in this paper we study the related interesting question of when do we have inequalities between these proportions. We establish two-sided inequalities under the following three assumptions: (i) the interarrival time distributions are of type NBUE or NWUE, (ii) the process being observed have monotone sample paths between arrival epochs, and (iii) the state of the process does not depend on future jumps of the arrival process. These assumptions are typically true in all standard queueing models and hence our results have wide implications. Stochastic inequalities between limiting distributions of interest, when they exist, also follow easily from our main result.

A FREQUENTLY occurring scenario in stochastic models, especially in queueing theory, is that a counting process both observes and interacts with another continuous time stochastic process. A concrete example is the $GI/G/1$ queue. In this setting, the counting process could be the arrival process and the stochastic process being observed could be derived from any queueing characteristic of interest, e.g., number of customers in the system or the total work in the system. For this model, it is well-known that if the arrival process is Poisson, then the limiting stationary actual and virtual waiting time distributions are identical, when they both exist. Results of this nature have been discovered many times in other model contexts as well. Motivated by these results, researchers, including Harrison and Lemoine [1976], Stidham [1972], Strauch [1970], and Wolff [1970], have made many attempts to prove various versions of this type of result within a more general framework. However, an important drawback of these papers is that, in order to

Subject classification: 564 new or used, better or worse in expectations, 569 renewal arrivals, 692 inequalities between arrival average and time average.

Operations Research
Vol. 32, No. 4, July–August 1984

0030-364X/84/3204-0785 $01.25

guarantee the existence of limiting distributions, they impose restrictive assumptions on the systems under study (e.g., the processes involved are regenerative, stationary, and/or ergodic). A related and more fundamental property which is free of these assumptions is that the limiting proportion of arrivals finding the system in some *state* is equal to the limiting proportion of time the system spends there. This property is usually called Poisson arrivals see time averages (PASTA). This result has recently been proved by Wolff [1982] under a basic lack of anticipation assumption, which says that the process being observed cannot anticipate the future jumps of the Poisson process. For more discussion of these papers and the importance of PASTA, we refer the readers to Wolff's paper.

A closely related and natural question of significant interest is: when do we have inequalities, in ways to be made precise later, between these two types of limiting proportions? This problem has also received considerable attention in the literature. The related literature includes papers by König and Schmidt [1980], Marshall and Wolff [1971], Mori [1975], Rolski [1981], Schmidt [1982], Stoyan [1977], Whitt [1983], and many others. In this paper, inspired by Wolff's approach, we carry out a similar analysis of this problem and show that when the distribution of interarrival times is NBUE (new better than used in expectation), then the limiting proportion of arrivals finding the process of interest in some state is, w.p.1, less than the limiting proportion of time it spends there, provided that the two limiting proportions (or averages) exist. A reverse inequality also holds when the distribution of interarrival times is NWUE (new worse than used in expectation). However, passing from Poisson arrivals to renewal arrivals requires that some assumptions be imposed on the behavior of the process being observed between arrival epochs. Loosely put, we require that the state of the process being observed at time t depends *only* on the past arrival epochs and what *happened* at those time points and, furthermore, that the sample paths of the process must be nonincreasing between arrival epochs. The main line of argument is similar to that of Wolff. A continuous time martingale used in his proof is replaced here by a properly constructed discrete time semimartingale (i.e., a super- or submartingale).

The assumptions we make are typically true in most standard queueing models. For example, the results are applicable to multiple server queues, queues with finite waiting rooms, queues with variable or state dependent service rates, and so forth. The process of interest in these models could be the number of customers in the system, the number of busy servers, the familiar work load vector in multiple server queues, the indicator function of whether the system is busy or not, and so on. Whenever limiting distributions exist, our results will imply stochastic inequalities between them.

The organization of this paper is as follows. Section 1 presents some necessary notation and terminology, especially those related to martingales. The basic assumptions we make are also formally presented in this section. The main results are then described and proved in Section 2. The final section contains some general comments and an example showing that, in order to produce positive results, our assumptions on the sample paths cannot be eliminated.

1. DEFINITIONS AND NOTATIONS

Consider two stochastic processes $V = \{V(t),\ t \geq 0\}$ and $N = \{N(t),\ t \geq 0\}$ defined on a common probability space $(\Omega, \mathscr{F}, P)$. The state space of V is assumed to be a measurable space $(S, \mathscr{S})$ endowed with a partial order "$\leq$". A subset A of S is called an *upper set* if $x \in A$ and $x \leq y$ implies $y \in A$. A *lower set* is defined similarly. We assume the σ-algebra $\mathscr{S}$ is generated by the collection of all upper sets in S. The process N is a renewal counting process whose sample paths are, w.p.1, right-continuous nondecreasing step functions with unit jumps. The *arrival* times are defined by $T_n = \min\{t \geq 0\colon N(t) = n\}$, $n \geq 0$. We assume that the random variables $X_n \equiv T_n - T_{n-1}$ for $n \geq 1$ are independent and identically distributed with mean $1/\lambda$, $0 < \lambda < \infty$. Furthermore, they have finite second moments. It is helpful to think of T_n for $n \geq 1$ as arrival times of customers to a queue (possibly with multiple servers) and $V(t)$ as the *state* (e.g., total work or number in system) of the queue at time t.

For each $t \geq 0$, denote by $\mathscr{F}_t$ the σ-algebra generated by $\{(N(s), V(s)),\ 0 \leq s \leq t\}$. Let $\mathscr{F}_n$ for $n \geq 0$ (for typographical ease, we do not use $\mathscr{F}_{T_n}$) denote the σ-algebra consisting of sets $A \in \mathscr{F}$ such that $A \cap \{T_n \leq t\} \in \mathscr{F}_t$ for all $t \geq 0$. In other words, $\mathscr{F}_n$ consists of events that are prior to T_n. We now make the following basic assumptions about the relationship between V and N:

ASSUMPTION 1. *The sample paths of* $\{V(t + T_n),\ 0 \leq t < X_{n+1}\}$ *are nonincreasing in t, w.p.*1, *for every* $n \geq 0$.

ASSUMPTION 2. *For any* $r \geq 0$, $s \geq 0$, $n \geq 1$, *and* $m \geq 1$, *we have* $P\{X_n > r + s,\ V(t_i + T_{n-1}) \in B_i,\ i = 1, \cdots, m \mid A \cap \{X_n > s\}\} = P\{X_n > r + s \mid A \cap \{X_n > s\}\} P\{V(t_i + T_{n-1}) \in B_i,\ i = 1, \cdots, m \mid A \cap \{X_n > s\}\}$, *for all* $A \in \mathscr{F}_{n-1}$, $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m \leq s$ *and* $B_i \in \mathscr{S}$, $i = 1, \cdots, m$.

ASSUMPTION 3. *For all* $A \in \mathscr{F}_{n-1}$, $n \geq 1$, *and* $s \geq 0$, $P\{A \cap \{X_n > s\}\} = P\{A\}P\{X_n > s\}$.

Note that what happens at the arrival epochs T_n, $n \geq 0$, is completely unspecified. For queueing applications, the state of the system usually goes *up* at arrival epochs. This information is not needed for our purposes here. Also, whether the sample paths of V are right-continuous or left-continuous will not be an issue here. In fact, we did not even assume a topological structure on S.

We next develop some standard notation and terminology related to martingales. Let $Y = \{Y(t), t \in T\}$ be a real-valued stochastic process defined on $(\Omega, \mathscr{F}, P)$. A family of sub σ-algebras $\{\mathscr{F}_t, t \in T\}$ is *increasing* if $\mathscr{F}_t \subset \mathscr{F}_s$ for all $t < s$ (assume T is linearly ordered). The process Y is *adapted* to an increasing family $\{\mathscr{F}_t, t \in T\}$ if $Y(t)$ is $\mathscr{F}_t -$ measurable for every $t \in T$. A *stopping time* τ relative to $\{\mathscr{F}_t, t \in T\}$ is a T-valued function defined on Ω such that $\{\omega: \tau(\omega) \le t\} \in \mathscr{F}_t$. The collection $\{(Y(t), \mathscr{F}_t), t \in T\}$ is called a *martingale* if, and only if, the following conditions hold:

(i) $\{\mathscr{F}_t, t \in T\}$ is increasing;
(ii) Y is adapted to $\{\mathscr{F}_t, t \in T\}$;
(iii) $\int_\Omega |Y(t)| dP < \infty$ for all $t \in T$;
(iv) $\int_A Y(s)dP = \int_A Y(t)dP$ for all $A \in \mathscr{F}_s$ and $s < t$.

A collection $\{(Y(t), \mathscr{F}_t), t \in T\}$ is called a *submartingale* (*supermartingale*) if, and only if, (i), (ii), (iii) hold and (iv) is true with $\le$ ($\ge$) replacing the equality.

Finally, we recall that a distribution function F with support in $[0, \infty)$ is said to be NBUE (NWUE) if the mean μ of F is finite and, with $\bar{F} \equiv 1 - F$, $\int_0^\infty [\bar{F}(t + x)/\bar{F}(t)]dx \le (\ge)\ \mu$ for all $t \ge 0$ such that $\bar{F}(t) > 0$. A random variable is said to be NBUE (NWUE) if its distribution function has that property.

2. THE MAIN RESULT

Throughout this paper, let I_B denote the indicator function of a set B. Then, for any interval $[0, t]$ and any upper set $B \in \mathscr{S}$, the proportion of time the process V spends in set B is defined by

$$A_B(t) = t^{-1} \int_0^t I_B(V(s))ds.$$

We next define a family of *exit* random variables $\{\xi_i, i \ge 1\}$ by

$$\xi_i = \sup\{0 \le t < X_i: V(t + T_{i-1}) \in B\},$$

where the supremum over an empty set is taken to be 0. In words, the variable ξ_i for $i \ge 1$ is the time until exit, if it occurs, from set B within the ith piece $[T_{i-1}, T_i)$ of the process V. The proportion of arrivals finding V in state B is now given by

$$C_B(t) = \textstyle\sum_{i=1}^{N(t)} I_{\{\xi_i = X_i\}}/N(t)$$

for any interval $[0, t]$. Observe that the random variables $A_B(t)$, $C_B(t)$, and ξ_i's are well-defined because of Assumption 1.

We now state our main result.

THEOREM 1. *Assume that the distribution of X_i's are of type NBUE (NWUE) and $A_B(t)$, $C_B(t)$ converge, w.p.1, to $A_B(\infty)$, $C_B(\infty)$ respectively; then,*

$$C_B(\infty) \leq (\geq) \ A_B(\infty) \quad \text{w.p.1.}$$

Remark 1. We hasten to point out that the tempting conjecture that $A_B(t)$ and $C_B(t)$ will converge and diverge together is *not* true without further assumptions, as witnessed by the following example. Let x_i for $i \geq 1$ be a sequence of real numbers with $0 \leq x_i < 1$ for all $i \geq 1$ defined so that $\sum_{i=1}^{n} x_i/n$ does not converge. Consider an arrival process N whose interarrival times are identically equal to 1 (which is NBUE). Define

$$V(t) = \begin{cases} 1 & \text{if} \quad i \leq t < i + x_{i+1} \quad \text{for some} \quad i \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $B = \{1\}$, $A_B(t)$ does not converge, but $C_B(t) = 0$ for all $t \geq 0$.

Remark 2. Another important point is that the limits $A_B(\infty)$ and $C_B(\infty)$ need not be constants w.p.1. They may converge to different values along different sample paths with positive probability.

From now on, we restrict our attention to the NBUE case only. The NWUE case is obtained by reversing various inequalities at appropriate places. We next establish two lemmas before proving Theorem 1.

LEMMA 1. *If X_i's are NBUE, then*

$$\lambda \int_A \max[X_n - \xi_n, 0] dP \leq \int_A I_{\{X_n > \xi_n\}} dP$$

for all $A \in \mathscr{F}_{n-1}$ and $n \geq 1$.

Proof. For any $A \in \mathscr{F}_{n-1}$, $n \geq 1$, we have

$$\begin{aligned} \int_A \max[X_n - \xi_n, 0] dP &= \int_{A \cap \{X_n > \xi_n\}} (X_n - \xi_n) dP \\ &= P(A \cap \{X_n > \xi_n\}) \, E(X_n - \xi_n \mid A \cap \{X_n > \xi_n\}) \\ &= P(A \cap \{X_n > \xi_n\}) \int_0^\infty P\{X_n - \xi_n > r \mid A \\ &\qquad \cap \{X_n > \xi_n\}\} dr \\ &= P(A \cap \{X_n > \xi_n\}) \int_0^\infty \int_0^\infty P\{X_n > \\ &\qquad r + s \mid \xi_n = s, A \cap \{X_n > s\}\} dP\{\xi_n \\ &\qquad \leq s \mid A \cap \{X_n > \xi_n\}\} dr \end{aligned}$$

$$= P(A \cap \{X_n > \xi_n\}) \int_0^\infty \int_0^\infty P\{X_n > r + s \mid A \cap \{X_n > s\}\} dP\{\xi_n \le s \mid A \cap \{X_n > \xi_n\}\} dr$$

$$= P(A \cap \{X_n > \xi_n\}) \int_0^\infty \int_0^\infty P\{X_n > r + s \mid \cdot X_n > s\} dP\}\xi_n \le s \mid A \cap \{X_n > \xi_n\}\} dr$$

$$= P(A \cap \{X_n > \xi_n\}) \int_0^\infty \int_0^\infty P\{X_n > r + s \mid \cdot X_n > s\} dr\, dP\}\xi_n \le s \mid A \cap \{X_n > \xi_n\}\}$$

$$\le (1/\lambda)\, P(A \cap \{X_n > \xi_n\})$$

$$= (1/\lambda) \int_A I_{\{X_n > \xi_n\}} dP.$$

The fifth equality above follows from Assumption 2 and the sixth equality follows from Assumption 3. The inequality is due to the NBUE condition.

We now come to the key construction. Let

$$Z_i = \lambda \int_{T_{i-1}}^{T_i^-} I_B(V(s))ds - I_{\{X_i = \xi_i\}}$$

for all $i \ge 1$ and let $R_n = \sum_{i=1}^n Z_i$ for all $n \ge 1$. Then

LEMMA 2. *$\{(R_n, \mathscr{F}_n), n \ge 1\}$ is a discrete time submartingale.*

Proof. Conditions (i), (ii), and (iii) in Section 1 are easily verified. Therefore, it is sufficient to show that for all $A \in \mathscr{F}_n$ and $n \ge 1$,

$$\int_A R_n dP \le \int_A R_{n+1} dP,$$

which is equivalent to $\int_A Z_{n+1} dP \ge 0$. Now

$$\lambda \int_A \int_{T_n}^{T_{n+1}^-} I_B(V(s))ds dP$$

$$= \lambda \int_A \min[X_{n+1}, \xi_{n+1}] dP$$

$$= \lambda \int_A \{X_{n+1} - \max[X_{n+1} - \xi_{n+1}, 0]\} dP$$

$$\geq \lambda\left\{\int_A E(X_{n+1})dP - E(X_{n+1}) \int_A I_{\{X_{n+1}>\xi_{n+1}\}}dP\right\}$$

$$= \int_A [1 - I_{\{X_{n+1}>\xi_{n+1}\}}]dP$$

$$= \int_A I_{\{X_{n+1}=\xi_{n+1}\}}dP.$$

Note that the crucial first equality is due to Assumption 1. The inequality follows from Lemma 1.

Proof of Theorem 1. By a well-known decomposition of submartingales due to Doob (see e.g. Chung [1974, Theorem 9.3.2, p. 321]), we can write R_n, $n \geq 1$, as $R_n = M_n + Y_n$ where $M_n = \sum_{i=1}^n [Z_i - E(Z_i | \mathscr{F}_{i-1})]$, and $Y_n = \sum_{i=1}^n E(Z_i | \mathscr{F}_{i-1})$.

Now Lemma 2 implies $E(Z_i | \mathscr{F}_{i-1}) \geq 0$ and consequently $Y_n/n \geq 0$ w.p.1. Next, we show that $M_n/n \rightarrow 0$ w.p.1 as $n \rightarrow \infty$. Observe that, for all $A \in \mathscr{F}_{i-1}$ and $i \geq 1$,

$$\int_A E(Z_i | \mathscr{F}_{i-1})dP = \int_A Z_i dP$$

$$\leq \int_A \lambda X_i dP$$

$$= \int_A 1\ dP$$

which implies $E(Z_i | \mathscr{F}_{i-1}) \leq 1$ w.p.1. These results together with the obvious inequality $-1 \leq Z_i \leq \lambda X_i$ imply

$$-2 \leq Z_i - E(Z_i | \mathscr{F}_{i-1}) \leq \lambda X_i,$$

and consequently $|Z_i - E(Z_i | \mathscr{F}_{i-1})| \leq \lambda X_i + 2$. Therefore,

$$E\{[Z_i - E(Z_i | \mathscr{F}_{i-1})]^2\} \leq E\{(\lambda X_i + 2)^2\} = 8 + \lambda^2 E(X_i^2) \equiv a < \infty.$$

Summarizing, we have

$$\sum_{i=1}^\infty E[(Z_i - E(Z_i | \mathscr{F}_{i-1}))^2]/i^2 \leq \sum_{i=1}^\infty a/i^2 < \infty. \tag{1}$$

Therefore, by the same strong law for martingales used by Wolff (Feller [1971, Theorem 3, p. 243]), we conclude that $M_n/n \rightarrow 0$ w.p.1. Finally, the proof is completed by observing that

$$A_B(\infty) - C_B(\infty) = \lim_{t\to\infty} \left\{ t^{-1} \int_0^t I_B(V(s))ds - [N(t)]^{-1} \sum_{i=1}^{N(t)} I_{\{X_i=\xi_i\}} \right\}$$

$$= \lim_{t\to\infty} \left\{ \lambda[N(t)]^{-1} \int_0^{T^-_{N(t)}} I_B(V(s))ds \right.$$

$$\left. - [N(t)]^{-1} \sum_{i=1}^{N(t)} I_{\{X_i=\xi_i\}} \right\}$$

$$= \lim_{t\to\infty} \sum_{i=1}^{N(t)} Z_i/N(t) = \lim_{t\to\infty} Y_{N(t)}/N(t) \geq 0 \text{ w.p.1.}$$

Note that we used the assumption that all series involved converge w.p.1.

Remark 3. In (1), we needed the assumption that the second moment of the interarrival times is finite. This condition is always true for NBUE arrivals because the second moment can be bounded from above by the second moment of an exponential random variable. We do not know whether it is possible to remove this condition in the NWUE case.

Let m_1 and m_2 be two probability measures defined on $(S, \mathscr{S})$. We say that $m_1 \leq^{\mathrm{st}} m_2$ if $m_1(B) \leq m_2(B)$ for all upper sets $B \in \mathscr{S}$. The following result is important in applications.

COROLLARY 1. *Assume, in addition to the assumptions in Theorem* 1, *that* $\lim_{n\to\infty} P\{V(T_n^-) \in B\}$ *and* $\lim_{t\to\infty} P\{V(t) \in B\}$ *both exist and are equal to* $m_1(B)$ *and* $m_2(B)$, *respectively, for every upper set* $B \in \mathscr{S}$. *Then either* $m_1 \leq^{\mathrm{st}} m_2$ *or* $m_2 \leq^{\mathrm{st}} m_1$, *according to whether the distribution of* X_i*'s are NBUE or NWUE. Furthermore, we have* $m_1(B) = \int_\Omega C_B(\infty)dP$ *and* $m_2(B) = \int_\Omega A_B(\infty)dP$.

Proof. This result is a direct consequence of Theorem 1 obtained by applying first the dominated convergence theorem and then Fubini's theorem.

Remark 4. For almost all queueing models, the conditions of Corollary 1 are satisfied. Furthermore, $A_B(\infty)$ and $C_B(\infty)$ are typically constants w.p.1.

Remark 5. A referee has pointed out that a different version of Corollary 1 can be proved by a method in Schmidt without assuming the finiteness of second moments of interarrival times. Also, only stochastic monotonicity of sample paths between arrivals is required in his proof. However, Schmidt made the stronger assumption that both V and N are stationary and ergodic.

3. CONCLUDING REMARKS

We start with some remarks on the possibility of weakening our assumptions. First, similar results will follow if we assume the sample paths of V to be *nondecreasing* between arrival epochs in Assumption 1 (of course, B will then be members of the collection of lower sets in $\mathscr{S}$). However, if we allow the sample paths to go both ways between arrivals then, in general, such inequalities are not possible. The following example illustrates this point. Let the interarrival times be 1 identically and let

$$V(t) = \begin{cases} 1 & \text{if } i + 1/3 \le t < i + 2/3 \quad \text{for some} \quad i \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $B = \{1\}$, $C_B(\infty) \le A_B(\infty)$ w.p.1. But if we define another process $\{V'(t), t \ge 0\}$ by $V'(t) = 1 - V(t)$, then, with obvious notation, $C_B'(\infty) \ge A_B'(\infty)$ w.p.1 is true. In fact, by constructing properly weighted versions of V and V', it is clear that *neither* inequalities need to be true in general.

Next, we note that the assumption of identically distributed interarrival times can be weakened. Suppose that the mean of X_i for $i = 1, 2, \cdots$ is $1/\lambda_i$; then a weighted version of Theorem 1 is true under suitable conditions. Specifically, let $\lambda(s) = \sum_{i=1}^{\infty} \lambda_i I_{[T_{i-1}, T_i)}(s)$ and consider the following arrival rate weighted time average:

$$\bar{A}_B(t) = \int_0^t I_B(V(s))\lambda(s)ds \Big/ \int_0^t \lambda(s)ds.$$

Then, observe that

$$\begin{aligned} &\lim_{t\to\infty} \textstyle\sum_{i=1}^{N(t)} \lambda_i \displaystyle\int_{T_{i-1}}^{T_i^-} I_B(V(s))ds/N(t) \\ &\quad = \lim_{t\to\infty} \left[\textstyle\sum_{i=1}^{N(t)} \lambda_i \displaystyle\int_{T_{i-1}}^{T_i^-} I_B(V(s))ds \Big/ \int_0^{T_{N(t)}^-} \lambda(s)ds \right] \\ &\qquad \cdot \lim_{t\to\infty} \left[\int_0^{T_{N(t)}^-} \lambda(s)ds/N(t) \right] \\ &\quad = \lim_{t\to\infty} \bar{A}_B(T_{N(t)}^-) \cdot \lim_{t\to\infty} \left[\textstyle\sum_{i=1}^{N(t)} \lambda_i X_i/N(t) \right] \\ &\quad = \bar{A}_B(\infty) \end{aligned}$$

provided that $\lim_{t\to\infty} [\sum_{i=1}^{N(t)} \lambda_i X_i/N(t)] = 1$ w.p.1. But, this result follows from the strong law of large numbers for independent (but not necessarily identically distributed) random variables (for example, see Chung [1974, Theorem 5.4.1 and its corollary]) if we assume

$$\sum_{i=1}^{\infty} \lambda_i^2 E(X_i^2)/i^2 < \infty. \tag{2}$$

Therefore, by replacing all λ's in the proof of Theorem 1 by λ_i's, it is clear that $C_B(\infty) \le$ (or $\ge$) $\bar{A}_B(\infty)$ w.p.1 whenever (2) is true.

The core of Wolff's proof of PASTA is the construction of a continuous time martingale, which is possible since the arrival process is Poisson. For renewal arrivals, such a construction can *not* be expected because the Poisson process is the only renewal counting process having the independent increments property. In effect, this difficulty is circumvented here by discovering a properly constructed *embedded* discrete time semimartingale hidden between arrivals. For processes with monotone sample paths between arrivals, this approach seems to be more natural since it does reveal a natural point of departure when one moves away from the Poisson assumption. And in fact, the Poisson arrival case is precisely the boundary between NBUE and NWUE classes of arrival processes.

In view of the amount of work, often involving quite ingenious devices, that has been devoted to problems of this nature, it is rather surprising that the general proof turns out to be so simple once the right mode of attack is seen. In conclusion, we have observed, once again, the principle that general results usually have very simple explanations. In this instance, the bare piece of information needed is that the mean residual times of the interarrival intervals are bounded by their means.

ACKNOWLEDGMENT

I would like to sincerely thank Professor Ronald W. Wolff of The University of California at Berkeley for sending me his paper "Poisson Arrivals See Time Averages" prior to its publication; his paper renewed my interest in the topic of this paper and brought an end to a 5-year-old query.

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