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ON JOHNSON'S TWO-MACHINE FLOW SHOP WITH RANDOM PROCESSING TIMES

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A set of n jobs is to be processed by two machines in series that are separated by an infinite waiting room; each job requires a (known) fixed amount of processing from each machine. In a classic paper, Johnson gave a simple rule for ordering of the set of jobs to minimize the time until the system becomes empty, i.e., the makespan. This paper studies a stochastic generalization of this problem in which job processing times are independent random variables. Our main result is a sufficient condition on the processing time distributions that implies that the makespan becomes stochastically smaller when two adjacent jobs in a given job sequence are interchanged. We also give an extension of the main result to job shops.

Consider the following model: jobs $\{1, 2, \ldots, n\}$ are all available for processing by two machines. Job i for $i = 1, \ldots, n$ must be processed by the first machine for a time interval of length $A_i \geq 0$, and then by the second machine for another time interval of length $B_i \geq 0$. No machine can process more than one job at any time; and the two machines are separated by a waiting room of infinite capacity. The problem of interest is to find a permutation of these n jobs that minimizes the total time required to complete all work in the system, the so-called makespan. This model is well-known and is usually referred to in the scheduling literature as a flow shop. It was first formulated and solved by Johnson (1954) for the case of *deterministic* job processing times. His solution, known as Johnson's rule, states that

$$\text{Job } i \text{ should precede job } j \quad \text{if and only if} \quad \min[A_i, B_j] \leq \min[A_j, B_i]. \quad (1)$$

Johnson's important result motivated additional studies on stochastic generalizations of the same problem that assumed that job processing times are independent *random variables*. In the case of exponentially distributed processing times with $E(A_i) = 1/a_i$ and $E(B_i) = 1/b_i$ for $i = 1, 2, \ldots, n$, Talwar (1967) conjectured an optimal scheduling rule with respect to the criterion of minimizing the *expected* makespan. His conjecture, henceforth referred to as Talwar's rule, was

$$\text{Job } i \text{ should precede job } j \quad \text{if and only if} \quad a_i - b_i \geq a_j - b_j. \quad (2)$$

Three years later, Bagga (1970) gave an incomplete proof of this conjecture. A complete proof was subsequently given by Cunningham and Dutta (1973).

As pointed out by Talwar and several other researchers, there is an interesting connection between Johnson's rule and Talwar's rule; namely, they both state that job i should precede job j if and only if $E(\min[A_i, B_j]) \leq E(\min[A_j, B_i])$. This connection, however, appears to be rather mysterious in that it does not seem to be readily explainable in terms of the known and *different* proofs of the two results. The papers cited previously established the optimality of Johnson's rule and Talwar's rule through adjacent pairwise interchange arguments. The authors first find explicit expressions for either the makespan or the expected makespan and then study the effects on the desired criteria of switching two adjacent jobs in a given sequence. The amount of algebra involved in their analyses was considerable. The precise explanation as to why simple rules such as (1) and (2) should work became lost in the lengthy derivations. A considerably shortened proof of the optimality of (2) was recently given by Weiss (1982) (see also Ross 1983, p. 117), but again his proof does not shed new light on the connection between the two rules. In the same paper, Weiss also posed the open question of whether the schedule determined by Talwar's rule minimizes the makespan *stochastically*; a random variable X is said to be stochastically less than another random variable Y, denoted by $X \leq^{st} Y$, if $P\{X > t\} \leq P\{Y > t\}$ for all t.

The purpose of this paper is to provide a precise explanation of the connection between Johnson's rule and Talwar's rule. As one might expect, such an

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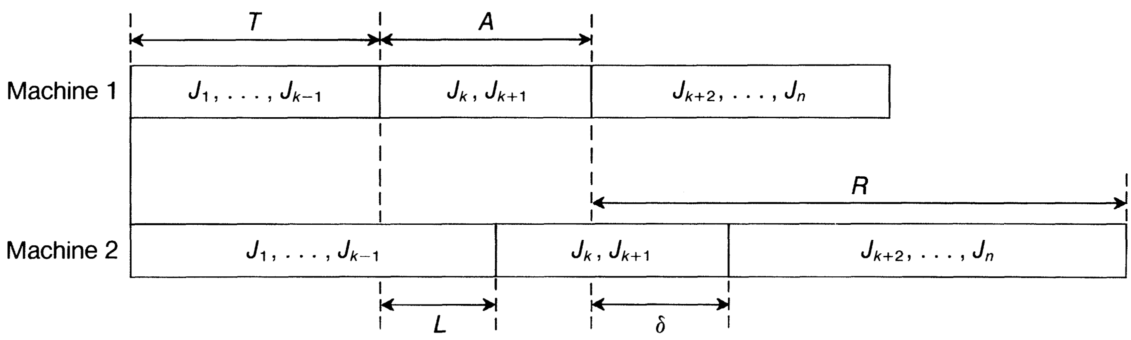

Figure 1. Extension of job shops.

explanation ought to be based on a *unified* proof of the optimality of (1) and (2). We shall therefore carry out a *single* adjacent pairwise interchange argument that uses all required conditions when they are needed. This approach leads to our main result, Theorem 1, which gives a weak sufficient condition on the processing time distributions, in terms of conditional stochastic order of random variables, such that the makespan becomes stochastically smaller when two adjacent jobs are interchanged. This condition includes both Johnson's rule and Talwar's rule as special cases; and, in particular, it answers Weiss's open question in the affirmative.

Several other more recent papers also contain stochastic ordering results for the makespan under various assumptions. Frostig and Adiri (1985) considered a 3-machine flow shop with independent random job processing times and proved that a job sequence in which processing times on the first and third machines are in monotone nondecreasing and nonincreasing order of likelihood ratio, respectively, and on the second machine are identically distributed, minimizes the makespan stochastically; in the special case of zero processing times on the second machine, their result is a special case of our Theorem 1. Pinedo (1982) considered stochastic flow shops with m machines and proved, among other things, that any SEPT-LEPT job sequence minimizes the expected makespan (in fact his argument can be adapted easily to establish stochastic ordering) when the processing times of a job on different machines are identically distributed and, furthermore, the processing times for different jobs are nonoverlapping (see his paper for definitions of SEPT-LEPT sequences and nonoverlapping random variables); his result does not follow from our Theorem 1. Pinedo and Ross (1982) also found stochastic ordering results for makespans in 2-machine open shops in which the order of processing of each job on the machines is immaterial.

The rest of this paper is organized as follows. Section 1 presents the main adjacent pairwise interchange argument as well as implications of the main result. Section 2 generalizes the flow shop results to job shops in which jobs go through the two machines in both directions. The appendix contains two preservation properties of likelihood ratio ordering.

1. Main Results

Our goal in this section is to find conditions on the processing time distributions so that an initial ordering of the set of jobs stochastically minimizes the makespan in a 2-machine flow shop with independent random job processing times. It is important to note that we are considering only the set of so-called *permutation* schedules. In the case of deterministic processing times, it is known (and easily shown) that a policy is never optimal if positive idle times are inserted when there are jobs waiting, and that an optimal policy can be found with the same ordering of jobs on both machines. An easy conditioning argument then shows that consideration of permutation schedules is sufficient when job processing times are random. This paper will not consider *dynamic* policies where decisions can be made at any instant in time based on past information.

The plan of attack is to study the effect on the makespan when two adjacent jobs are interchanged. Consider two job sequences:

$$S_1: J_1, J_2, \ldots, J_k, J_{k+1}, \ldots, J_n$$
and $$S_2: J_1, J_2, \ldots, J_{k+1}, J_k, \ldots, J_n,$$

where S_2 is obtained from S_1 by switching jobs J_k and J_{k+1} for some $1 \leq k \leq n-1$. The following notations will be used (see Figure 1):

T = time until jobs $J_1, \ldots, J_{k-1}$ complete their processing on machine 1 $(= \sum_{i=1}^{k-1} A_i)$,

L = additional time after time T necessary for machine 2 to complete the processing of jobs $J_1, \ldots, J_{k-1}$,

A = total processing time of J_k and J_{k+1} on machine 1 ($= A_k + A_{k+1}$),

$R_1(R_2)$ = additional time necessary to empty the system after time $T + A$ under sequence $S_1(S_2)$, and

$\delta_1(\delta_2)$ = additional time after time $T + A$ necessary for machine 2 to complete the processing of jobs $J_1, \ldots, J_{k+1}$ under sequence $S_1(S_2)$.

Then, by definition, the makespans M_1 and M_2 under sequences S_1 and S_2, respectively, are given by

$$M_i = T + A + R_i \tag{3}$$

for $i = 1, 2$. Let $(X \mid \cdot)$ denote a "conditional" random variable; then our main result can be described as follows:

Theorem 1. *A sufficient condition for* $M_1 \leqslant^{st} M_2$ *is*

$$(\min[A_k, B_{k+1}] \mid A_k + A_{k+1} = a \text{ and } B_k + B_{k+1} = b)$$
$$\leqslant^{st} (\min[A_{k+1}, B_k] \mid A_k + A_{k+1} = a \text{ and } B_k + B_{k+1} = b) \tag{4}$$

for all $a, b \geqslant 0$ *for which the conditional distributions of the random variables involved are well-defined.*

To facilitate the proof of Theorem 1, we first present two lemmas. Notice that the random variables T and A are independent of each other, whereas R_i for $i = 1, 2$ critically depends on T, A, *and* the particular sequence under consideration. More specifically, we have the following result:

Lemma 1. *There exists an increasing (= nondecreasing) function r such that* $R_i = r(\delta_i)$ *for both* $i = 1, 2$.

Lemma 2. $\delta_1 = B_k + B_{k+1} - \min[A_{k+1}, B_k, A - L]$. $\delta_2 = B_k + B_{k+1} - \min[A_k, B_{k+1}, A - L]$.

Proof. Consider sequence S_1 first. Let I_k and I_{k+1} be the idle times before the processings of jobs J_k and J_{k+1}, respectively, on machine 2, i.e.,

$$I_k = \max[A_k - L, 0] \tag{5}$$

and

$$I_{k+1} = \max[A_k + A_{k+1} - L - I_k - B_k, 0]. \tag{6}$$

It then follows that

$$\begin{aligned}\delta_1 &= L + I_k + B_k + I_{k+1} + B_{k+1} - A \\ &= L + B_k + B_{k+1} + \max[A - L - B_k, A_k - L, 0] - A \\ &= B_k + B_{k+1} + \max[-B_k, -A_{k+1}, L - A] \\ &= B_k + B_{k+1} - \min[A_{k+1}, B_k, A - L],\end{aligned}$$

where the second equality is due to (5) and (6). The proof of the equality for δ_2 is similar.

Proof of Theorem 1. Clearly, the desired conclusion would follow by unconditioning if we can show that

$$(M_1 \mid L = \ell, A_k + A_{k+1} = a, B_k + B_{k+1} = b)$$
$$\leqslant^{st} (M_2 \mid L = \ell, A_k + A_{k+1} = a, B_k + B_{k+1} = b)$$

for all $\ell, a, b \geqslant 0$. Now, from (3), we have

$$(M_i \mid L = \ell, A_k + A_{k+1} = a, B_k + B_{k+1} = b)$$
$$= (T \mid L = \ell) + a + (r(\delta_i) \mid L = \ell, A_k + A_{k+1} = a, B_k + B_{k+1} = b)$$

for $i = 1, 2$. The crucial fact here is that the last three terms are *conditionally* independent! From Lemma 1 and the fact that stochastic ordering is preserved upon taking increasing functions (see e.g. Ross, p. 155), it is therefore sufficient to show that

$$(\delta_1 \mid L = \ell, A_k + A_{k+1} = a, B_k + B_{k+1} = b)$$
$$\leqslant^{st} (\delta_2 \mid L = \ell, A_k + A_{k+1} = a, B_k + B_{k+1} = b),$$

which is in turn implied by Lemma 2 and the condition stated in the theorem; hence the proof is complete.

Remark 1. (4) in Theorem 1 is not a necessary condition for $M_1 \leqslant^{st} M_2$. A counterexample can be easily constructed based on Theorem 1 in Pinedo (1982).

The content of condition (4) is perhaps not quite clear at first sight. One may regard it as being too strong to be of practical value because $\min[A_k, B_{k+1}]$ and $\min[A_{k+1}, B_k]$ must be stochastically ordered for *all* possible realizations of $A_k + A_{k+1}$ and $B_k + B_{k+1}$. We next demonstrate that (4) is actually quite weak by giving several special cases for which condition (4) does hold.

Proposition 1. *Condition* (4) *in Theorem* 1 *is implied by either one of the following two conditions:*

(i) $(A_k \mid A_{k+1} = a) \leqslant^{st} (A_{k+1} \mid A_k + A_{k+1} = a)$ *and* $(B_k \mid B_k + B_{k+1} = b) \geqslant^{st} (B_{k+1} \mid B_k + B_{k+1} = b)$ *for all* $a, b \geqslant 0$,

(ii) $\min[A_k, B_{k+1}] \leqslant \min[A_{k+1}, B_k]$ *with probability* 1.

Proof. (i) Note that $(A_k \mid A_k + A_{k+1} = a) = (A_k \mid A_k + A_{k+1} = a,\ B_k + B_{k+1} = b)$ since the processing times are assumed to be independent. (ii) Immediate.

Proposition 2. *Suppose that the processing times are all exponentially distributed with $E(A_i) = 1/a_i$ and $E(B_i) = 1/b_i$ for $i = 1, 2, \ldots, n$. Then* (4) *holds whenever $a_k - b_k \geq a_{k+1} - b_{k+1}$.*

Proof. It is easy to see that (4) is equivalent to

$$P\{A_k > t, B_{k+1} > t, A_k + A_{k+1} = a, B_k + B_{k+1} = b\}$$
$$\leq P\{A_{k+1} > t, B_k > t, A_k + A_{k+1} = a, B_k + B_{k+1} = b\};$$

here and in the remainder of the proof the notation $P\{\cdot\}$ designates either a probability or a density. Now, for any $0 \leq t \leq \min[a, b]$ and $a, b \geq 0$,

$$
\begin{aligned}
&P\{A_k > t, B_{k+1} > t, A_k + A_{k+1} = a, B_k + B_{k+1} = b\}\\
&\quad = P\{A_k > t, A_k + A_{k+1} = a\}\, P\{B_{k+1} > t, B_k + B_{k+1} = b\}\\
&\quad = P\{A_k > t\} P\{A_k + A_{k+1} = a \mid A_k > t\}\\
&\qquad \cdot P\{B_{k+1} > t\} P\{B_k + B_{k+1} = b \mid B_{k+1} > t\}\\
&\quad = \exp\{-(a_k + b_{k+1})t\} P\{A_k + A_{k+1} = a \mid A_{k+1} > t\}\\
&\qquad \cdot P\{B_k + B_{k+1} = b \mid B_k > t\}\\
&\quad \leq \exp\{-(a_{k+1} + b_k)t\} P\{A_k + A_{k+1} = a \mid A_{k+1} > t\}\\
&\qquad \cdot P\{B_k + B_{k+1} = b \mid B_k > t\}\\
&\quad = P\{A_{k+1} > t, B_k > t, A_k + A_{k+1} = a, B_k + B_{k+1} = b\}.
\end{aligned}
$$

To explain the third equality, we observe, by the memoryless property of exponential distributions, that

$$
\begin{aligned}
&P\{A_k + A_{k+1} = a \mid A_k > t\}\\
&\quad = P\{(A_k - t) + A_{k+1} = a - t \mid A_k > t\}\\
&\quad = P\{A_k + A_{k+1} = a - t\}
\end{aligned}
$$

and, hence, is symmetric with respect to the indices k and $k + 1$. The same property also holds for $P\{B_k + B_{k+1} = b \mid B_{k+1} > t\}$. Finally, the previous inequality is due to the stated condition of the proposition; the proof is thus complete.

Remark 2. Note that (ii) of Proposition 1 includes Johnson's rule as a special case and that Proposition 2 answers Weiss's open question. Thus condition (4) *explains* the connection between Johnson's rule and Talwar's rule.

A nonnegative random variable X, with density f_X, is said to be smaller than another nonnegative random variable Y, with density f_Y, in likelihood ratio, denoted by $X \leq^L Y$, if

$$f_Y(x)/f_X(x) \leq f_Y(y)/f_X(y) \tag{7}$$

for all $0 \leq x \leq y$. As is well-known, $X \leq^L Y$ implies $X \leq^{st} Y$. This stronger ordering turns out to be related to (i) of Proposition 1.

Proposition 3. *Let X and Y be two independent nonnegative random variables with densities f_X and f_Y respectively. If $X \leq^L Y$, then $(X \mid X + Y = s) \leq^L (Y \mid X + Y = s)$ for all $s \geq 0$.*

Proof. Given $X \leq^L Y$, the ratio

$$\frac{f_{(Y \mid X+Y=s)}(t)}{f_{(X \mid X+Y=s)}(t)} = \frac{f_Y(t) f_X(s - t)}{f_X(t) f_Y(s - t)}$$

is nondecreasing in t.

Corollary 1. *Condition* 4 *is implied by $A_k \leq^L A_{k+1}$ and $B_k \geq^L B_{k+1}$.*

Proof. Combine Proposition 3 and Proposition 1(i).

Remark 3. Counterexamples can be constructed to show that the converse of Proposition 3 does not hold.

Since likelihood ratio ordering is of some relevance to our results, we point out the fact that Gamma, Poisson, and Beta distributions, and others, are all likelihood ratio ordered according to their parameter values (see Ferguson 1967, p. 209). In an attempt to construct likelihood ratio ordered random variables from known ones, we also found two isolated preservation results; they will be given in the Appendix.

Proposition 3 resembles a notion of uniform conditional stochastic order defined and discussed in Keilson and Sumita (1982) and Whitt (1980); $(X \mid X + Y = s) \leq^{st} (Y \mid X + Y = s)$ for all s, however, is not a special case of the uniform conditional stochastic order discussed in their papers.

We now turn our attention to the problem of finding an optimal schedule. To establish that a job sequence $(J_1, J_2, \ldots, J_n)$ is stochastically optimal, verifying that (4) holds for all $1 \leq k \leq n - 1$ is by itself not sufficient; it is sufficient if the ordering defined by (4) is transitive (see Baker 1974, p. 44, for a discussion and additional references). Unfortunately, (4) does not in general define a transitive ordering; counterexamples can be readily constructed. Therefore, the reader should be aware of the necessity of giving a separate transitivity argument when applying Theorem 1. As an example, suppose that $A_1 \leq^L A_2 \leq^L \cdots \leq^L A_n$ and $B_1 \geq^L B_2 \geq^L \cdots \geq^L B_n$. As is known and easily shown, likelihood ratio ordering is transitive; therefore, Corollary 1 implies that the makespan under the job sequence (J_1, $J_2, \ldots, J_n$) is stochastically minimal. This result is also contained in Frostig and Adiri's paper.

2. An Extension to Job Shops

A 2-machine job shop model can be described as follows. There are two sets of jobs, Γ_1 and Γ_2 with $|\Gamma_1| = m$ and $|\Gamma_2| = n$, available for processing. The jobs in $\Gamma_1(\Gamma_2)$ require processing first on machine 1(2) and then on machine 2(1). Let A_i and B_i (C_i and D_i) denote the processing times of job i in set $\Gamma_1(\Gamma_2)$ on machines 1 and 2 (2 and 1), respectively. We assume the processing times are independent random variables. Let $J_1, J_2, \ldots, J_m$ be a permutation of jobs in Γ_1 and let $K_1, K_2, \ldots, K_n$ be a permutation of jobs in Γ_2. We will consider schedules of the following type:

On machine 1: process jobs in Γ_1 according to a permutation $J_1, J_2, \ldots, J_m$ first and then jobs from Γ_2 in any order.

On machine 2: process jobs in Γ_2 according to a permutation $K_1, K_2, \ldots, K_m$ first and then jobs from Γ_1 in any order.

Remark 4. Jackson (1956) studied the job shop problem with deterministic processing times. He observed that one can restrict attention to this class of schedules and that an optimal schedule can be produced by applying Johnson's rule to Γ_1 and Γ_2 separately. As in the flow shop case, a conditioning argument, again, shows that it is sufficient to consider this class of schedules when processing times are random.

The problem of interest is to find a permutation J_1, $J_2, \ldots, J_m$ of Γ_1 and a permutation $K_1, K_2, \ldots, K_n$ of Γ_2 for which the makespan (time to system emptiness) is *stochastically* minimal. The purpose of this section is to point out that the same method of analysis employed in Section 1 is also applicable to the more general job shop model.

Pinedo (1981) considered job shops with exponentially distributed processing times. He proved that the *dynamic* policy of always assigning jobs in Γ_1 and Γ_2 according to Talwar's rule minimizes the *expected* makespan. Since, for exponential processing times, the optimality of an initial permutation schedule implies that the same policy is also dynamically optimal, Pinedo's result is a special case of Theorem 2, to be stated next.

Again, we examine the effect on the makespan of interchanging two adjacent jobs in either Γ_1 or Γ_2. Consider two sets of permutations:

P_1: $J_1, J_2, \ldots, J_k, J_{k+1}, \ldots, J_m$ for Γ_1 and $K_1, K_2, \ldots, K_n$ for Γ_2

P_2: $J_1, J_2, \ldots, J_{k+1}, J_k, \ldots, J_m$ for Γ_1 and $K_1, K_2, \ldots, K_n$ for Γ_2

for some $1 \leqslant k \leqslant m - 1$. As in Section 1, let

$T = \sum_{i=1}^{k-1} A_i$,

L = additional time after T necessary for machine 2 to complete the processing of jobs $J_1, \ldots, J_{k-1}$ and jobs in Γ_2,

$A = A_k + A_{k+1}$,

$R_1(R_2)$ = additional time necessary to complete *all* processing requirements at machine 2 after time $T + A$ under $P_1(P_2)$,

$\bar{R}_1(\bar{R}_2)$ = additional time necessary to complete *all* processing requirements at machine 1 after time $T + A$ under $P_1(P_2)$.

The first important observation we want to make is that $\bar{R}_1 = \bar{R}_2$ with probability 1 and hence, from now on, we will denote both by $\bar{R}$. It then follows that the makespans, M_1 and M_2, under P_1 and P_2 are given by

$$M_i = T + A + \max[\bar{R}, R_i] \tag{9}$$

for $i = 1, 2$. We now state and prove the main result in this section.

Theorem 2. *In a job shop model,* $M_1 \leqslant^{st} M_2$ *if Condition* 4 *in Theorem* 1 *is satisfied.*

Proof. The key observation is that, in addition to L, A, and $B_k + B_{k+1}$, we also need to condition on the values of T. From (9), we then have (for all t, ℓ, a, $b \geqslant 0$):

$$\begin{aligned}(M_i \mid T = t, L = \ell, A = a, B_k + B_{k+1} = b) \\ = t + a \\ + (\max[\bar{R}, R_i] \mid T = t, \\ L = \ell, A = a, B_k + B_{k+1} = b).\end{aligned}$$

Note that $\bar{R}$ and R_i are *not* conditionally independent because R_i might still depend on departure times of jobs in Γ_1 from machine 1 after time $t + a$. To remove this dependence, we need further conditioning. Conditioning on

$$\bar{T} \equiv \sum_{i=k+2}^{m} A_i = \bar{t}(\geqslant 0),$$

we have

$$\begin{aligned}(\max[\bar{R},R_1] \mid T=t, L=\ell, A=a, B_k+B_{k+1}=b, \text{and } \bar{T}=\bar{t})\\ =\max\begin{bmatrix}(\bar{R}\mid T=t, L=\ell, A=a, \text{and } \bar{T}=\bar{t}),\\ (R_1\mid L=\ell, A=a, B_k+B_{k+1}=b, \text{and } \bar{T}=\bar{t})\end{bmatrix}\\ \leqslant^{\text{st}}\max\begin{bmatrix}(\bar{R}\mid T=t, L=\ell, A=a, \text{and } \bar{T}=\bar{t}),\\ (R_2\mid L=\ell, A=a, B_k+B_{k+1}=b, \text{and } \bar{T}=\bar{t})\end{bmatrix}\\ =(\max[\bar{R},R_2] \mid T=t, L=\ell, A=a, B_k+B_{k+1}=b,\\ \text{and } \bar{T}=\bar{t})\end{aligned}$$

since $\bar{R}$ and R_i are *now* conditionally independent, and, furthermore,

$$\begin{aligned}(R_1 \mid L=\ell, A=a, B_k+B_{k+1}=b, \text{and } \bar{T}=\bar{t})\\ \leqslant^{\text{st}}(R_2 \mid L=\ell, A=a, B_k+B_{k+1}=b, \text{and } \bar{T}=\bar{t})\end{aligned}$$

by the same argument used in the proof of Theorem 1. The proof is now completed by successive unconditioning.

Appendix

We present two preservation properties of likelihood ratio ordering (Karlin and Rinott 1980 obtain more general results), and give an example of how these results can be helpful (see Corollary 1 of this paper) in recognizing situations when random variables are likelihood ratio ordered.

Proposition A.1. *Let $f_1, f_2, \ldots, f_n$ be a collection of density functions satisfying $f_1 \leqslant^L f_2 \leqslant^L \cdots \leqslant^L f_n$ (defined according to (7)) and let $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\boldsymbol{\beta} \equiv (\beta_1, \beta_2, \ldots, \beta_n)$ be two probability vectors. Suppose that $\boldsymbol{\alpha}$ is smaller than $\boldsymbol{\beta}$ in the sense of discrete likelihood ratio, i.e.,*

$$\beta_i/\alpha_i \leqslant \beta_j/\alpha_j$$

for all $1 \leqslant i \leqslant j \leqslant n$. Then

$$\sum_{i=1}^n \alpha_i f_i \leqslant^L \sum_{i=1}^n \beta_i f_i.$$

Proof. We need to establish

$$\frac{\sum_{i=1}^n \beta_i f_i(x)}{\sum_{i=1}^n \alpha_i f_i(x)} \leqslant \frac{\sum_{i=1}^n \beta_i f_i(y)}{\sum_{i=1}^n \alpha_i f_i(y)}$$

for all $0 \leqslant x \leqslant y$. Multiplying by the denominators and canceling equal terms shows that this inequality is equivalent to

$$\sum_{i=1}^n \sum_{\substack{j=1\\ j\neq i}}^n \beta_i\alpha_j f_i(x) f_j(y) \leqslant \sum_{i=1}^n \sum_{\substack{j=1\\ j\neq i}}^n \beta_i\alpha_j f_i(y) f_j(x),$$

or

$$\begin{aligned}\sum_{i=1}^n \sum_{\substack{j=1\\ j>i}}^n [\beta_i\alpha_j f_i(x) f_j(y) + \beta_j\alpha_i f_j(x) f_i(y)]\\ \leqslant \sum_{i=1}^n \sum_{\substack{j=1\\ j>i}}^n [\beta_i\alpha_j f_i(y) f_j(x) + \beta_j\alpha_i f_j(y) f_i(x)].\end{aligned}$$

Now, for each fixed pair (i, j) with $i < j$, we have

$$\begin{aligned}\beta_i\alpha_j f_i(y) f_j(x) + \beta_j\alpha_i f_j(y) f_i(x)\\ - \beta_i\alpha_j f_i(x) f_j(y) - \beta_j\alpha_i f_j(x) f_i(y)\\ = (\beta_i\alpha_j - \beta_j\alpha_i)(f_i(y) f_j(x) - f_i(x) f_j(y))\end{aligned}$$

which is nonnegative because both terms are nonpositive by assumption.

Proposition A.2. *Let X_1, X_2 and Y be three nonnegative random variables, where Y is independent of both X_1 and X_2; also let $X_1(X_2)$ have density function $f_1(f_2)$ and Y have density function g. Then $X_1 \leqslant^L X_2$ and g is log-concave imply that $X_1 + Y \leqslant^L X_2 + Y$.*

Proof. We need to establish

$$\frac{\int_0^\infty g(t-x) f_2(x)\,dx}{\int_0^\infty g(t-x) f_1(x)\,dx} \geqslant \frac{\int_0^\infty g(s-x) f_2(x)\,dx}{\int_0^\infty g(s-x) f_1(x)\,dx}$$

for all $0 \leqslant s \leqslant t$, or equivalently,

$$\begin{vmatrix}\int_0^\infty g(s-x) f_1(x)\,dx & \int_0^\infty g(s-x) f_2(x)\,dx\\ \int_0^\infty g(t-x) f_1(x)\,dx & \int_0^\infty g(t-x) f_2(x)\,dx\end{vmatrix} \geqslant 0.$$

Next, by the well-known basic composition formula (Karlin 1968, p. 17), the left-hand side is equal to

$$\iint_{x_1<x_2} \begin{vmatrix} g(s-x_1) & g(s-x_2)\\ g(t-x_1) & g(t-x_2)\end{vmatrix}\begin{vmatrix} f_1(x_1) & f_2(x_1)\\ f_1(x_2) & f_2(x_2)\end{vmatrix} dx_2\,dx_2.$$

The conclusion now follows if we note that the first determinant is nonnegative since g is log-concave, and that the second determinant is nonnegative since $X_1 \leqslant^L X_2$.

Corollary A.1. *If $X_1 \leqslant^L Y_1$ and $X_2 \leqslant^L Y_2$, where X_1 is independent of X_2 and Y_1 is independent of Y_2, then* *the following statements are true*:

(i) *If* X_1 *and* Y_2 *have log-concave densities, then* $X_1 + X_2 \leq^L Y_1 + Y_2$.
(ii) *If* X_2 *and* Y_1 *have log-concave densities, then* $X_1 + X_2 \leq^L Y_1 + Y_2$.

Proof. The following chain of inequalities establishes (i):

$$X_1 + X_2 \leq^L X_1 + Y_2 \leq^L X_2 + Y_2.$$

The proof of (ii) is similar.

Example A.1. Let $E(\lambda_1, \lambda_2, \ldots, \lambda_n)$ denote the convolution of n exponential distributions with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively; assume without loss of generality that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Since exponential densities are log-concave, Corollary A.1 implies that $E(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq^L E(\mu_1, \mu_2, \ldots, \mu_n)$ (defined in an obvious way) whenever $\lambda_i \geq \mu_i$ for $i = 1, 2, \ldots, n$. Next, for $0 \leq q \leq p \leq 1$ and $p + q = 1$, we have, from Proposition A.1,

$$pE(\lambda_1, \lambda_2, \ldots, \lambda_n) + qE(\mu_1, \mu_2, \ldots, \mu_n)$$
$$\leq^L qE(\lambda_1, \lambda_2, \ldots, \lambda_n) + pE(\mu_1, \mu_2, \ldots, \mu_n).$$

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