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Analyzing household brand switching: A stochastic model

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Abstract: In a recent study, Kahn, Morrison and Wright showed that for exponentially distributed interpurchase times (of individuals), the household choice process approaches a zero-order process as the number of individuals in the household increases, even though the purchasing behavior of each individual is first order. We show that their result holds for any arbitrary interpurchase-time distribution that has a density over some interval. Thus, additional support is provided for their conclusion that the empirically observed zero-order choice behavior at the household level may not convey much information about individuals' choice behavior. We also derive a general formula that determines exactly the degree of dependence between two successive household purchases for any given family size and (well-behaved) interpurchase-time distribution.

Keywords: Interpurchase time; New better than used in expectation; Panel data; Renewal theory; Zero-order process

1. Introduction

Brand choice is an important aspect of the purchasing behavior of a household. Generally, a buyer has to choose one of several brands existing in the market place on successive purchase occasions. On each occasion, the selected brand may either depend on or be completely independent of choices made on previous occasions. Empirical examinations of households' successive purchases of specific products reveal that frequent brand switching is typical for frequently-purchased, low-priced products.

Many researchers have proposed stochastic models for analyzing the multibrand purchasing behavior of households for frequently purchased products (e.g., Ehrenberg, 1972; Bass, 1974; Wierenga, 1974; Bass, Jeuland and Wright, 1976; Goodhardt, Ehrenberg and Chatfield, 1984). A key element in the development of such models has been the extensive use of panel data, a longitudinal history of household purchases. An excellent summary of studies using panel data to study the multibrand buying behavior has been given in Bass et al. (1984, p. 270). These studies find that the purchasing behavior of a substantial number of households in the panel on a given occasion is independent of their past purchases. Such independence of choice probability is often termed as constituting a zero-order process of multibrand buying.

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An important fact concerning the use of panel data in studying multibrand buying is that the unit of observation is a *household* and not a particular individual. In other words, the panel data is an aggregation of all family members' purchase histories. Hence, an interesting question in connection with the use of panel data is: "To what extent does household purchasing behavior depend on that of individual family members?". This question can be stated differently as: "What is the effect of aggregating individual family members' purchasing behavior to the household level?". Kahn, Morrison and Wright (1986), herein after referred to as KMW, recently formulated a model to answer this question. They consider a household consisting of individuals who have first order (Markovian) purchasing behavior and whose interpurchase times are independent and identically distributed (i.i.d.) exponential random variables. Their main result shows that for exponentially distributed interpurchase times, the extent of dependence between successive purchases decreases as the number of individuals in the household increases.

An intuitive explanation of KMW's result is the following: As the number of individuals in a household increases, the likelihood for two successive purchases made by the household to be from the same individual decreases and therefore, since the brand choice processes for different individuals are assumed to be independent, the degree of dependence between successive *household* purchases must also decrease. In fact, one might conjecture that the above explanation should be valid for any 'well-behaved' interpurchase-time distribution and not necessarily only for the exponential distribution. Various studies have identified situations where it would be more appropriate to use distributions other than exponential for modeling interpurchase times (e.g., see Herniter, 1971; Banerjee and Bhattacharya, 1976; Lawrence, 1980; Dunn, Reader and Wrigley, 1983). One major purpose of this paper is to show that the above-stated conjecture is indeed true and therefore our results provide additional support for KMW's conclusion.

Our analyses rely on renewal theoretic arguments. We first uncover an interesting probabilistic interpretation for the measure that was used by KMW to study the extent to which the household brand choice process is a zero-order process. This interpretation naturally leads to a general formula for the degree of dependence between two successive household purchases. We also bound the speed at which the aggregated household brand choice process approaches a zero-order process for the wide class of NBUE (New Better than Used in Expectation) interpurchase-time distributions. In fact, we show that the speed of convergence to zero in this general case is no worse than the speed of convergence when the interpurchase times are exponentially distributed. Finally, we provide a probabilistic proof of the conjecture stated in the previous paragraph.

The rest of the paper is organized as follows: Section 2 contains a detailed description of the model. Section 3 presents the main results in the form of four theorems. Finally, Section 4 contains some discussion.

2. The model

Consider a household consisting of n individuals indexed by $i = 1, 2, \ldots, n$. We assume that the purchase behaviors for different individuals in the household are independent and that each individual has i.i.d. interpurchase times with distribution function F and finite mean λ. To avoid pathological situations, we assume in addition that F has a density over some interval.

The market is viewed as a two brand market, indexed by $j = 0, 1$. The brand under consideration is denoted by 1 and the other brand(s) by 0. We further assume that individual i makes his/her purchases independently according to a first-order brand choice process, i.e., a two-state Markov chain $\{X_k^i,\ k \geq 1\}$, where X_k^i denotes the brand bought by individual i on the k-th purchase occasion, $i = 1, 2, \ldots, n$. Let the i-th individual's transition probability matrix P_i be of the form

$$P_i = \begin{bmatrix} P_i(1\,|\,1) & P_i(0\,|\,1) \\ P_i(1\,|\,0) & P_i(0\,|\,0) \end{bmatrix}$$

where $P_i(\ell \mid j)$ is the conditional probability that the i-th individual chooses brand ℓ on his $(k+1)$-st purchase occasion given that he chose brand j on the k-th occasion, $\ell = 0, 1$ and $j = 0, 1$. The stationary probabilities corresponding to P_i will be denoted by π_{ji}, $j = 0, 1$.

Similarly, at the household level, let $\{Z_k, k \geq 1\}$ denote the household's brand choice process where

$$Z_k = \begin{cases} 1 & \text{if brand 1 is bought at the } k\text{-th household purchase occasion,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the process $\{Z_k, k \geq 1\}$ is an aggregation of individual brand choice processes $\{X_k^i, k \geq 1\}$, $i = 1, 2, \ldots, n$, and thus, it is in general *not* a Markov chain. However, for this process there exists a transition rate (rather than probability) matrix P, where

$$P = \begin{bmatrix} P(1 \mid 1) & P(0 \mid 1) \\ P(1 \mid 0) & P(0 \mid 0) \end{bmatrix}$$

where $P(\ell \mid j)$ denotes the transition rate from state j to state ℓ for $j = 0, 1$ and $\ell = 0, 1$; and the transition rate $P(\ell \mid j)$ is defined as the proportion of times the process next enters state ℓ after leaving state j.

Now, if the household brand switching process were zero-order, i.e., successive brand choices were i.i.d., then the rows of P would be identical, implying that $P(1 \mid 1)$-$P(1 \mid 0)$. KMW study a quantity D defined by

$$D = P(1 \mid 1) - P(1 \mid 0)$$

to measure the degree of dependence between two consecutive purchases at the household level. Since a more natural measure of dependence between successive purchases is the correlation between Z_k and Z_{k+1}, we shall show in Section 3 that the measure D is, in fact, the correlation coefficient between Z_k and Z_{k+1}, and we shall examine the behavior of D as the number of individuals in the household increases. In the remainder of this section, we derive expressions that relate the elements of P to that of P_i for $i = 1, 2, \ldots, n$ *and* the interpurchase-time distribution F, needed for our analysis in Section 3.

First, we observe that at any purchase occasion *in steady state,* the distribution of the random variable Y_e representing the time to next purchase for all individuals *other than* the one making the current purchase is given by the equilibrium distribution F_e (the assumption of F having a density over some interval is needed here), defined for $t \geq 0$ by

$$F_e(t) = \frac{1}{\lambda} \int_0^t [1 - F(u)] \, du$$

(see Ross, 1983, p. 76). We note that Y and Y_e are drawn independently from two different distributions of interpurchase times F and F_e respectively.

To facilitate understanding we begin by considering the case of two individuals. For expositional clarity, we denote the two individuals by A and B. If a person makes two consecutive purchases, then his/her interpurchase time Y is less than the other person's equilibrium interpurchase time Y_e. On the other hand, if the person making the purchase on the second occasion is different from the one making the first, then Y is greater than Y_e.

Now although $P(1 \mid 1)$ is a transition rate, it can be viewed as a conditional 'probability', so that

$$P(1 \mid 1) = P(Z_{k+1} = 1 \mid Z_k = 1) = P(Z_{k+1} = 1, Z_k = 1)/P(Z_k = 1),$$

where the 'probabilities' $P\{Z_{k+1} = 1, Z_k = 1\}$ and $P\{Z_k = 1\}$ are limiting proportions defined by

$$P\{Z_k = 1\} = \lim_{m \to \infty} \frac{1}{m} \sum_{h=1}^{m} I_{\{Z_h = 1\}}$$

and

$$P\{Z_k = 1,\ Z_{k+1} = 1\} = \lim_{m \to \infty} \frac{1}{m} \sum_{h=1}^{m} I_{\{Z_h = 1,\, Z_{h+1} = 1\}}$$

where I_S denotes the indicator function of a given event S, i.e.,

$$I_{\{Z_i = 1\}} = \begin{cases} 1 & \text{if } Z_i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and where

$$I_{\{Z_i = 1,\, Z_{i+1} = 1\}} = \begin{cases} 1 & \text{if } Z_i = 1 \text{ and } Z_{i+1} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

To calculate the numerator of $P(1 \mid 1)$, we consider four mutually exclusive events as follows:

Event:	*Probability:*
A makes two consecutive purchases	$\frac{1}{2}\pi_{1A} P(Y < Y_e)\ P_A(1 \mid 1)$,
B makes two consecutive purchases	$\frac{1}{2}\pi_{1B} P(Y < Y_e)\ P_B(1 \mid 1)$,
A buys first and B buys next	$\frac{1}{2}\pi_{1A} P(Y > Y_e)\ \pi_{1B}$,
B buys first and A buys next	$\frac{1}{2}\pi_{1B} P(Y > Y_e)\ \pi_{1A}$.

For example, the probability of the first event is obtained as follows: The factor $\frac{1}{2}$ is due to the fact that A purchases half of the time; π_{1A} is the steady state probability of A buying brand 1 (it can be viewed as the long-run proportion of times individual A purchases brand 1); $P(Y < Y_e)$ is the probability that A makes two consecutive purchases, and $P_A(1/1)$ is the probability that A purchases brand 1 again. The probabilities for the other events are obtained similarly.

Noting that $P(Z_k = 1) = \frac{1}{2}(\pi_{1A} + \pi_{1B})$, we therefore have

$$P(1 \mid 1) = \frac{[\pi_{1A} P_A(1 \mid 1) + \pi_{1B} P_B(1 \mid 1)] P(Y < Y_e) + 2\pi_{1A}\pi_{1B} P(Y > Y_e)}{\pi_{1A} + \pi_{1B}}. \tag{2.1}$$

Similarly,

$$P(1 \mid 0) = \frac{[\pi_{0A} P_A(1 \mid 0) + \pi_{0B} P_B(1 \mid 0)] P(Y < Y_e) + [\pi_{0A}\pi_{1B} + \pi_{0B}\pi_{1A}] P(Y > Y_e)}{\pi_{0A} + \pi_{0B}}. \tag{2.2}$$

We now outline the derivations of $P(\ell \mid j)$'s for the general case of n individuals, $n \geq 2$. In order for the same individual to make two consecutive purchases, his interpurchase time Y must be less than $Y_{e,n-1}$, the minimum of $n-1$ independent random variables each distributed as Y_e. Denote the probability of this event by α_{n-1}; i.e., let $\alpha_{n-1} = P(Y < Y_{e,n-1})$.

Similarly, if two different persons make the purchases then Y must be greater than $Y_{e,n-1}$. Consequently, the probability of this event will be $1 - \alpha_{n-1}$. Moreover, given that the individual who makes the first purchase does not make the next purchase, it is equally likely for the next purchase to be made by any of the remaining $n-1$ individuals. Following this argument we get

$$P(1 \mid 1) = \frac{\alpha_{n-1}\left[\sum_{i=1}^{n} \pi_{1i} P_i(1 \mid 1)\right] + \frac{1}{n-1}(1 - \alpha_{n-1})\left[\sum_{i \neq j} \pi_{1i}\pi_{1j}\right]}{\sum_{i=1}^{n} \pi_{1i}} \tag{2.3}$$

and

$$P(1|0) = \frac{\alpha_{n-1}\left[\sum_{i=1}^{n} \pi_{0i} P_i(1|0)\right] + \frac{1}{n-1}(1-\alpha_{n-1})\left[\sum_{i \neq j} \pi_{0i}\pi_{1j}\right]}{\sum_{i=1}^{n} \pi_{0i}}. \tag{2.4}$$

Expressions (2.3) and (2.4) are generalizations of related formulas in KMW, in that they are derived for a general interpurchase-time distribution. To evaluate $P(1|1)$ and $P(1|0)$ further, we must make specific assumptions on the interpurchase-time distribution F. We now give the following examples.

Example 1. Exponential interpurchase times: $F(t) = 1 - \exp(-t/\lambda)$, $\lambda > 0$.

For F exponential, it is easy to show that $Y_{e,n-1}$ is exponentially distributed with parameter $1/[\lambda(n-1)]$. Hence

$$\alpha_{n-1} = \lambda / [\lambda + \lambda(n-1)] = 1/n.$$

Substituting (2.5) into (2.3) and (2.4), we have

$$P(1|1) = (1/n)\left[\sum_{i=1}^{n} \pi_{1i} P_i(1|1) + \sum_{i \neq j} \pi_{1i}\pi_{1j}\right] \Big/ \sum_{i=1}^{n} \pi_{1i}$$

and

$$P(1|0) = (1/n)\left[\sum_{i=1}^{n} \pi_{0i} P_i(1|0) + \sum_{i \neq j} \pi_{0i}\pi_{ij}\right] \Big/ \sum_{i=1}^{n} \pi_{0i}.$$

Expressions (2.6) and (2.7) agree with equations (4) and (5) in KMW (p. 267).

Example 2. Erlang interpurchase times: $Y \sim \text{Erlang}(M, M/\lambda)$, $M \geq 1$.

Consider the case of two individuals. Since Y can be viewed as a sum of M i.i.d. exponentially distributed 'phases', it is not difficult to see (e.g., Ross, 1985, p. 209) that

$$\begin{aligned} P(Y > Y_e) &= \sum_{j=1}^{M} P(Y > Y_e \mid Y_e \text{ is in phase } j)\, P(Y_e \text{ is in phase } j) \\ &= \sum_{j=1}^{M} \sum_{i=j}^{M+j-1} \binom{M+j-1}{i} \left(\frac{1}{2}\right)^{M+j-1} \left(\frac{1}{M}\right). \end{aligned}$$

For example, if $M = 2$, then $P(Y > Y_e) = \frac{5}{8}$ and $P(Y < Y_e) = \frac{3}{8}$. The above expression can then be substituted into (2.1) and (2.2) to find D, given the transition probabilities. The coefficient of variation for the Erlang family is $1/\sqrt{M}$ which implies that it is less variable than the exponential distribution, for which the coefficient of variation is one.

Example 3. Hyperexponential interpurchase times: $F(t) = \sum_{i=1}^{M} a_i F_i(t)$ where $F_i(t) = 1 - \exp(-t/\lambda_i)$, a_i's are nonnegative numbers satisfying $\sum_{i=1}^{M} a_i = 1$, and λ_i's are positive real numbers.

From the definition of F_e, we have

$$\begin{aligned} P(Y_e > y) &= \left\{ \int_y^{\infty} \sum_{i=1}^{M} a_i \exp(-t/\lambda_i)\, \mathrm{d}t \Big/ \sum_{i=1}^{M} a_i \lambda_i \right\} \\ &= \sum_{i=1}^{M} a_i \lambda_i \exp(-y/\lambda_i) \Big/ \sum_{i=1}^{M} a_i \lambda_i = \sum_{i=1}^{M} b_i \exp(-y/\lambda_i), \end{aligned}$$

where $b_i = (a_i\lambda_i)/\sum_{i=1}^{M} a_i\lambda_i$, $i = 1, 2, \ldots, M$. Therefore, by conditioning on the 'active components' of Y_e and Y, we obtain

$$P(Y_e < Y) = \sum_{i=1}^{M}\sum_{j=1}^{M}\left(\frac{\lambda_j}{\lambda_i + \lambda_j}\right)a_i b_j.$$

We can now obtain an explicit expression for D for two individuals. Generalization to the case of n individuals is straightforward and thus omitted.

Hyperexponential distributions can be used to model situations where the random variables under consideration are believed to be more 'variable' than the exponential, e.g., when the coefficient of variation is greater than 1. (We are not aware of any documented use of this distribution in the marketing literature.)

3. Main results

In this section, we present our main results in the form of four theorems. Theorem 1 gives an interesting probabilistic interpretation for D (defined in the previous section). Theorem 2 gives an explicit formula for D for any given family size n and (well-behaved) interpurchase-time distribution F. Theorem 3 establishes a bound on D for the wide class of NBUE interpurchase-time distributions. And finally, Theorem 4 deals with the limiting behavior of D as the number of individuals in the household increases.

Theorem 1. *D is equal to the correlation coefficient between Z_k and Z_{k+1}. That is,*

$$D = \rho(Z_k, Z_{k+1}), \tag{3.1}$$

where $\rho(Z_k, Z_{k+1})$ is the correlation coefficient between Z_k and Z_{k+1}, given by

$$\rho(Z_k, Z_{k+1}) = \mathrm{Cov}(Z_k, Z_{k+1})/\mathrm{Var}(Z_k).$$

Proof. To simplify notation, we shall, for any $k \geq 1$, let $P(j) = P(Z_k = j)$ for $j = 0, 1$ and $P(i, j) = P(Z_{k+1} = i, Z_k = j)$ for $i = 0, 1$; $j = 0, 1$.
By definition,

$$E(Z_k) = 1\cdot P(Z_k = 1) + 0\cdot P(Z_k = 0) = P(Z_k = 1) = P(1) \tag{3.2}$$

and

$$\begin{aligned} E(Z_k Z_{k+1}) &= 1\cdot P(Z_k = 1, Z_{k+1} = 1) + 0\cdot\big[P(Z_k = 1, Z_{k+1} = 0) + P(Z_k = 0, Z_{k+1} = 1) \\ &\qquad + P(Z_k = 0, Z_{k+1} = 0)\big] \\ &= P(Z_k = 1, Z_{k+1} = 1) = P(1, 1). \end{aligned} \tag{3.3}$$

The variance of Z_k, by definition, is

$$\begin{aligned} \mathrm{Var}(Z_k) &= E(Z_k^2) - [E(Z_k)]^2 = E(Z_k) - [E(Z_k)]^2 = E(Z_k)[1 - E(Z_k)] \\ &= P(1)[1 - P(1)] = P(1)P(0). \end{aligned} \tag{3.4}$$

Also, by stationarity, $E(Z_{k+1}) = E(Z_k)$ and $\mathrm{Var}(Z_{k+1}) = \mathrm{Var}(Z_k)$. The quantity D can then be expressed as

$$\begin{aligned} D &= P(1\,|\,1) - P(1\,|\,0) = \frac{P(0)P(1,1) - P(1)P(1,0)}{P(0)P(1)} \\ &= \frac{[1-P(1)]P(1,1) - P(1)P(1,0)}{P(0)P(1)} = \frac{P(1,1) - P(1)[P(1,1)+P(1,0)]}{P(0)P(1)} \\ &= \frac{P(1,1) - P(1)P(1)}{P(0)P(1)} = \frac{P(Z_k=1,\, Z_{k+1}=1) - P(Z_k=1)P(Z_{k+1}=1)}{[P(Z_k=1)P(Z_k=0)]^{1/2}[P(Z_{k+1}=1)P(Z_{k+1}=0)]^{1/2}} \\ &= \frac{E(Z_k Z_{k+1}) - E(Z_k)E(Z_{k+1})}{\left[E(Z_k^2) - (E(Z_k))^2\right]^{1/2}\left[E(Z_{k+1}^2) - (E(Z_{k+1}))^2\right]^{1/2}} \\ &= \frac{\mathrm{Cov}(Z_k,\, Z_{k+1})}{(\mathrm{Var}(Z_k))^{1/2}(\mathrm{Var}(Z_{k+1}))^{1/2}} = \rho(Z_k,\, Z_{k+1}), \end{aligned}$$

completing our proof.

Theorem 1 is a very general result; it depends neither on the interpurchase-time distribution nor on the number of individuals in the household. Of particular importance is the fact that it enables us to obtain an explicit expression for D, stated in the next theorem, via a probabilistic argument.

Theorem 2.

$$D = \left\{ \frac{1}{n}\alpha_{n-1}\left[\sum_{i=1}^{n} \mathrm{Cov}(X_k^i,\, X_{k+1}^i) + \sum_{i=1}^{n} \pi_{1i}^2\right] + \frac{1}{n(n-1)}(1-\alpha_{n-1})\left[\sum_{i\neq j} \pi_{1i}\pi_{1j}\right] - \frac{1}{n^2}\left[\sum_{i=1}^{n} \pi_{1i}\right]^2 \right\} \Big/ \mathrm{Var}(Z_k). \tag{3.5}$$

Proof. We begin by evaluating (3.1). The covariance term $\mathrm{Cov}(Z_k,\, Z_{k+1})$ is calculated by conditioning on the pair of individuals who make the k-th and $(k+1)$-st purchases which we denote by $(I_k,\, I_{k+1})$, where I_k (I_{k+1}) is equal to i if the i-th individual makes the k-th (($k+1$)-st) purchase, $i = 1, 2, \dots, n$. For the case of 2 individuals, say A and B, the pairs would be (A, A), (A, B), (B, A), and (B, B). In general, there would be n^2 possible pairs of $(I_k,\, I_{k+1})$.

First, we find the joint distribution of $(I_k,\, I_{k+1})$. Using the argument presented in Section 2 for deriving (2.3) and (2.4), we have

$$P(I_{k+1}=i,\, I_k=i) = P(I_k=i)P(I_{k+1}=i\,|\,I_k=i) = (1/n)\alpha_{n-1} \tag{3.6}$$

and

$$\begin{aligned} P(I_{k+1}=j,\, I_k=i) &= P(I_k=i)P(I_{k+1}\neq i\,|\,I_k=i)P(I_{k+1}=j\,|\,I_k=i,\, I_{k+1}\neq i) \\ &= (1/n)(1-\alpha_{n-1})[1/(n-1)] \quad \text{for } i\neq j. \end{aligned} \tag{3.7}$$

It then follows from a well-known conditional covariance formula (Barlow and Proschan, 1975, p. 30) that

$$\mathrm{Cov}(Z_k,\, Z_{k+1}) = E[\mathrm{Cov}(Z_k,\, Z_{k+1}\,|\,I_k,\, I_{k+1})] + \mathrm{Cov}[E(Z_k\,|\,I_k,\, I_{k+1}),\, E(Z_{k+1}\,|\,I_k,\, I_{k+1})]. \tag{3.8}$$

Noting that the two processes $\{X_k^i,\ k \geq 1\}$ and $\{X_k^j,\ k \geq 1\}$ are independent when $i \neq j$, the first term above can be evaluated as

$$\begin{aligned} E[\text{Cov}(Z_k, Z_{k+1}|I_k, I_{k+1})] &= \sum_{i=1}^{n} \text{Cov}(X_k^i, Z_{k+1}^i) P(I_k = i, I_{k+1} = i) \\ &\quad + \sum_{i \neq j} \text{Cov}(X_k^i, X_{k+1}^j) P(I_k = i, I_{k+1} = j) \\ &= \sum_{i=1}^{n} \text{Cov}(X_k^i, X_{k+1}^i) P(I_k = i, I_{k+1} = i). \end{aligned}$$

Substituting (3.6) into the above expression, we get

$$E[\text{Cov}(Z_k, Z_{k+1}|I_k, I_{k+1})] = \frac{1}{n}\alpha_{n-1}\left[\sum_{i=1}^{n} \text{Cov}(X_k^i, X_{k+1}^i)\right]. \tag{3.9}$$

Next, by definition of covariance and properties of conditional expectations, the second term in (3.8) is equal to

$$E[E(Z_k|I_k)E(Z_{k+1}|I_{k+1})] - E(Z_k)E(Z_{k+1}), \tag{3.10}$$

where

$$E(Z_{k+1}) = E(Z_k) = P(Z_k = 1) = \sum_{i=1}^{n} P(Z_k = 1|I_k = i)P(I_k = i) = \left[\sum_{i=1}^{n} \pi_{1i}\right]\left(\frac{1}{n}\right). \tag{3.11}$$

Again, the first term in (3.10) is evaluated by conditioning on possible pairs of (I_k, I_{k+1}) and using (3.6) and (3.7); it reduces, after some algebra, to

$$\frac{1}{n}\alpha_{n-1}\left[\sum_{i=1}^{n} \pi_{1i}^2\right] + \frac{1}{n(n-1)}(1 - \alpha_{n-1})\left[\sum_{i \neq j} \pi_{1i}\pi_{1j}\right]. \tag{3.12}$$

Using (3.9), (3.11) and (3.12), (3.8) can now be written as

$$\begin{aligned} \text{Cov}(Z_k, Z_{k+1}) &= \frac{1}{n}\alpha_{n-1}\left[\sum_{i=1}^{n} \text{Cov}(X_k^i, X_{k+1}^i) + \sum_{i=1}^{n} \pi_{1i}^2\right] \\ &\quad + \frac{1}{n(n-1)}(1 - \alpha_{n-1})\left[\sum_{i \neq j} \pi_{1i}\pi_{1j}\right] - \frac{1}{n^2}\left(\sum_{i=1}^{n} \pi_{1i}\right)^2. \end{aligned} \tag{3.13}$$

Finally, substitution of (3.13) into (3.1) yields (3.5); and the proof is complete.

For example, if F is exponential then $\alpha_{n-1} = 1/n$ and using this value of α_{n-1} in (3.5), we get after some algebra the following results:

$$\text{Cov}(Z_k, Z_{k+1}) = \frac{1}{n^2} \sum_{i=1}^{n} \text{Cov}(X_k^i, X_{k+1}^i) \tag{3.14}$$

and

$$D = \left[\sum_{i=1}^{n} \text{Cov}(X_k^i, X_{k+1}^i)\right] \Bigg/ \left[\left(\sum_{i=1}^{n} \pi_{1i}\right)\left(\sum_{i=1}^{n} \pi_{0i}\right)\right]. \tag{3.15}$$

For moderate or small family sizes and well-behaved interpurchase-time distributions, one could compute the values of D from Theorem 2 explicitly, in order to get a good feel as to how 'close' a particular household brand choice process is to zero-order process.

In view of KMW's main result that for exponentially distributed interpurchase times, $|D| \le 1/n$, it is, however, more interesting to ask the following questions:

a) Is exponential distribution a realistic assumption for the interpurchase-time distribution?
b) Is it possible to obtain similar bounds for D when the interpurchase-time distribution is not exponential?

As a response to the first question, Herniter (1971) argued that the assumption of exponentially distributed interpurchase times is perhaps not very realistic because individuals are unlikely to make another purchase immediately following a purchase. After analyzing histograms of interpurchase times for several different frequently-purchased products, he recommended that the Erlang family of density functions was more appropriate to describe individuals' interpurchase timing behavior. Chatfield and Goodhardt (1973) suggested and provided empirical support for the use of Erlang-2 as a better approximation of interpurchase-time distribution at the household level. Jeuland, Bass and Wright (1980) also advocated the superiority of the Erlang-2 distribution in describing the purchase patterns of a product category. On the other hand, Banerjee and Bhattacharya (1976) and Lawrence (1980) used the generalized inverse Gaussian distribution and the lognormal distribution respectively to describe purchase frequency rate. Dunn, Reader and Wrigley (1983) in an exhaustive study concluded that for heavy buyers of a product category, a distribution more 'regular' than exponential is appropriate. Recently, Gupta (1988) while analyzing scanner panel data for regular ground coffee, has provided empirical evidence that the interpurchase-time distribution is Erlang-2.

To answer the second question, we will show that it is possible to provide a bound for D for the very general class of NBUE (New Better than Used in Expectation) interpurchase times. The NBUE assumption can be interpreted as: the mean time to next purchase of the person making the current purchase is greater than or equal to that of any other person; i.e., Y is an NBUE random variable if and only if $E(Y-t \mid Y>t) \le E(Y)$ for all $t \ge 0$. It is important to note that Erlang and hence also exponential random variables are NBUE.

Theorem 3. *The speed of convergence of D to zero is no worse than* O$(1/n)$ *for the class of NBUE interpurchase-time distributions.*

Proof. It is well known (Ross 1983, p. 273) that Y is NBUE if and only if Y is stochastically larger than Y_e, i.e.,

$$Y \sim \text{NBUE} \Leftrightarrow P(Y>t) \ge P(Y_e > t) \quad \text{for all } t \ge 0,$$

and hence $P(Y < Y_{e,n-1}) \le P(Y_e < Y_{e,n-1}) = 1/n$. This fact, together with (3.5), implies that

$$\begin{aligned}
\limsup |D| &\le \limsup \left\{ \frac{1}{cn} \alpha_{n-1} \left| \sum_{i=1}^{n} \operatorname{Cov}\left(X_k^i, X_{k+1}^i\right) + \sum_{i=1}^{n} \pi_{1i}^2 - \frac{1}{n-1} \sum_{i \ne j} \pi_{1j}\pi_{1j} \right| + \frac{1}{c} \left| -\frac{1}{n} \sum_{i=1}^{n} \pi_{li}^2 \right| \right\} \\
&\le \limsup \left\{ \frac{1}{cn^2} \sum_{i=1}^{n} \left| \operatorname{Cov}\left(X_k^i, X_{k+1}^i\right) \right| + \frac{2}{cn^2} \sum_{i=1}^{n} \pi_{1i}^2 + \frac{1}{c(n-1)n^2} \sum_{i \ne j} \pi_{1i}\pi_{1j} \right\} \\
&= \limsup \left(\frac{1}{cn} + \frac{2}{cn} + \frac{1}{cn} \right) = \limsup \left(\frac{4}{cn} \right),
\end{aligned}$$

completing our proof.

We are now ready to state and prove the conjecture stated in the introduction.

Theorem 4. *When the number of individuals in a household increases, D converges to zero.*

Proof. First, observe that $P(Y < Y_e) < 1$ and hence $\alpha_{n-1} \to 0$ as $n \to \infty$. Next, assuming that lim inf $\text{Var}(Z_k)$ equals a positive constant c, (3.5) implies that

$$\begin{aligned}\limsup |D| &\le \frac{1}{c}\limsup \left| \frac{1}{n(n-1)} \left[\sum_{i \neq j} \pi_{1i}\pi_{1j} \right] - \frac{1}{n^2}\left(\sum_{i=1}^{n} \pi_{1i} \right)^2 \right| \\ &\le \frac{1}{c}\limsup \left| -\frac{1}{n^2} \sum_{i=1}^{n} \pi_{1i}^2 \right| \qquad \text{(after replacing } n-1 \text{ by } n\text{)} \\ &= 0,\end{aligned}$$

completing our proof.

Theorem 4 shows that for large n, the aggregated process $\{Z_k, k \ge 1\}$ looks like a zero-order process, irrespective of the interpurchase-time distribution and of the brand switching behaviors of individual family members. (Note that zero-order behavior implies $D = 0$ but the reverse implication is not true in general.) In other words, zero-order brand switching behavior at the household level is a somewhat expected phenomenon, especially when the number of 'effective' individuals in the household is large; and thus it may not convey much information about individuals' choice behavior.

4. Discussion

This paper generalizes the results of KMW and shows that their results hold even when the interpurchase times have a well-behaved, arbitrary distribution. Thus, we provide additional support to their main conclusion that the empirically observed zero-order brand switching behavior at the household level does not convey much information about individuals' switching behavior. One should therefore be careful when attempting to extrapolate household brand switching behavior to that of individuals.

We realize that it is not very realistic to assume that all members of a household have the same mean interpurchase time. For expositional clarity, we have made this assumption. Future research may relax this assumption and study its consequences. Also, an implicit assumption in our formulation is that n represents the number of individuals in the household participating in making purchases in the product category under consideration.

A potentially useful outcome of our analysis is the possibility of a more powerful statistical test for the zero-order hypothesis. Our Theorem 1 can be used by future researchers to develop such a test which could then be employed in empirical work.

Our results also suggest that more empirical work should be done on product categories where only one member of a household is a consumer of the product. Clearly, in this instance, the aggregation issue would be irrelevant, and tests of the zero-order hypothesis might yield useful information.

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