

The Assignment Problem: An Example

A company has 4 machines available for assignment to 4 tasks. Any machine can be assigned to any task, and each task requires processing by one machine. The time required to set up each machine for the processing of each task is given in the table below.

| | TIME (Hours) | | | |
|-----------|--------------|--------|--------|--------|
| | Task 1 | Task 2 | Task 3 | Task 4 |
| Machine 1 | 13 | 4 | 7 | 6 |
| Machine 2 | 1 | 11 | 5 | 4 |
| Machine 3 | 6 | 7 | 2 | 8 |
| Machine 4 | 1 | 3 | 5 | 9 |

The company wants to minimize the total setup time needed for the processing of all four tasks.

If we think of the setup times as transportation costs and define

$$x_{ij} = \begin{cases} 1 & \text{if machine } i \text{ is assigned to process task } j, \\ 0 & \text{if machine } i \text{ is not assigned to process task } j, \end{cases}$$

where $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$, then it is easily seen that what we have is a balanced transportation problem with 4 sources (representing the machines), 4 sinks (representing the tasks), a single unit of supply from each source (representing the availability of a machine), and a single unit of demand at each sink (representing the processing requirement of a task). This particular class of transportation problems is called the *assignment problems*. These problems can, of course, be solved by the streamlined Simplex algorithm. This is sketched below.

The transportation tableau for this problem is given below.

| | | Sinks | | | | |
|---------|---|-------|----|---|---|---|
| | | 1 | 2 | 3 | 4 | |
| Sources | 1 | 13 | 4 | 7 | 6 | 1 |
| | 2 | 1 | 11 | 5 | 4 | 1 |
| | 3 | 6 | 7 | 2 | 8 | 1 |
| | 4 | 1 | 3 | 5 | 9 | 1 |
| | | 1 | 1 | 1 | 1 | |

Using the least-cost method, an initial basic feasible solution can be easily obtained; this is shown in the tableau below.

| | | Sinks | | | | |
|----------------|----|-----------------|-----------------|-----------------|-----------------|-----|
| | | 1* | 2* | 3* | 4* | |
| Sources | 1 | 13 | 4 | 7 | 6 | 1 0 |
| | 2* | 1 | 11 | 5 | 4 | 1 1 |
| | 3* | 6 | 7 | 2 | 8 | 1 1 |
| | 4* | 1 | 3 | 5 | 9 | 1 1 |
| | | $\not\emptyset$ | $\not\emptyset$ | $\not\emptyset$ | $\not\emptyset$ | |

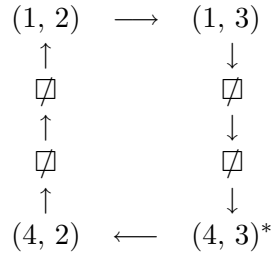
These assignments are made in the following order: $x_{41} = 1$, $x_{33} = 1$, $x_{42} = 0$, $x_{12} = 1$, $x_{24} = 1$, $x_{14} = 0$, and $x_{13} = 0$. Notice that a standard feature of any basic feasible solution in an assignment problem is that it is degenerate.

Next, we will use the u - v method to conduct the optimality test. The modifiers associated with the current solution, based on the initial assignment $u_1 = 0$, are shown in the tableau below.

| | | Sinks | | | | | Modifier |
|----------------|---|--------------|-----------|-----------|-----------|---|------------|
| | | 1 | 2 | 3 | 4 | | |
| Sources | 1 | 13 | 4 | 7 | 6 | 1 | $u_1 = 0$ |
| | 2 | 1 | 11 | 5 | 4 | 1 | $u_2 = -2$ |
| | 3 | 6 | 7 | 2 | 8 | 1 | $u_3 = -5$ |
| | 4 | 1 | 3 | 5 | 9 | 1 | $u_4 = -1$ |
| | | 1 | 1 | 1 | 1 | | |
| Modifier | | $v_1 = 2$ | $v_2 = 4$ | $v_3 = 7$ | $v_4 = 6$ | | |

It follows that the reduced costs associated with the nonbasic cells are: $\bar{c}_{11} = 11$, $\bar{c}_{21} = 1$, $\bar{c}_{22} = 9$, $\bar{c}_{23} = 0$, $\bar{c}_{31} = 9$, $\bar{c}_{32} = 8$, $\bar{c}_{34} = 7$, $\bar{c}_{43} = -1$, and $\bar{c}_{44} = 4$. Since $\bar{c}_{43} = -1$, the current solution is not optimal, and the entering cell is cell (4, 3).

The stepping-stone path associated with cell (4, 3) is shown below.



Notice that both x_{13} and x_{42} are degenerate basic variables; therefore, the maximum possible increase for the entering variable x_{43} is 0. This implies that we are about to conduct a pivot that corresponds to the simple declaration that x_{43} replaces either x_{13} or x_{42} (but not both) as a degenerate basic variable in the basis. Since all three variables are equal to 0, there is no numerical change in the current solution. Such a pivot is called a *degenerate pivot*.

Let us choose (arbitrarily) to let x_{13} exit the current basis. Then, after conducting a pivot according to the above path and updating the modifiers (with the initial assignment $u_4 = 0$), we obtain the new tableau below.

| | | Sinks | | | | | |
|----------|---|--------------|-----------|-----------|-----------|---|------------|
| | | 1 | 2 | 3 | 4 | | |
| | 1 | 13 | 4 | 7 | 6 | | |
| 1 | | | 1 | | 0 | 1 | $u_1 = 1$ |
| | 2 | 1 | 11 | 5 | 4 | | |
| 2 | | | | | 1 | 1 | $u_2 = -1$ |
| | 3 | 6 | 7 | 2 | 8 | | |
| 3 | | | | 1 | | 1 | $u_3 = -3$ |
| | 4 | 1 | 3 | 5 | 9 | | |
| 4 | | 1 | 0 | 0 | | 1 | $u_4 = 0$ |
| | | 1 | 1 | 1 | 1 | | |
| Modifier | | $v_1 = 1$ | $v_2 = 3$ | $v_3 = 5$ | $v_4 = 5$ | | |

It is easily seen that the reduced costs associated with the nonbasic cells are: $\bar{c}_{11} = 11$, $\bar{c}_{13} = 1$, $\bar{c}_{21} = 1$, $\bar{c}_{22} = 9$, $\bar{c}_{23} = 1$, $\bar{c}_{31} = 8$, $\bar{c}_{32} = 7$, $\bar{c}_{34} = 6$, and $\bar{c}_{44} = 4$. Since all of these are positive, we conclude that the current solution is (uniquely) optimal. Thus, it is optimal to assign Machine 1 to Task 2, Machine 2 to Task 4, Machine 3 to Task 3, and Machine 4 to Task 1. The total setup time associated with this solution is 11 hours. This completes the solution of the problem.

As noted earlier, every basic feasible solution in an assignment problem is degenerate. Since degeneracy is known to impede progress toward an optimal solution, other algorithms have been developed for the solution of assignment problems. These other algorithms are

designed with the avoidance of large numbers of degenerate pivots in mind; therefore, they are more efficient than the streamlined Simplex algorithm. We will not pursue the details here.