The Streamlined Simplex Method: An Example

Consider the (minimization) transportation problem below.

		Sinks				
		1	2	3	4	
	1	10	0	20	11	15
Sources	2	12	7	9	20	25
	3	0	14	16	18	5
		5	15	15	10	

We will solve this problem using the streamlined Simplex algorithm for transportation problems. In the first phase, we will apply the Vogel's method to construct an initial basic feasible solution; and in the second phase, where the task is to iterate toward an optimal solution, we will apply the u-v method to conduct optimality tests. Since the ideas behind these methods have already been explained in detail, our intent is to use this example to illustrate the complete algorithm. We will, therefore, be brief.

The first observation is that the total supply and the total demand are both equal to 45. Therefore, we have a balanced transportation problem, and there is no need to add either a dummy source or a dummy sink. In addition, note that there are 3 sources and 4 sinks; therefore, the number of basic variables in every basic feasible solution is 6 ($= 3 + 4 - 1$).

We now begin the construction of an initial basic feasible solution, using the Vogel's method. An inspection of the c_{ij}'s in the tableau above shows that the row penalties are: 10, 2, and 14; and that the column penalties are: 10, 7, 7, and 7. These are shown on the margins of the tableau below.

		Sinks					
		1	2	3	4		Penalty
	1	10	0	20	11	15	10
Sources	2	12	7	9	20	25	2
	3	0	14	16	18	5	14
		5	15	15	10		
Penalty		10	7	7	7		

Since the highest penalty comes from row 3, the first entering cell is cell (3, 1). After assigning 5 as x_{31} and going through a round of updates, we obtain the new tableau below.

Sinks

		1	2	3	4		Penalty
	1	[10]	[0]	[20]	[11]	15	10
Sources	2	[12]	[7]	[9]	[20]	25	2
	3*	[0] 5	[14]	[16]	[18]	~~5~~ ~~0~~	~~1~~4
		~~5~~ 0	15	15	10		
Penalty		~~10~~ 2	7	~~7~~ 11	~~7~~ 9		

Note that we have removed row 3, while leaving column 1 available for further assignment. Moreover, we have also updated the penalties.

Observe that column 3 now has the highest penalty; therefore, the next entering cell is cell (2, 3). After assigning 15 as x_{23} and updating in the same manner, we obtain the tableau below.

Sinks

		1	2	3*	4		Penalty
	1	[10]	[0]	[20]	[11]	15	10
Sources	2	[12]	[7]	[9] 15	[20]	~~25~~ 10	~~2~~ 5
	3*	[0] 5	[14]	[16]	[18]	~~5~~ ~~0~~	~~1~~4
		~~5~~ 0	15	~~15~~ ~~0~~	10		
Penalty		~~10~~ 2	7	~~7~~ ~~11~~	~~7~~ 9		

Since row 1 has the highest penalty, the next entering cell is cell (1, 2). After assigning 15 as x_{12} and updating, we obtain the tableau below.

		Sinks 1	2	3*	4		Penalty
	1*	[10]	[0] 15	[20]	[11]	~~15~~ ~~0~~	~~10~~
Sources	2	[12]	[7]	[9] 15	[20]	~~25~~ 10	~~2~~ 5
	3*	[0] 5	[14]	[16]	[18]	~~5~~ ~~0~~	~~14~~
		~~5~~ 0	~~15~~ 0	~~15~~ ~~0~~	10		
Penalty		~~10~~ 2	7	~~7~~ ~~11~~	~~7~~ 9		

With only three cells in row 2 remaining, we simply assign, sequentially, 0 as x_{22}, 0 as x_{21}, and 10 as x_{24}. This yields the initial basic feasible solution given in the tableau below.

		Sinks 1*	2*	3*	4*		Penalty
	1*	[10]	[0] 15	[20]	[11]	~~15~~ ~~0~~	~~10~~
Sources	2	[12] 0	[7] 0	[9] 15	[20] 10	~~25~~ ~~10~~ 0	~~2~~ 5
	3*	[0] 5	[14]	[16]	[18]	~~5~~ ~~0~~	~~14~~
		~~5~~ ~~0~~ ~~0~~	~~15~~ ~~0~~ ~~0~~	~~15~~ ~~0~~	~~10~~ ~~0~~		
Penalty		~~10~~ 2	7	~~7~~ ~~11~~	~~7~~ 9		

Note that we have entered two degenerate basic variables, in cells (2, 1) and (2, 2); and that, after entering $x_{24} = 10$, we chose to remove column 4, as opposed to removing row 2. At this point, we have completed the first phase of the algorithm. It is prudent to confirm that we indeed have 6 basic variables.

We now begin the second phase of the algorithm.

The first step is to calculate the set of modifiers associated with the above initial basic feasible solution. The results are shown in the tableau below.

Sinks

		1	2	3	4		Modifier
	1	[10]	[0] 15	[20]	[11]	15	$u_1 = -7$
Sources	2	[12] 0	[7] 0	[9] 15	[20] 10	25	$u_2 = 0$
	3	[0] 5	[14]	[16]	[18]	5	$u_3 = -12$
		5	15	15	10		
Modifier		$v_1 = 12$	$v_2 = 7$	$v_3 = 9$	$v_4 = 20$		

The detailed calculations are as follows. We begin by entering the assignment $u_2 = 0$. This choice is motivated by the fact that all four cells in row 2 are basic. Indeed, with $u_2 = 0$, we immediately have $v_1 = 12$, $v_2 = 7$, $v_3 = 9$, and $v_4 = 20$ (which are simply copies of c_{21}, c_{22}, c_{23}, and c_{24}). Since cell (3, 1) in column 1 is basic, the fact that $c_{31} = 0$ and $v_1 = 12$ implies that $u_3 = -12$. Finally, since cell (1, 2) in column 2 is basic, our last assignment is $u_1 = 0 - 7 = -7$.

With the modifiers given, the reduced costs in the nonbasic cells can now be calculated as follows:

$$\begin{aligned}\bar{c}_{11} &= 10 - (-7) - 12 \\ &= 5\,,\end{aligned}$$

$$\begin{aligned}\bar{c}_{13} &= 20 - (-7) - 9 \\ &= 18\,,\end{aligned}$$

$$\begin{aligned}\bar{c}_{14} &= 11 - (-7) - 20 \\ &= -2\,,\end{aligned}$$

$$\begin{aligned}\bar{c}_{32} &= 14 - (-12) - 7 \\ &= 19\,,\end{aligned}$$

$$\begin{aligned}\bar{c}_{33} &= 16 - (-12) - 9 \\ &= 19\,,\end{aligned}$$

and

$$\begin{aligned}\bar{c}_{34} &= 18 - (-12) - 20 \\ &= 10\,.\end{aligned}$$

Since $\bar{c}_{14}$ is negative, the next entering cell is cell (1, 4). The stepping-stone path associated with this cell is:

$$\begin{array}{ccccc} (1,\,2) & \longrightarrow & \not\square & \longrightarrow & (1,\,4)^{*} \\ \uparrow & & & & \downarrow \\ (2,\,2) & \longleftarrow & \not\square & \longleftarrow & (2,\,4) \end{array}$$

After conducting a pivot according to this path and updating the modifiers, we obtain the new solution below.

		Sinks 1	2	3	4		Modifier
	1	[10]	[0] 5	[20]	[11] 10	15	$u_1 = -7$
Sources	2	[12] 0	[7] 10	[9] 15	[20]	25	$u_2 = 0$
	3	[0] 5	[14]	[16]	[18]	5	$u_3 = -12$
		5	15	15	10		
Modifier		$v_1 = 12$	$v_2 = 7$	$v_3 = 9$	$v_4 = 18$		

It is easily seen that the reduced costs associated with the nonbasic cells are: $\bar{c}_{11} = 5$, $\bar{c}_{13} = 18$, $\bar{c}_{24} = 2$, $\bar{c}_{32} = 19$, $\bar{c}_{33} = 19$, and $\bar{c}_{34} = 12$. Since all of these are positive, we conclude, finally, that the current solution is (uniquely) optimal. The objective-function value of this solution is 315. This completes the second phase of the algorithm.