

Progressing toward an Optimal Solution

After having constructed an initial basic feasible solution, our next task is to progress toward an optimal solution. Again, we will describe two methods for doing this. The first one is called the stepping-stone method; and the second, the u - v method.

The Stepping-Stone Method

Again, we will use the previous example to illustrate the method. The transportation tableau associated with the basic feasible solution produced by the least-cost method is given below.

		Sinks			
		1	2	3	
Sources	1	3	2	0	45
			15	30	
	2	1	5	0	
	50	10			
3	5	4	0	35	
		35			
		50	60	30	

Our aim is to iterate toward an optimal solution, starting with this solution.

A little bit of reflection should convince you that the present scenario is essentially the same as that at the start of Phase II of the standard Simplex method. There is, however, a logistical difference, namely that the standard Simplex tableau associated with the current solution is not explicitly available to us. Therefore, we need to develop a different set of procedures for generating informations that are necessary for the execution of the Simplex algorithm. In particular, it is critical that we have corresponding mechanisms for conducting optimality tests and for performing pivots.

We begin with the question of whether or not the current solution is optimal. In the standard Simplex method, the optimality test is based on a reading of the coefficients of the nonbasic variables in the zeroth row of the Simplex tableau; that is, it is based on a reading of the reduced costs. Since the reduced costs are not explicitly available in a given transportation tableau, our first task is to develop a method for (re)constructing them.

Recall that the reduced cost associated with a nonbasic variable is defined to be the amount by which the objective-function value degrades if we increase (nominally) the value of that nonbasic variable by 1 (while holding all other nonbasic variables at 0). We will apply this definition to constructively generate the reduced costs associated with all nonbasic variables.

Consider the nonbasic variable x_{11} , which is located in cell (1, 1). A cell that contains a nonbasic variable will be referred to as a *nonbasic cell*. Now, imagine an increase in the value of x_{11} from 0 to δ , where δ is nonnegative. Since the sum of the x_{ij} values in row 1, i.e., the row sum $x_{11} + x_{12} + x_{13}$, must be maintained at 45 to preserve feasibility, such an increase will necessitate a corresponding decrease in x_{12} and/or x_{13} . Notice that both x_{12} and x_{13} are basic variables. A cell that contains a basic variable will be referred to as a *basic cell*. We will consider first the basic cell (1, 3). Observe that if we attempt to decrease the value of x_{13} from 30 to $30 - \delta$, then, since the column sum $x_{13} + x_{23} + x_{33}$ must be maintained at 30, at least one of the variables x_{23} and x_{33} in that column must undergo a corresponding increase. Since both x_{23} and x_{33} are nonbasic, their values are to remain at 0; it follows that a decrease in x_{13} is not permitted. This leads us back to basic cell (1, 2). Again, since the column sum $x_{12} + x_{22} + x_{32}$ must be maintained at 60, a decrease in x_{12} will necessitate a corresponding increase in x_{22} and/or x_{32} . Notice, however, that an increase in x_{32} will force a corresponding decrease in the nonbasic cells (3, 1) and (3, 3); therefore, an increase in x_{32} is not permitted. It follows that the only sequence of permissible revisions, at this point, is to decrease the value of x_{12} from 15 to $15 - \delta$ and then to match that decrease with an increase of the value of x_{22} from 10 to $10 + \delta$. Next, the row-sum requirement for row 2 shows that we should now decrease the value of x_{21} from 50 to $50 - \delta$. Finally, an examination of column 1 shows that we have managed to navigate through a full cycle of revisions, in the sense that this last decrease is balanced by the original increase in x_{11} .

In summary, the above discussion reveals that a nominal increase in x_{11} from 0 to δ can be “accommodated” by sequentially decreasing the value of x_{12} from 15 to $15 - \delta$, increasing the value of x_{22} from 10 to $10 + \delta$, and finally decreasing the value of x_{21} from 50 to $50 - \delta$. (All other x_{ij} ’s, basic or not, retain their original values.) This sequence of revisions can be explicitly indicated as in the tableau below.

		Sinks					
		1	2	3			
1	3	$+\delta$	2	$15 - \delta$	0	30	45
2	1	$50 - \delta$	5	$10 + \delta$	0		60
3	5		4	35	0		35
		50	60	30			

An inspection of this tableau shows that the only cells “visited” by this sequence of revisions are: (1, 1), (1, 2), (2, 2), and (2, 1). The layout of these cells can be described in the form

of a path, as shown below.

$$\begin{array}{ccc} (1, 1)^* & \longrightarrow & (1, 2) \\ \uparrow & & \downarrow \\ (2, 1) & \longleftarrow & (2, 2) \end{array}$$

Thus, the path begins with cell $(1, 1)$, which is marked with an asterisk to indicate that it is the entering nonbasic cell. We then visit basic cells $(1, 2)$, $(2, 2)$, and $(2, 1)$ in succession; and finally, we return to cell $(1, 1)$ from the last stop, cell $(2, 1)$. The path just described is an example of what is called a *stepping-stone path*. This name originates from the fact that we are stepping through a sequence of basic cells, and that we can think of these basic cells as “stones” in a pond—the pond being the entire tableau. (Nonbasic cells are not considered as stones; therefore, if one (mis)steps on a nonbasic cell, then one falls into water and gets wet.)

Note that, as indicated by the arrows, the order of visits in the stepping-stone path above is clockwise. It is easily seen that reversing this direction to a counter-clockwise order will have no impact on the revised values of the x_{ij} 's. It follows that the direction of visits is irrelevant.

A notable property of a stepping-stone path is that in the transportation tableau, it will always make a 90-degree turn after stepping on a cell. This is a consequence of the fact that we are alternately maintaining the row-sum (or supply) constraints and the column-sum (or demand) constraints.

Another important observation is that exactly one stepping-stone path can originate from the nonbasic cell $(1, 1)$. You should convince yourself about the validity of this claim by going through a careful review of the discussion above. This claim, in fact, holds in general; that is, every nonbasic cell has exactly one associated stepping-stone path.

Our discussion above can also be summarized more formally as follows. By increasing the value of δ , i.e., by bringing x_{11} into the basis, we can generate a family of solutions of the form:

$$(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) = (\delta, 15 - \delta, 30, 50 - \delta, 10 + \delta, 0, 0, 35, 0).$$

Geometrically, this corresponds to an attempt to follow an edge of the feasible region, starting from the corner-point solution

$$(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) = (0, 15, 30, 50, 10, 0, 0, 35, 0).$$

Since all variables are to remain nonnegative, we need to require that both $15 - \delta$ and $50 - \delta$ stay nonnegative. It follows that the value of δ should not exceed 15. With $\delta = 15$, we obtain the new solution

$$(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) = (15, 0, 30, 35, 25, 0, 0, 35, 0);$$

and this means that we have arrived at a corner-point solution that is adjacent to the previous one. In other words, what we have done is the equivalent of an ordinary Simplex pivot.

We are now ready to derive the reduced cost associated with x_{11} . From the above tableau, we see that for any given δ , the difference between the new objective-function value and the original objective-function value can be computed by adding individual differences associated with cells on the stepping-stone path. Specifically, the individual differences are: $3 \times \delta$ for cell (1, 1), $2 \times (-\delta)$ for cell (1, 2), $5 \times \delta$ for cell (2, 2), and $1 \times (-\delta)$ for cell (2, 1). It follows that with $\delta = 1$, the overall difference can be summed up as:

$$\begin{aligned} 3 \times \delta + 2 \times (-\delta) + 5 \times \delta + 1 \times (-\delta) &= 3 \times 1 + 2 \times (-1) + 5 \times 1 + 1 \times (-1) \\ &= 3 - 2 + 5 - 1 \\ &= 5. \end{aligned}$$

Thus, a 1-unit increase in x_{11} results in a 5-unit increase (which is a degradation) in the objective-function value. In other words, the reduced cost associated with x_{11} equals 5. Since this reduced cost is positive, it is not desirable to bring x_{11} into the basis.

Suppose a variable x_{ij} is nonbasic in a solution; then, we will denote the reduced cost associated with x_{ij} by \bar{c}_{ij} . In this notation, the above calculation can be summarized more-compactly as:

$$\begin{aligned} \bar{c}_{11} &= c_{11} - c_{12} + c_{22} - c_{21} \\ &= 3 - 2 + 5 - 1 \\ &= 5, \end{aligned}$$

where the c_{ij} 's are picked up sequentially from the cells on the stepping-stone path.

There are three other nonbasic cells in the above tableau, namely (2, 3), (3, 1), and (3, 3). We now continue on to an examination of these cells. Since the underlying ideas have been explained in detail, we will be brief.

The stepping-stone path associated with cell (2, 3) is:

$$\begin{array}{ccc} (1, 2) & \longrightarrow & (1, 3) \\ \uparrow & & \downarrow \\ (2, 2) & \longleftarrow & (2, 3)^* \end{array}$$

After picking up the c_{ij} 's (and alternating their signs) in the order (2, 3), (2, 2), (1, 2), and (1, 3), the reduced cost associated with x_{23} can be computed as:

$$\begin{aligned} \bar{c}_{23} &= 0 - 5 + 2 - 0 \\ &= -3. \end{aligned}$$

The fact that this reduced cost is negative implies that the current solution is not optimal.

The stepping-stone path associated with cell (3, 1) is:

$$\begin{array}{ccc} (2, 1) & \longrightarrow & (2, 2) \\ \uparrow & & \downarrow \\ (3, 1)^* & \longleftarrow & (3, 2) \end{array}$$

By stepping through cells on this path (starting with cell (3, 1)), the reduced cost associated with x_{31} can be computed as:

$$\begin{aligned} \bar{c}_{31} &= 5 - 1 + 5 - 4 \\ &= 5. \end{aligned}$$

Since \bar{c}_{31} is positive, it is not desirable to bring x_{31} into the basis.

Finally, the stepping-stone path associated with cell (3, 3) is:

$$\begin{array}{ccc} (1, 2) & \longrightarrow & (1, 3) \\ \uparrow & & \downarrow \\ \square & & \square \\ \uparrow & & \downarrow \\ (3, 2) & \longleftarrow & (3, 3)^* \end{array}$$

Here, the symbol “ \square ” (an icon of a crossed-out cell) between cells (3, 2) and (1, 2) denotes the fact that we do not intend to make any revision in cell (2, 2); the interpretation of the \square between cells (1, 3) and (3, 3) is similar. In the language of the pond analogy, this means that we are “jumping” over cells (2, 2) and (2, 3). (In this connection, we note that, in general, the shape of a stepping-stone path can be very complex. In particular, it does not have to be rectangular.) By stepping through cells on this path, we obtain:

$$\begin{aligned} \bar{c}_{33} &= 0 - 4 + 2 - 0 \\ &= -2. \end{aligned}$$

The fact that \bar{c}_{33} is negative indicates, again, that the current solution is not optimal.

Since the value of \bar{c}_{23} is more negative than that of \bar{c}_{33} , we should now bring x_{23} into the basis. In the language of the standard Simplex algorithm, this means that we should execute a pivot in the x_{23} -column. However, since we are working with a transportation tableau, this pivot will have to be implemented in a different manner. Observe that bringing x_{23} into the basis means that we are interested in solutions of the form indicated by the tableau

below.

		Sinks			
		1	2	3	
Sources	1	3	2	0	45
			$15 + \delta$	$30 - \delta$	
	2	1	5	0	
		50	$10 - \delta$		60
3	5	4	0		35
		35			
		50	60	30	

An inspection of this tableau shows that the x_{ij} values contained in cells (2, 2) and (1, 3), namely $10 - \delta$ and $30 - \delta$ are being decreased. Since all variables must remain nonnegative, it follows that the maximum possible value for δ is 10 (this corresponds to a ratio test in the ordinary Simplex algorithm); and that x_{22} is the leaving variable. We will, therefore, let $\delta = 10$ in the above tableau; and doing so takes us to the tableau below.

		Sinks			
		1	2	3	
Sources	1	3	2	0	45
			25	20	
	2	1	5	0	
		50			60
3	5	4	0		35
		35			
		50	60	30	

Note that cell (2, 2) is now left blank, indicating that x_{22} is nonbasic in the new solution.

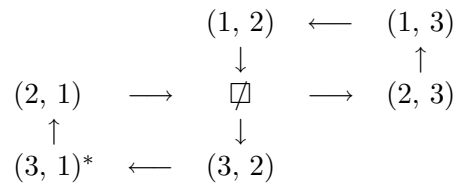
The next task, of course, is to test the new solution for optimality. The nonbasic cells in the new tableau are: (1, 1), (2, 2), (3, 1), and (3, 3). The stepping-stone paths associated with these cells are:

$$\begin{array}{ccccc}
 (1, 1)^* & \longrightarrow & \square & \longrightarrow & (1, 3) \\
 \uparrow & & & & \downarrow \\
 (2, 1) & \longleftarrow & \square & \longleftarrow & (2, 3)
 \end{array}$$

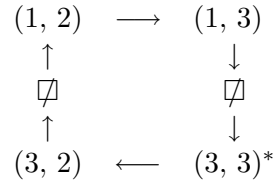
for cell (1, 1);

$$\begin{array}{ccc}
 (1, 2) & \longrightarrow & (1, 3) \\
 \uparrow & & \downarrow \\
 (2, 2)^* & \longleftarrow & (2, 3)
 \end{array}$$

for cell (2, 2);



for cell (3, 1); and finally,



for cell (3, 3). Therefore, the corresponding (new) reduced costs are:

$$\begin{aligned}
 \bar{c}_{11} &= 3 - 0 + 0 - 1 \\
 &= 2,
 \end{aligned}$$

$$\begin{aligned}
 \bar{c}_{22} &= 5 - 2 + 0 - 0 \\
 &= 3,
 \end{aligned}$$

$$\begin{aligned}
 \bar{c}_{31} &= 5 - 1 + 0 - 0 + 2 - 4 \\
 &= 2,
 \end{aligned}$$

and

$$\begin{aligned}
 \bar{c}_{33} &= 0 - 4 + 2 - 0 \\
 &= -2.
 \end{aligned}$$

Since \bar{c}_{33} remains negative, the current solution is not optimal. Therefore, we will next let x_{33} enter the basis.

Bringing x_{33} into the basis means that we will consider solutions of the form below.

		Sinks			
		1	2	3	
Sources	1	3	2	0	45
			25 + δ	20 - δ	
	2	1	5	0	10
		50			
3	5	4	0		35
			35 - δ	+ δ	
		50	60	30	

An inspection of cells (1, 3) and (3, 2) shows that x_{33} can be boosted up to 20; and that at this level, x_{13} exits the basis. Execution of this pivot now yields the new solution below.

		Sinks			
		1	2	3	
Sources	1	3	2	0	45
		45			
	2	1	5	0	
	50		10		
3	5	4	0	35	
	15		20		
	50	60	30		

After constructing a new set of stepping-stone paths for cells (1, 1), (1, 3), (2, 2), and (3, 1), details of which we omit, the new reduced costs are:

$$\begin{aligned}\bar{c}_{11} &= 3 - 2 + 4 - 0 + 0 - 1 \\ &= 4,\end{aligned}$$

$$\begin{aligned}\bar{c}_{13} &= 0 - 0 + 4 - 2 \\ &= 2,\end{aligned}$$

$$\begin{aligned}\bar{c}_{22} &= 5 - 0 + 0 - 4 \\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\bar{c}_{31} &= 5 - 1 + 0 - 0 \\ &= 4.\end{aligned}$$

Since all reduced costs are positive, we conclude, finally, that the current solution is optimal. The fact that these reduced costs are strictly positive also implies that there are no other optimal solutions.

In conclusion, the optimal solution produced by the above procedure, which is called the *stepping-stone method*, is:

$$(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) = (0, 45, 0, 50, 0, 10, 0, 15, 20).$$

This basic feasible solution has an objective-function value of 200.

The u - v Method

Recall that in every iteration of the stepping-stone method, one has to construct a stepping-stone path for every nonbasic cell. This aspect of the method seems laborious. In fact, it can be shown that the amount of effort involved in this task grows exponentially as a function of problem size. Thus, the required computational effort for large-scaled problems will be prohibitive. It is therefore desirable to look for a more-efficient alternative. The u - v method is one alternative whose computational effort grows only linearly.

Again, we will use the previous numerical example to motivate the idea behind the u - v method. Recall that the solution produced by the least-cost method is given in the tableau below.

		Sinks			
		1	2	3	
Sources	1	3	2	0	45
2	1	5	0		60
3	5	4	0		35
		50	60	30	

For reasons that will become clear shortly, let us suppose that the c_{ij} 's in this tableau are modified to a new set of values, as specified in the tableau below.

		Sinks				
		1	2	3		
Sources	1	5	0	0	45	
2	0	0	-3		60	
3	5	0	-2		35	
		50	60	30		

Observe that a distinct feature of this tableau is that the c_{ij} 's associated with the basic cells are all equal to 0. (That some of the costs in this tableau are negative is not a concern; this is because negative costs can be interpreted as profits.) To understand the implication of this feature, let us compute the reduced cost for cell (1, 1). Recall that the stepping-stone path for cell (1, 1) is:

$$\begin{array}{ccc}
 (1, 1)^* & \longrightarrow & (1, 2) \\
 \uparrow & & \downarrow \\
 (2, 1) & \longleftarrow & (2, 2)
 \end{array}$$

Therefore,

$$\begin{aligned}\bar{c}_{11} &= c_{11} - c_{12} + c_{22} - c_{21} \\ &= 5 - 0 + 0 - 0 \\ &= 5.\end{aligned}$$

In other words, since every basic cell along this stepping-stone path has its associated c_{ij} equal to 0, we have $\bar{c}_{11} = c_{11}$. A little bit of reflection now reveals that, in fact, we have

$$\bar{c}_{ij} = c_{ij}$$

for every nonbasic cell, regardless of the shape of the stepping-stone paths. It follows that whenever a given set of c_{ij} 's and a given basic feasible solution, together, "happen" to have the distinct feature described above, then the reduced costs can be read directly from the tableau without any need for constructing the stepping-stone paths.

While this remarkable feature is certainly desirable, it seems unrealistic to expect such "luck" in a given tableau. Surprisingly, it turns out that we actually have a lot of flexibility in specifying the c_{ij} values in a problem. What this means is that it is possible to modify a given set of c_{ij} 's in ways that preserve the identity of the optimal solution.

To see how this is accomplished, let us consider the first tableau again. Suppose all three costs in column 1 are revised downward by 1 unit. That is, let $c_{11} = 3 - 1 = 2$, $c_{21} = 1 - 1 = 0$, and $c_{31} = 5 - 1 = 4$. Clearly, as a result of this downward revision, the total contribution to the objective-function value from the three variables in the first column, namely x_{11} , x_{21} , and x_{31} , must undergo a corresponding reduction. This reduction can be explicitly calculated as:

$$\begin{aligned}(3 \times x_{11} + 1 \times x_{21} + 5 \times x_{31}) &- (2 \times x_{11} + 0 \times x_{21} + 4 \times x_{31}) \\ &= x_{11} + x_{21} + x_{31} \\ &= 50,\end{aligned}$$

where the second equality is a consequence of the demand constraint $x_{11} + x_{21} + x_{31} = 50$ at Sink 1. Notice that the outcome of this calculation is independent of the specific values of x_{11} , x_{21} , and x_{31} . In other words, *every* feasible solution will have its objective-function value reduced by 50. It follows that, indeed, if a solution is optimal prior to these cost revisions, then the same solution will remain optimal after the revisions.

Similarly, as a consequence of the requirement that $x_{11} + x_{12} + x_{13} = 45$, the identity of the optimal solution is preserved after downward revisions in all three costs in row 1 by 1 unit.

Continuation of this argument shows that, in fact, we can modify the costs in every row and every column in this manner without any fear of "losing" the identity of the optimal

solution. For this reason, we shall say that two sets of costs are *equivalent* if they are related to each other in this manner.

Armed with this newly-found flexibility, let us return to the “distinct feature” noted earlier. The question now is: Is it possible to modify the original c_{ij} ’s to arrive at an equivalent set of costs that has this distinct feature? We will show that the answer is yes.

The idea is to work with a sequence of variably-sized reductions (as opposed to 1-unit reductions) in the costs in the transportation tableau, first row-by-row and then column-by-column. Specifically, let

u_i = the size of a reduction in every cost in row i , where $i = 1, 2, 3$.

v_j = the size of a reduction in every cost in column j , where $j = 1, 2, 3$.

We shall refer to the u_i ’s and the v_j ’s as the *modifiers*. Thus, u_1 is the modifier for row 1, v_1 is the modifier for column 1, and so on. We will also allow these modifiers to assume negative values.

That the letters “ u ” and “ v ” are used to denote the modifiers is why we refer to this method as the *u - v method*.

Clearly, after cycling through these six modifications, the original cost in cell (i, j) will be reduced twice, the first time by u_i and the second time by v_j . It follows that the revised cost for cell (i, j) is equal to $c_{ij} - u_i - v_j$. This is explicitly shown in the tableau below.

		Sinks				
		1	2	3	Modifier	
Sources	1	$3 - u_1 - v_1$	$2 - u_1 - v_2$	$0 - u_1 - v_3$	45	u_1
			15	30		
	2	$1 - u_2 - v_1$	$5 - u_2 - v_2$	$0 - u_2 - v_3$	60	u_2
	50	10				
3	$5 - u_3 - v_1$	$4 - u_3 - v_2$	$0 - u_3 - v_3$	35	u_3	
		35				
		50	60	30		
Modifier		v_1	v_2	v_3		

Notice that we have also indicated the modifier for every row and every column at the right and bottom margins of this tableau. It is helpful to visualize the revised cost in cell (i, j) as being equal to the original c_{ij} subtracted first by the modifier located at the right margin of row i and then by the modifier located at the bottom margin of column j .

Now, in light of the distinct feature above, our goal is to choose a set of values for the u_i 's and the v_j 's to achieve the outcome $c_{ij} - u_i - v_j = 0$, or equivalently $c_{ij} = u_i + v_j$, in *every basic cell*. That is, we would like to see:

$$2 = u_1 + v_2,$$

$$0 = u_1 + v_3,$$

$$1 = u_2 + v_1,$$

$$5 = u_2 + v_2,$$

and

$$4 = u_3 + v_2.$$

It follows that the desired values for the modifiers can be obtained by solving this system of 5 linear equations in 6 unknowns. Note that an important feature of this equation system is that exactly two variables appear in every equation. We now show that this feature greatly reduces the solution effort.

Since the number of variables is greater than the number of equations by 1, we have one extra "degree of freedom." This means that we can choose to assign an arbitrary value to any one of the modifiers. Let us assign, say, a 0 to u_1 . Since u_1 appears in the first two equations, which are $2 = u_1 + v_2$ and $0 = u_1 + v_3$, this initial assignment immediately implies that we have $v_2 = 2 - u_1 = 2 - 0 = 2$ and $v_3 = 0 - u_1 = 0 - 0 = 0$, respectively. Since v_2 appears in the last two equations, which are $5 = u_2 + v_2$ and $4 = u_3 + v_2$, the just-assigned value for v_2 , in turn, implies that we have $u_2 = 5 - v_2 = 5 - 2 = 3$ and $u_3 = 4 - v_2 = 4 - 2 = 2$. Finally, since u_2 appears in the only remaining equation, namely $1 = u_2 + v_1$, the just-assigned value for u_2 further implies that $v_1 = 1 - u_2 = 1 - 3 = -2$. This completes the solution process.

The set of modifiers we obtained can also be entered into the tableau as follows.

		Sinks					
		1	2	3			
Sources	1	$3 - u_1 - v_1$	$2 - u_1 - v_2$	$0 - u_1 - v_3$	45	$u_1 = 0$	
			15	30			
	2	$1 - u_2 - v_1$	$5 - u_2 - v_2$	$0 - u_2 - v_3$			50
3	$5 - u_3 - v_1$	$4 - u_3 - v_2$	$0 - u_3 - v_3$		35	35	$u_3 = 2$
		50	60	30			
Modifier		$v_1 = -2$	$v_2 = 2$	$v_3 = 0$			

You should verify visually that for every basic cell (say cell (i, j)), we indeed have $c_{ij} = u_i + v_j$. That is, we have succeeded in producing a set of modifiers that transforms the original costs into an equivalent set of costs that has the desired distinct feature.

What about the revised costs in the nonbasic cells? An inspection of the above tableau shows that the revised cost in cell (1, 1), for example, is equal to $3 - u_1 - v_1 = 3 - 0 - (-2) = 5$; and that the revised costs in other nonbasic cells can be similarly computed. The results of these calculations (including those for the basic cells) are given in the tableau below.

		Sinks			
		1	2	3	
Sources	1	5	0	0	45
			15	30	
	2	0	0	-3	
	50	10			
3	5	0	-2	35	
		35			
	50	60	30		

Notice that this tableau is precisely the one that motivated our development of the u - v method at the beginning of this discussion.

Finally, recall that with the distinct feature in place, the reduced cost associated with a nonbasic cell is simply the cost in that cell, without any need for constructing an explicit stepping-stone path. Since the reduced costs are denoted by \bar{c}_{ij} , it follows that for every nonbasic cell (say cell (i, j)), we have

$$\bar{c}_{ij} = c_{ij} - u_i - v_j.$$

Our conclusion, therefore, is that all of the reduced costs associated with a transportation tableau can be generated by an application of the u - v method. A little bit of reflection should also convince you that this procedure is computationally more efficient than the stepping-stone method.

The rest of the solution procedure for this transportation problem proceeds in the same manner as what was done earlier using the stepping-stone method. Note, however, that after the selection of an entering cell based on a computed set of modifiers, it is still necessary to construct the stepping-stone path associated with that particular cell to conduct a pivot. For completeness, we provide a brief summary below.

An inspection of the above tableau shows that the entering cell is (2, 3). The stepping-stone path associated with this cell is:

$$\begin{array}{ccc}
 (1, 2) & \longrightarrow & (1, 3) \\
 \uparrow & & \downarrow \\
 (2, 2) & \longleftarrow & (2, 3)^*
 \end{array}$$

After conducting a pivot according to this path, we obtain the new solution below.

		Sinks				
		1	2	3		
Sources	1	3	2	0	45	$u_1 = 0$
			25	20		
	2	1	5	0		
	50		10			
3	5	4	0	35	$u_3 = 2$	
		35				
		50	60	30		
Modifier		$v_1 = 1$	$v_2 = 2$	$v_3 = 0$		

Notice that a new set of modifiers has been specified on the right and bottom margins of this tableau. These are computed as follows. We begin by entering the assignment $u_1 = 0$ (which is arbitrarily chosen) at the right margin of row 1. Observe that cells (1, 2) and (1, 3) in row 1 are basic. Therefore, with u_1 given as 0, we should assign $v_2 = 2 - 0 = 2$ and $v_3 = 0 - 0 = 0$. These new assignments are entered at the bottom margins of column 2 and column 3, respectively. Since cell (3, 2) in column 2 is basic, the fact that $c_{32} = 4$ and $v_2 = 2$ implies that $u_3 = 4 - 2 = 2$, which is entered at the right margin of row 3. Similarly, since cell (2, 3) in column 3 is basic, the fact that $c_{23} = 0$ and $v_3 = 0$ implies that $u_2 = 0 - 0 = 0$, which is entered at the right margin of row 2. Finally, since cell (2, 1) in row 2 is basic, the only remaining assignment is $v_1 = 1 - 0 = 1$, entered at the bottom of column 1.

With the set of new modifiers given, the reduced costs in the nonbasic cells can now be calculated as follows:

$$\begin{aligned} \bar{c}_{11} &= c_{11} - u_1 - v_1 \\ &= 3 - 0 - 1 \\ &= 2, \end{aligned}$$

$$\begin{aligned} \bar{c}_{22} &= c_{22} - u_2 - v_2 \\ &= 5 - 0 - 2 \\ &= 3, \end{aligned}$$

$$\begin{aligned} \bar{c}_{31} &= c_{31} - u_3 - v_1 \\ &= 5 - 2 - 1 \\ &= 2, \end{aligned}$$

and

$$\begin{aligned}\bar{c}_{33} &= c_{33} - u_3 - v_3 \\ &= 0 - 2 - 0 \\ &= -2.\end{aligned}$$

Note that these reduced costs are in agreement with those produced by the stepping-stone method earlier.

Since \bar{c}_{33} is negative, the next entering cell is $(3, 3)$. The stepping-stone path associated with this cell is:

$$\begin{array}{ccc} (1, 2) & \longrightarrow & (1, 3) \\ \uparrow & & \downarrow \\ \square & & \square \\ \uparrow & & \downarrow \\ (3, 2) & \longleftarrow & (3, 3)^* \end{array}$$

After conducting a pivot according to this path and updating the modifiers, we obtain the new solution below.

		Sinks						
		1	2	3				
Sources	1	3	2	0	45	$u_1 = 0$		
	2	1	5	0			60	$u_2 = 2$
	3	5	4	0				
		50	60	30				
Modifier		$v_1 = -1$	$v_2 = 2$	$v_3 = -2$				

The reduced costs associated with the nonbasic cells can be calculated as follows.

$$\begin{aligned}\bar{c}_{11} &= 3 - 0 - (-1) \\ &= 4,\end{aligned}$$

$$\begin{aligned}\bar{c}_{13} &= 0 - 0 - (-2) \\ &= 2,\end{aligned}$$

$$\begin{aligned}\bar{c}_{22} &= 5 - 2 - 2 \\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\bar{c}_{31} &= 5 - 2 - (-1) \\ &= 4.\end{aligned}$$

Since all of these are positive, we conclude, finally, that the current solution is (uniquely) optimal.