

# The Fundamental Insight

We will begin with a review of matrix multiplication, which is needed for the development of the fundamental insight.

A matrix is simply an array of numbers. If a given array has  $m$  rows and  $n$  columns, then it is called an  $m \times n$  (or  $m$ -by- $n$ ) matrix. As examples, the array

$$\mathbf{A}_1 = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 2 \end{bmatrix}$$

is a  $2 \times 3$  matrix; and the array

$$\mathbf{A}_2 = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 3 & 2 \\ 5 & 1 & -2 \end{bmatrix}$$

is a  $3 \times 3$  matrix.

A number can be viewed as a  $1 \times 1$  matrix. We are, of course, familiar with the multiplication of numbers. The multiplication between two matrices is an operation that is an extension of the multiplication operation between two numbers (i.e., two  $1 \times 1$  matrices). This extension, however, will not be applicable to all pairs of matrices. More specifically, let  $\mathbf{A}$  and  $\mathbf{B}$  be two matrices; then, the product of these two matrices, denoted by  $\mathbf{A} \times \mathbf{B}$ , is defined only when the number of columns in  $\mathbf{A}$  is the same as the number of rows in  $\mathbf{B}$ . Since the matrices  $\mathbf{A}_1$  and  $\mathbf{A}_2$  above satisfy this condition, we will first construct the product of these two matrices as a specific numerically example.

Denote by  $\mathbf{A}_3$  the product of  $\mathbf{A}_1$  and  $\mathbf{A}_2$ . The number of rows in  $\mathbf{A}_3$  will be identical to that of  $\mathbf{A}_1$ ; and the number of columns in  $\mathbf{A}_3$  will be identical to that of  $\mathbf{A}_2$ . That is,  $\mathbf{A}_3$  will have 2 rows and 3 columns. We will construct the rows in  $\mathbf{A}_3$  one by one. The first row in  $\mathbf{A}_3$  will be determined by entries in the first row in  $\mathbf{A}_1$  and all of the rows in  $\mathbf{A}_2$ , as follows. Denote the entries in the first row of  $\mathbf{A}_1$  by  $a_{11}$ ,  $a_{12}$ , and  $a_{13}$ ; and denote the rows in  $\mathbf{A}_2$  by  $R_1$ ,  $R_2$ , and  $R_3$ . Then, the first row in  $\mathbf{A}_3$  is defined to be the outcome of the row operations  $a_{11} \times R_1 + a_{12} \times R_2 + a_{13} \times R_3$ . Explicitly, these operations yield

$$1 \times \begin{bmatrix} 2 & 0 & 4 \end{bmatrix} + (-1) \times \begin{bmatrix} -1 & 3 & 2 \end{bmatrix} + 0 \times \begin{bmatrix} 5 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 2 \end{bmatrix}.$$

In a similar way, the second row of  $\mathbf{A}_3$  will be determined by entries in the second row in  $\mathbf{A}_1$  and all of the rows in  $\mathbf{A}_2$  as follows.

$$(-2) \times \begin{bmatrix} 2 & 0 & 4 \end{bmatrix} + 3 \times \begin{bmatrix} -1 & 3 & 2 \end{bmatrix} + 2 \times \begin{bmatrix} 5 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 11 & -6 \end{bmatrix}.$$

Hence, the product of  $\mathbf{A}_1$  and  $\mathbf{A}_2$  is given by

$$\mathbf{A}_3 = \begin{bmatrix} 3 & -3 & 2 \\ 3 & 11 & -6 \end{bmatrix}.$$

Although the above numerical illustration of the definition of matrix multiplication is actually sufficient for our modest purposes here, we will quickly summarize these calculations in a formal definition, for the sake of completeness. Let  $\mathbf{C}$  denote the product of two matrices  $\mathbf{A}$  and  $\mathbf{B}$ . Suppose  $\mathbf{A}$  is  $m \times k$  and  $\mathbf{B}$  is  $k \times n$ ; and let  $a_{ij}$  and  $b_{ij}$  denote the respective entries at the intersection of the  $i$ th row and the  $j$ th column in  $\mathbf{A}$  and  $\mathbf{B}$ . Then,  $\mathbf{C}$  is defined as the  $m \times n$  matrix whose entry at the intersection of its  $i$ th row and its  $j$ th column is specified by

$$c_{ij} \equiv \sum_{l=1}^k a_{il}b_{lj}.$$

What is the connection between matrix multiplication and the Simplex algorithm? Observe that in the calculation of the product of  $\mathbf{A}_1$  and  $\mathbf{A}_2$ , entries in the first row of  $\mathbf{A}_1$ , namely 1,  $-1$ , and 0, serve as individual multipliers to  $R_1$ ,  $R_2$ , and  $R_3$ ; and entries in the second row of  $\mathbf{A}_1$ , namely  $-2$ , 3, and 2, serve as a second set of multipliers to  $R_1$ ,  $R_2$ , and  $R_3$ . With this perspective, we see that the language of matrix multiplication offers a very compact description of sets of row operations.

Indeed, our next step is to show that each pivot in the Simplex algorithm is equivalent to pre-multiplying a given tableau by an appropriately-chosen matrix. To understand what this statement means, we will revisit the linear program below, and use it as a concrete example.

$$\begin{array}{ll} \text{Maximize} & z \\ \text{Subject to:} & \\ & z - 4x_1 - 3x_2 = 0 \quad (0) \\ & 2x_1 + 3x_2 + s_1 = 6 \quad (1) \\ & -3x_1 + 2x_2 + s_2 = 3 \quad (2) \\ & 2x_2 + s_3 = 5 \quad (3) \\ & 2x_1 + x_2 + s_4 = 4 \quad (4) \\ & x_1, x_2, s_1, s_2, s_3, s_4 \geq 0. \end{array}$$

Ignoring the variable names, the initial tableau for this problem is:

1	-4	-3	0	0	0	0	0
0	2	3	1	0	0	0	6
0	-3	2	0	1	0	0	3
0	0	2	0	0	1	0	5
0	2	1	0	0	0	1	4

We will view this tableau as a matrix with 5 rows and 8 columns; and we will refer to it as  $\mathbf{T}_I$ . Recall that the pivot element in the ensuing pivot is the entry “2”, located at the intersection of the last row and the second column; and that the specific row operations performed in this pivot are:  $2 \times R_4 + R_0$ ,  $(-1) \times R_4 + R_1$ ,  $(3/2) \times R_4 + R_2$ ,  $0 \times R_4 + R_3$ , and  $(1/2) \times R_4$ . Consider the first set of operations, namely  $2 \times R_4 + R_0$ . Observe that this “recipe” can be rewritten as  $1 \times R_0 + 0 \times R_1 + 0 \times R_2 + 0 \times R_3 + 2 \times R_4$ . In other words, we can explicitly indicate the nonparticipation of  $R_1$ ,  $R_2$ , and  $R_3$  in these operations by introducing three new multipliers that are equal to 0. This augmented recipe provides a more-complete description of the operations, in that the level of participation of *every* row in  $\mathbf{T}_I$  is explicitly indicated via an associated multiplier. Similarly, the operations  $(-1) \times R_4 + R_1$  can be rewritten as  $0 \times R_0 + 1 \times R_1 + 0 \times R_2 + 0 \times R_3 + (-1) \times R_4$ ; the operations  $(3/2) \times R_4 + R_2$ , as  $0 \times R_0 + 0 \times R_1 + 1 \times R_2 + 0 \times R_3 + (3/2) \times R_4$ ; the operations  $0 \times R_4 + R_3$ , as  $0 \times R_0 + 0 \times R_1 + 0 \times R_2 + 1 \times R_3 + 0 \times R_4$ ; and the operation  $(1/2) \times R_4$ , as  $0 \times R_0 + 0 \times R_1 + 0 \times R_2 + 0 \times R_3 + (1/2) \times R_4$ .

Thus, to produce the new  $R_0$ , we perform the matrix multiplication

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & -4 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 & 0 & 0 & 6 \\ 0 & -3 & 2 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 5 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 2 & 8 \end{bmatrix};$$

and similarly, to produce the remaining new rows, we perform the matrix multiplications:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & -4 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 & 0 & 0 & 6 \\ 0 & -3 & 2 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 5 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 1 & 0 & 0 & -1 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 3/2 \end{bmatrix} \times \begin{bmatrix} 1 & -4 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 & 0 & 0 & 6 \\ 0 & -3 & 2 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 5 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 7/2 & 0 & 1 & 0 & 3/2 & 9 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -4 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 & 0 & 0 & 6 \\ 0 & -3 & 2 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 5 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 1 & 0 & 5 \end{bmatrix},$$

and

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1/2 \end{bmatrix} \times \begin{bmatrix} 1 & -4 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 & 0 & 0 & 6 \\ 0 & -3 & 2 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 5 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1/2 & 0 & 0 & 0 & 1/2 & 2 \end{bmatrix}.$$

In fact, if we combine all five sets of multipliers into a single matrix, i.e., if we let

$$\mathbf{P}_1 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \end{bmatrix},$$

then all of the above operations can be consolidated into a single matrix multiplication, namely

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \end{bmatrix} \times \begin{bmatrix} 1 & -4 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 & 0 & 0 & 6 \\ 0 & -3 & 2 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 5 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 2 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 7/2 & 0 & 1 & 0 & 3/2 & 9 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 5 \\ 0 & 1 & 1/2 & 0 & 0 & 0 & 1/2 & 2 \end{bmatrix};$$

and the outcome of this multiplication is precisely the tableau produced by the first pivot in the Simplex algorithm.

Now, if we denote the matrix at the right-hand side of the above display as  $\mathbf{T}_1$ , then what we have is that

$$\mathbf{P}_1 \times \mathbf{T}_I = \mathbf{T}_1.$$

In words, this means that the Simplex tableau  $\mathbf{T}_1$ , obtained after the first pivot, is simply the product of the matrix  $\mathbf{P}_1$  and the initial tableau  $\mathbf{T}_I$ .

Continuing in this manner, and using similar notation, we see that the combined effects of  $k$  consecutive pivots, starting with the initial tableau  $\mathbf{T}_I$ , can be conceptualized as

$$\mathbf{P}_k \times \cdots \times \mathbf{P}_1 \times \mathbf{T}_I = \mathbf{T}_k.$$

In particular, if we are interested in the final tableau, which we denote by  $\mathbf{T}_F$ , then we have

$$\mathbf{P} \times \mathbf{T}_I = \mathbf{T}_F,$$

where  $\mathbf{P}$  is defined as the product of the string of matrices (of the form  $\mathbf{P}_k \times \cdots \times \mathbf{P}_1$ ) that correspond to individual pivots that, together, lead to the final tableau. In other words, we have arrived at the conclusion that there exists a matrix  $\mathbf{P}$  such that  $\mathbf{P} \times \mathbf{T}_I = \mathbf{T}_F$ ; and that the matrix  $\mathbf{P}$  fully captures the combined effects of all successive pivots. This remarkably simple fact will be referred to as the *fundamental insight*.

Observe, however, that the fundamental insight is essentially useless unless the matrix  $\mathbf{P}$  is somehow available to us. So, the next question is: How does one determine the entries in  $\mathbf{P}$ ? A little bit of reflection now leads us to the (unfortunate) realization that the Simplex algorithm itself can, in fact, be viewed as a procedure for generating the matrix  $\mathbf{P}$ . In other words, there is no free lunch. However, we will show that the story becomes quite different if we had gone through the solution of a linear program once.

Let us again return to the linear program above. Since we did solve that problem to optimality in an earlier section, the final tableau for this problem is available to us. This tableau is reproduced below.

$z$	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	
1	0	0	1/2	0	0	3/2	9
0	0	1	1/2	0	0	-1/2	1
0	0	0	-7/4	1	0	13/4	11/2
0	0	0	-1	0	1	1	3
0	1	0	-1/4	0	0	3/4	3/2

In our current notation, this means that we have

$$\mathbf{T}_F = \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 0 & 3/2 & 9 \\ 0 & 0 & 1 & 1/2 & 0 & 0 & -1/2 & 1 \\ 0 & 0 & 0 & -7/4 & 1 & 0 & 13/4 & 11/2 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & -1/4 & 0 & 0 & 3/4 & 3/2 \end{bmatrix}.$$

Therefore, in the relation  $\mathbf{P} \times \mathbf{T}_I = \mathbf{T}_F$ , both  $\mathbf{T}_I$  and  $\mathbf{T}_F$  are explicitly known to us. This naturally suggests that we might be able to identify  $\mathbf{P}$  more easily. It turns out that this indeed is possible.

We will next describe two additional properties of matrix multiplications that will help us identify  $\mathbf{P}$ .

An  $n \times n$  square matrix is called an identity matrix if all of its diagonal entries are equal to 1 and all of its off-diagonal entries are equal to 0. Such a matrix will be denoted by  $\mathbf{I}_n$ . For example, a  $3 \times 3$  identity matrix assumes the form:

$$\mathbf{I}_3 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easily seen that if we are given a matrix  $\mathbf{A}$  with  $n$  columns, then  $\mathbf{A} \times \mathbf{I}_n = \mathbf{A}$ . That is, (post) multiplying a given matrix by  $\mathbf{I}_n$  will not change the identity of that matrix. (This is why we call  $\mathbf{I}_n$  the identity matrix.)

Next, we will revisit the matrices  $\mathbf{A}_1$  and  $\mathbf{A}_2$ , and use them to illustrate a slightly different way to describe a matrix multiplication. First, we will view the matrix  $\mathbf{A}_2$  as a collection of three  $3 \times 1$  matrices, or columns. These columns are explicitly given by

$$\mathbf{c}_1 \equiv \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}, \mathbf{c}_2 \equiv \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \text{ and } \mathbf{c}_3 \equiv \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}.$$

Next, recall that the first column in  $\mathbf{A}_3$  (the product of  $\mathbf{A}_1$  and  $\mathbf{A}_2$ ) is

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

It is easily seen that this column can be computed via  $\mathbf{A}_1 \times \mathbf{c}_1$ , i.e.,

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix};$$

and similarly, that the remaining two columns in  $\mathbf{A}_3$  can be computed via  $\mathbf{A}_1 \times \mathbf{c}_2$  and  $\mathbf{A}_1 \times \mathbf{c}_3$ . More formally, we have

$$\begin{aligned} \mathbf{A}_3 &\equiv \mathbf{A}_1 \times \mathbf{A}_2 \\ &= \mathbf{A}_1 \times \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}_1 \times \mathbf{c}_1 & \mathbf{A}_1 \times \mathbf{c}_2 & \mathbf{A}_1 \times \mathbf{c}_3 \end{bmatrix}; \end{aligned}$$

that is, the matrix  $\mathbf{A}_3$  can be generated one column at a time, by executing a sequence of matrix products of the form  $\mathbf{A}_1 \times \mathbf{c}_j$  with  $j = 1, 2, 3$ .

We now return to our example. Observe that in the initial tableau, the columns associated with the variables  $z$ ,  $s_1$ ,  $s_2$ ,  $s_3$ , and  $s_4$  constitute a  $5 \times 5$  identity matrix,  $\mathbf{I}_5$ . Therefore, if

we focus our attention on these five columns only, then, as a consequence of  $\mathbf{P} \times \mathbf{T}_I = \mathbf{T}_F$  (the fundamental insight) and the alternative description of matrix multiplication above, the corresponding five columns in the final tableau must equal to  $\mathbf{P} \times \mathbf{I}_5$ . Since  $\mathbf{P} \times \mathbf{I}_5 = \mathbf{P}$ , we can now identify  $\mathbf{P}$  as the matrix defined by these five columns in the final tableau. More explicitly, this simply means that since

$$\begin{aligned} \mathbf{P} \times & \begin{bmatrix} 1 & ? & ? & 0 & 0 & 0 & 0 & ? \\ 0 & ? & ? & 1 & 0 & 0 & 0 & ? \\ 0 & ? & ? & 0 & 1 & 0 & 0 & ? \\ 0 & ? & ? & 0 & 0 & 1 & 0 & ? \\ 0 & ? & ? & 0 & 0 & 0 & 1 & ? \end{bmatrix} \\ = & \begin{bmatrix} 1 & ? & ? & 1/2 & 0 & 0 & 3/2 & ? \\ 0 & ? & ? & 1/2 & 0 & 0 & -1/2 & ? \\ 0 & ? & ? & -7/4 & 1 & 0 & 13/4 & ? \\ 0 & ? & ? & -1 & 0 & 1 & 1 & ? \\ 0 & ? & ? & -1/4 & 0 & 0 & 3/4 & ? \end{bmatrix}, \end{aligned}$$

where the ?'s represent ignored entries, we must have

$$\mathbf{P} = \begin{bmatrix} 1 & 1/2 & 0 & 0 & 3/2 \\ 0 & 1/2 & 0 & 0 & -1/2 \\ 0 & -7/4 & 1 & 0 & 13/4 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -1/4 & 0 & 0 & 3/4 \end{bmatrix}.$$

In conclusion, we have shown that an important consequence of the fundamental insight is that if a linear program has been solved to optimality once, then the matrix  $\mathbf{P}$  can be read out directly from the final tableau. Applications of this result to sensitivity analysis will be discussed in the latter part of this section.