

# Unrestricted and Other Variable Types

Consider the linear program:

$$\begin{array}{ll} \text{Minimize} & 2x_1 + 3x_2 - x_3 \\ \text{Subject to:} & \\ & x_1 - x_2 + 2x_3 \leq 10 \quad (1) \\ & -3x_1 + 2x_2 - 4x_3 = 6 \quad (2) \\ & x_1 + 9x_2 \geq 7 \quad (3) \\ & x_1 \leq 0, x_2 \geq 0, x_3 \text{ is unrestricted.} \end{array}$$

In this problem, the variable  $x_1$  is nonpositive and the variable  $x_3$  is unrestricted in sign. Before the launch of the two-phase procedure, we need to first convert the problem into one where all variables are nonnegative.

Handling nonpositive variables is quite easy. The idea is to define a new variable that equals the negative of the original variable. In this case, we can denote the new variable by  $x'_1$  (say) and let  $x'_1 \equiv -x_1$ . Since the original variable is nonpositive, it follows that the new variable  $x'_1$  is nonnegative. To implement this change of variable, we simply replace every instance of  $x_1$  in the given problem by  $-x'_1$ .

More generally, a variable may sometimes have a given constant upper bound. For example, suppose the requirement  $x_1 \leq 0$  is revised to  $x_1 \leq 3$ . Observe that the latter can be rewritten as  $x_1 - 3 \leq 0$ , which suggests that if we define  $\bar{x}_1 \equiv x_1 - 3$ , then  $\bar{x}_1$  is a nonpositive variable. Hence, we can replace every instance of  $x_1$  in the given problem by  $\bar{x}_1 + 3$  to create an equivalent problem with  $\bar{x}_1 \leq 0$ . Notice that doing so will create an extra constant in the objective function. The constant will not have any impact on the Simplex algorithm, but it is recommended that it be set aside temporarily until the end of the solution procedure.

Of course, a variable with a constant lower bound, say  $x_1 \geq 5$ , can be handled similarly: Replace every instance of  $x_1$  by  $\bar{x}_1 + 5$ , where  $\bar{x}_1 \geq 0$ .

We now turn our attention to the handling of unrestricted variables,  $x_3$  in this case. The idea is to rewrite such a variable as the difference between two nonnegative variables. That is, we can introduce two nonnegative variables, say  $x_3^+$  and  $x_3^-$ , and let  $x_3 \equiv x_3^+ - x_3^-$ .

To see how this works, consider a numerical example. Suppose the value of  $x_3$  is to equal 5; then, the assignments  $x_3^+ = 8$  and  $x_3^- = 3$  will yield a difference of 5. Such assignments are clearly not unique. Other examples are:  $x_3^+ = 18$  and  $x_3^- = 13$ ; and  $x_3^+ = 11$  and  $x_3^- = 6$ . In fact, such assignments are innumerable.

Of particular interest is the assignments  $x_3^+ = 5$  and  $x_3^- = 0$ . Notice that in this case, we have  $x_3^+ = \max[x_3, 0]$  and  $x_3^- = -\min[x_3, 0]$ . When these two relations hold, we say that  $x_3^+$  is the *positive part* of  $x_3$  and  $x_3^-$  is the *negative part* of  $x_3$ . Thus, if the variable  $x_3$  equals 5, then its positive part is 5 and its negative part is 0.

As another numerical example, suppose  $x_3$  is to equal  $-5$ . Then, the positive part of  $x_3$ , denoted by  $x_3^+$ , equals 0; and the negative part of  $x_3$ , denoted by  $x_3^-$ , equals 5. Indeed,  $x_3 = 0 - 5 = -5$ ; and both the positive part and the negative part are nonnegative.

We now execute the proposed conversions for  $x_1$  and  $x_3$ . With  $x_1$  replaced by  $-x'_1$  and  $x_3$  replaced by  $x_3^+ - x_3^-$ , the given problem is equivalent to the linear program below.

$$\begin{array}{ll}
 \text{Minimize} & -2x'_1 + 3x_2 - x_3^+ + x_3^- \\
 \text{Subject to:} & \\
 & -x'_1 - x_2 + 2x_3^+ - 2x_3^- \leq 10 \quad (1) \\
 & 3x'_1 + 2x_2 - 4x_3^+ + 4x_3^- = 6 \quad (2) \\
 & -x'_1 + 9x_2 \geq 7 \quad (3) \\
 & x'_1 \geq 0, x_2 \geq 0, x_3^+ \geq 0, x_3^- \geq 0.
 \end{array}$$

Thus, the new problem has 4 variables, all of which are nonnegative; and it is now ready for the application of the two-phase procedure.

After solving the new problem, it is very easy to convert its optimal solution into one for the original problem. As an example, consider an arbitrarily chosen feasible solution to the new problem, say  $(x'_1, x_2, x_3^+, x_3^-) = (1, 1, 1, 1)$ . Since  $x_1 = -x'_1$  and  $x_3 = x_3^+ - x_3^-$ , the corresponding feasible solution for the original problem can be reconstructed as  $(x_1, x_2, x_3) = (-1, 1, 0)$ . A similar conversion can, of course, be executed for the optimal solution.

Finally, we observe that in the new problem, terms involving  $x_3^+$  and  $x_3^-$  always appear in the form of a pair, with opposite signs. This property has an interesting consequence. Recall that all of the operations in the Simplex algorithm are *row* operations. This implies that the above property will be preserved throughout the Simplex iterations. Now, suppose  $x_3^+$  is a basic variable in a tableau. Then, since every column associated with a basic variable must have its entries equal to 0 except for one that equals 1, it follows that the  $x_3^-$ -column

assumes the form

$$\begin{array}{|c|} \hline x_3^- \\ \hline 0 \\ \hline 0 \\ \hline \vdots \\ \hline 0 \\ \hline -1 \\ \hline 0 \\ \hline \vdots \\ \hline 0 \\ \hline \end{array}$$

where the coefficient  $-1$  is located at some row. Clearly, the variable associated with such a column cannot be basic. Therefore, if  $x_3^+$  happens to be basic, then  $x_3^-$  must be nonbasic; and conversely, if  $x_3^-$  happens to be basic, then  $x_3^+$  must be nonbasic. Of course, both of these variables can be nonbasic at the same time. In summary, the conclusion is that, in any basic feasible solution, at most one of the two values assigned to  $x_3^+$  and  $x_3^-$  can be positive. Remarkably, this is in complete agreement with the (intended) interpretation of  $x_3^+$  and  $x_3^-$  as the positive part and the negative part of the variable  $x_3$ .