Initialization: The Two-Phase Formulation

Consider again the linear program:

$$\begin{array}{lrrcrl} \text{Minimize} & 4x_1 & +x_2 & & & \\ \text{Subject to:} & & & & & \\ & 3x_1 & +x_2 & = & 3 & (1) \\ & 4x_1 & +3x_2 & \geq & 6 & (2) \\ & x_1 & +2x_2 & \leq & 3 & (3) \end{array}$$

$$x_1,\, x_2 \geq 0\,.$$

We will solve this problem using the two-phase method.

The only difference between the big-M method and the two-phase method is in the formulation of the objective function. As noted earlier, we will first focus on driving out the artificial variables. This is done in Phase I, where we work with the objective function

$$\text{Minimize} \quad A_1 + A_2\,.$$

That is, we will be minimizing the sum of all artificial variables, subject to the same set of constraints. If Phase I ends with a success, i.e., with all artificial variables driven to zero, then we will launch a second phase, Phase II, where we attempt to iterate toward an optimal solution of the original linear program, starting with the final basic feasible solution in Phase I. If, on the other hand, Phase I ends with a failure, i.e., with at least one positive artificial variable, then the original problem must be infeasible, and hence the second phase is not necessary.

Phase I

Let $z = A_1 + A_2$. After introducing $z - A_1 - A_2 = 0$ as equation (0), we obtain the tableau below.

Basic Variable	z	x_1	x_2	A_1	s_1	A_2	s_2	
	1	0	0	−1	0	−1	0	0
A_1	0	3	1	1	0	0	0	3
A_2	0	4	3	0	−1	1	0	6
s_2	0	1	2	0	0	0	1	3

Note that equations (1), (2), and (3) are copied verbatim from the previous big-M formulation. Since A_1 and A_2 do not have a zero coefficient in R_0, we will add a copy of both R_1 and R_2 into R_0. This yields the initial Simplex tableau given below.

Basic Variable	z	x_1	x_2	A_1	s_1	A_2	s_2	
	1	7	4	0	−1	0	0	9
A_1	0	3	1	1	0	0	0	3
A_2	0	4	3	0	−1	1	0	6
s_2	0	1	2	0	0	0	1	3

Thus, the starting basic feasible solution is $(x_1, x_2, A_1, s_1, A_2, s_2) = (0, 0, 3, 0, 6, 3)$, with a corresponding objective-function value of 9.

Since we are minimizing, the fact that both x_1 and x_2 have positive coefficients in R_0 implies that the current solution is not optimal. Moreover, since 7 is strictly larger than 4, x_1 is the entering variable, and the x_1-column is the pivot column.

A ratio test shows that R_1 is the pivot row, and we will therefore execute a pivot with the entry "3" as the pivot element. An examination of the x_1-column shows that we need to go through the following row operations: $(-7/3) \times R_1 + R_0$, $(1/3) \times R_1$, $(-4/3) \times R_1 + R_2$, and $(-1/3) \times R_1 + R_3$. After removing the column associated with A_1, which is the leaving artificial variable, these four sets of operations lead to the new tableau below.

Basic Variable	z	x_1	x_2	s_1	A_2	s_2	
	1	0	5/3	−1	0	0	2
x_1	0	1	1/3	0	0	0	1
A_2	0	0	5/3	−1	1	0	2
s_2	0	0	5/3	0	0	1	2

It follows that the new basic feasible solution is $(x_1, x_2, s_1, A_2, s_2) = (1, 0, 0, 2, 2)$, with a corresponding objective-function value of 2. Since the coefficient of x_2 in R_0 is positive, the current solution is not optimal. This marks the end of the first iteration.

Since the x_2-column is the pivot column, we compute the ratios: $1/(1/3)$, $2/(5/3)$, and $2/(5/3)$. It follows that the minimum ratio is equal to 6/5, which, interestingly, is attained at both R_2 and R_3. In such a situation, the standard Simplex algorithm calls for breaking the tie arbitrarily. Hence, in general, we could let either R_2 or R_3 be the pivot row. However, observe that the basic variable currently associated with R_2 is A_2, which is an artificial variable. Therefore, we will choose R_2 as the pivot row, so as to drive out A_2 as early as possible. (This choice, however, does not necessarily shorten the overall solution time.) After deleting the A_2-column and executing the row operations $(-1) \times R_2 + R_0$, $(-1/5) \times R_2 + R_1$, $(3/5) \times R_2$, and $(-1) \times R_2 + R_3$, we obtain the new tableau below.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	0	0	0	0
x_1	0	1	0	1/5	0	3/5
x_2	0	0	1	−3/5	0	6/5
s_2	0	0	0	1	1	0

The basic feasible solution associated with this tableau is $(x_1, x_2, s_1, s_2) = (3/5, 6/5, 0, 0)$, with a corresponding objective-function value of 0. Since both artificial variables have been driven out of the basis, the current solution is feasible to the original problem. This concludes Phase I.

Phase II

The first step in Phase II is to reintroduce the original objective function, which has not participated in the solution process up to this point. Let $z = 4x_1 + x_2$, which implies that the new equation (0) is $z - 4x_1 - x_2 = 0$. With the introduction of this new equation (0), the final tableau in Phase I becomes:

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	−4	−1	0	0	0
x_1	0	1	0	1/5	0	3/5
x_2	0	0	1	−3/5	0	6/5
s_2	0	0	0	1	1	0

At this point, the tableau is not yet ready for the Simplex algorithm, since x_1 and x_2 have nonzero coefficients in R_0. To rectify that, we add $4 \times R_1$ and R_2 into R_0. This yields the initial Simplex tableau below.

Basic Variable	z	x_1	x_2	s_1	s_2		Ratio Test	
	1	0	0	1/5	0	18/5		
x_1	0	1	0	1/5	0	3/5	$(3/5)/(1/5) = 3$	
x_2	0	0	1	−3/5	0	6/5	−	
s_2	0	0	0	1	1	0	$0/1 = 0$	← Minimum

Here, the initial basic feasible solution is $(x_1, x_2, s_1, s_2) = (3/5, 6/5, 0, 0)$, which is, of course, carried over from Phase I. The corresponding objective-function value is 18/5. Since the coefficient of s_1 in R_0 is positive, this solution is not optimal.

Next, we carry out a ratio test, with the s_1-column as the pivot column; this is shown at the right margin of the above tableau. Notice that the constant on the right-hand side of R_3 is zero, and hence it produces the minimum ratio 0.

An examination of the s_1-column shows that we should now execute the following row operations: $(-1/5) \times R_3 + R_0$, $(-1/5) \times R_3 + R_1$, $(3/5) \times R_3 + R_2$, and $1 \times R_3$. Doing so leads to the tableau below.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	0	0	−1/5	18/5
x_1	0	1	0	0	−1/5	3/5
x_2	0	0	1	0	3/5	6/5
s_1	0	0	0	1	1	0

Notice that both the basic feasible solution and its corresponding objective-function value did not change as a result of this pivot. Since s_2 has a negative coefficient in R_0, the current solution is optimal. This concludes Phase II, and the solution of the given linear program as well.

A few comments are in order.

Recall that the solutions associated with the last two tableaus are identical. That is, they both are $(x_1, x_2, s_1, s_2) = (3/5, 6/5, 0, 0)$. Although the *numerical* values in these solutions are the same, there is a conceptual difference between the two, namely that they correspond to, or are read from, tableaus with different basis. Specifically, the basis in the first tableau is x_1, x_2, and s_2; whereas in the second, x_1, x_2, and s_1. Thus, although s_2 and s_1 are basic in these two respective tableaus, they assume the value 0.

In terms of the mechanics of the Simplex algorithm, it is interesting to observe that the tied ratios in R_2 and R_3 just before the last pivot in Phase I is what led to the 0 on the right-hand side of equation (3) after that pivot.

A basic feasible solution is said to be *degenerate* if at least one of the basic variables in it assumes the value 0. Our example here shows that when a solution is degenerate, it is possible for two (or more) different-looking tableaus correspond to the same numerical solution. One consequence of the phenomenon of degeneracy is that the value of the objective function may not improve strictly from one iteration to the next. In other words, the Simplex algorithm may *stall.* When stalling occurs, the fear is that we may never progress to an optimal solution (even if one is known to exist). Since degeneracy may affect the termination of the Simplex algorithm, we will return to a further discussion about its origin, later in this section.

Another consequence of degeneracy is that it is possible for an *optimal* solution to fail the optimality test at the end of an iteration. This is exemplified by the initial tableau in Phase II.

If one completed all of the iterations under the previous big-M formulation, then it can be seen that the Simplex tableaus in these two solution procedures closely mirror each other. The difference is that, without the M's, the arithmetic in the two-phase method is slightly simpler. This is why we chose not to complete the details in the previous big-M formulation.