

A Characterization of the Corner-Point Solutions

Consider the previous example again. Our first step is to “rewrite” that problem as:

$$\begin{array}{rllll}
 \text{Maximize} & 4x_1 & +3x_2 & & & \\
 \text{Subject to:} & & & & & \\
 & 2x_1 & +3x_2 & +s_1 & & = 6 & (1) \\
 & -3x_1 & +2x_2 & & +s_2 & = 3 & (2) \\
 & & 2x_2 & & +s_3 & = 5 & (3) \\
 & 2x_1 & +x_2 & & & +s_4 = 4 & (4) \\
 & x_1, x_2 \geq 0 & \text{and } s_1, s_2, s_3, s_4 \geq 0, & & & &
 \end{array}$$

where we have (i) introduced four new variables, $s_1, s_2, s_3,$ and $s_4,$ (ii) converted all functional (inequality) constraints into equalities, and (iii) added nonnegativity requirements for the new variables.

We will first argue that this new problem is indeed equivalent to the previous one. Consider any feasible solution to the original problem. Let $(x_1, x_2) = (1, 1),$ for example. This solution is feasible because

$$\begin{array}{rcll}
 2 \times 1 & +3 \times 1 & = & 5 \leq 6 \\
 -3 \times 1 & +2 \times 1 & = & -1 \leq 3 \\
 & 2 \times 1 & = & 2 \leq 5 \\
 2 \times 1 & +1 \times 1 & = & 3 \leq 4,
 \end{array}$$

and the two values in $(1, 1)$ are both nonnegative. Note that for each functional constraint, there is a nonnegative “slack” between the evaluation on the left-hand side and the constant on the right-hand side. Explicitly, these slacks are given by $1 (= 6 - 5), 4 (= 3 - (-1)), 3 (= 5 - 2),$ and $1 (= 4 - 3),$ respectively. Now, if we let $s_1 = 1, s_2 = 4, s_3 = 3,$ and $s_4 = 1,$ then the *augmented solution* $(x_1, x_2, s_1, s_2, s_3, s_4) = (1, 1, 1, 4, 3, 1)$ is, by construction, a feasible solution to the new problem. Thus, by appropriately augmenting a given feasible solution to the original problem, we can always construct a corresponding feasible solution for the new problem. Next, consider any feasible solution to the new problem. Let $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 3, 5, 4),$ for example. This solution is feasible because

$$\begin{array}{rcll}
 2 \times 0 & +3 \times 0 & +6 & = 6 \\
 -3 \times 0 & +2 \times 0 & & +3 = 3 \\
 & 2 \times 0 & & +5 = 5 \\
 2 \times 0 & +1 \times 0 & & +4 = 4,
 \end{array}$$

and all six assigned values are nonnegative. Now, if we strip away the last four values in $(0, 0, 6, 3, 5, 4)$ and consider the solution $(x_1, x_2) = (0, 0),$ then, since these four values are

nonnegative, we obviously have

$$\begin{array}{rcl} 2 \times 0 & +3 \times 0 & \leq 6 \\ -3 \times 0 & +2 \times 0 & \leq 3 \\ & 2 \times 0 & \leq 5 \\ 2 \times 0 & +1 \times 0 & \leq 4, \end{array}$$

which means that the solution $(0, 0)$ is feasible to the original problem. Therefore, by removing the last four components of a given feasible solution to the new problem, we can always obtain a corresponding feasible solution to the original problem. In summary, we have shown that the feasible sets of the original and the new linear program are identical. Finally, since these two problems also have the same objective function, they are equivalent.

This discussion shows that a point in the feasible region can be described either in the original form of two coordinates (x_1, x_2) or in the augmented form with six coordinates $(x_1, x_2, s_1, s_2, s_3, s_4)$. We shall henceforth refer to the new variables $s_1, s_2, s_3,$ and s_4 as the *slack variables*, and the new problem as the *augmented problem*.

An important reason for constructing the augmented problem is that the value of the slack variable associated with a new equality constraint provides *explicit* information on the “tightness” of the corresponding original (inequality) constraint. We will illustrate this via a series of examples. Please refer to Figure LP-4 while going over these examples.

Consider the solution $(x_1, x_2) = (3/2, 1)$, which is feasible for the original problem. This solution has the corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (3/2, 1, 0, 11/2, 3, 0)$, where the values for the slack variables are determined by substituting $x_1 = 3/2$ and $x_2 = 1$ into equations (1), (2), (3), and (4). Notice that in the augmented solution, we have $s_1 = 0$ and $s_4 = 0$, which corresponds to the fact that this solution satisfies constraints (1) and (4) as equalities in the original problem. In other words, this solution is such that the original inequality constraints (1) and (4) are tight, or *binding*. Notice further that, in contrast, both s_2 and s_3 are strictly positive; hence, the inequality constraints (2) and (3) in the original problem are not binding. From Figure LP-4, we see that, in this example, we have a solution that corresponds to a corner point of the feasible region.

As a second example, consider the feasible solution $(x_1, x_2) = (1, 4/3)$ and its corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (1, 4/3, 0, 10/3, 7/3, 2/3)$. In this case, we have $s_1 = 0$ and the values of the other slack variables are all positive; therefore, constraint (1) in the original problem is binding, whereas the other functional constraints are not. Note that point $(1, 4/3)$ lies on an edge of the feasible region.

As a third example, consider the (candidate) solution $(x_1, x_2) = (2/3, 5/2)$ and its corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (2/3, 5/2, -17/6, 0, 0, 1/6)$. Since

$s_2 = 0$ and $s_3 = 0$, the corresponding original constraints (2) and (3) are binding. Note however that $s_1 = -17/6$, a negative value. This means that constraint (1) in the original problem is violated. Hence, while the point $(2/3, 5/2)$ lies at the intersection of two of the original constraint lines, it is not a feasible solution to the original problem.

In the above three examples, we focussed our attention on whether or not any given original functional constraint is binding. In the next two examples, we will also consider the relevance of the original nonnegativity constraints, $x_1 \geq 0$ and $x_2 \geq 0$.

Consider, as a fourth example, the solution $(x_1, x_2) = (0, 0)$, with corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 3, 5, 4)$. Since both x_1 and x_2 are equal to 0 and since all slack variables are positive, we see that $(0, 0)$ is feasible, and that the nonnegativity constraints $x_1 \geq 0$ and $x_2 \geq 0$ are the only binding constraints. This solution, like $(x_1, x_2) = (3/2, 1)$, corresponds to a corner point.

As our final example, consider the solution $(x_1, x_2) = (1, 1)$ and its corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (1, 1, 1, 4, 3, 1)$. This solution is clearly feasible. The fact that all variables (including x_1 and x_2) assume a positive value tells us that every point that is located in the vicinity of $(1, 1)$ is also feasible. More formally, let δ_1 and δ_2 be two arbitrary (not necessarily positive) values; then, as long as the absolute values of δ_1 and δ_2 are sufficiently small, every solution of the form $(x_1, x_2) = (1 + \delta_1, 1 + \delta_2)$ will also be feasible. In other words, at point $(1, 1)$, none of the six inequality constraints in the original problem is tight. A point such as $(1, 1)$ will be called an *interior point*.

We are now ready to revisit Procedure Corner Points.

Recall that in the first step of that procedure, we are to choose two of the six equations as defining equations and solve the selected pair of equations to obtain the coordinates of their intersection. In the setting of the augmented problem, this can be rephrased as follows. From our discussion above, we see that declaring an equation in the original problem as a defining equation is tantamount to assigning a value of zero to a variable in the augmented problem. For example, this means that: (i) if we are interested in solutions in the original problem that make the inequality $2x_1 + 3x_2 \leq 6$ binding, we could simply assign the value zero to the variable s_1 and consider augmented solutions of the form $(x_1, x_2, 0, s_2, s_3, s_4)$; (ii) if we are interested in solutions that make the inequality $x_1 \geq 0$ binding, we could let $x_1 = 0$ and consider augmented solutions of the form $(0, x_2, s_1, s_2, s_3, s_4)$; and (iii) if we are interested in solutions that make both $x_1 \geq 0$ and $2x_1 + 3x_2 \leq 6$ binding, we could let $x_1 = s_1 = 0$ and consider augmented solutions of the form $(0, x_2, 0, s_2, s_3, s_4)$. Thus, what we are looking for in the first step of Procedure Corner Points corresponds to the set of augmented solutions that have the value zero assigned to two of their coordinates. Formally, this results in the following restatement.

Revised Step 1: Choose an arbitrary pair of variables in the augmented problem, and assign the value zero to these variables. This reduces the functional constraints in the augmented problem to a set of four equations in four unknowns. Solve (if a unique solution exists) this equation system to obtain an explicit augmented solution.

A solution produced by this step will be called a *basic solution*.

By construction, an augmented solution produced by the above step necessarily satisfies all functional constraints in the augmented problem. This leaves us with the verification of the nonnegativity constraints, which will be done in the next step.

Revised Step 2: If all of the values in the augmented solution produced by the above Revised Step 1 are nonnegative, accept it as a corner-point solution; otherwise, discard the solution.

A solution produced by this step will be called a *basic feasible solution*.

Finally, as in the original Procedure Corner Points, repeating these two steps will produce, one by one, all of the corner-point solutions.

What we have established is that the set of corner-point solutions is characterized, or is identical, to the set of basic feasible solutions in the augmented problem. Because this characterization is algebraic (and hence is not dependent on a graphical representation of the feasible set), it is extremely important. In the next section, we will show that it forms the foundation of the Simplex method.

Due to their importance, we now conclude this section with a formal definition of the concepts of basic solutions and basic feasible solutions.

Consider a system of m linear equations in n unknowns, denoted by x_1, x_2, \dots, x_n , and assume that the system has the following format:

$$\begin{array}{cccccc} a_{11}x_1 & +a_{12}x_2 & + & \cdots & +a_{1n}x_n & = & b_1 \\ a_{21}x_1 & +a_{22}x_2 & + & \cdots & +a_{2n}x_n & = & b_2 \\ & & & \cdots & & & \\ a_{m1}x_1 & +a_{m2}x_2 & + & \cdots & +a_{mn}x_n & = & b_m, \end{array}$$

where the a_{ij} 's and the b_j 's are given constants. We shall assume that $m \leq n$, which means the number of variables is at least as many as the number of equations. Under this assumption, the equation system will typically have an infinite number of solutions. If we arbitrarily select $n - m$ variables and set their values to zero, then the system will be reduced to a set of m equations in m unknowns. The selected set of $n - m$ variables will be called *nonbasic variables*, since they do not "participate" in the reduced equation system; and

the remaining m variables will be called *basic variables*. Now, the reduced equation system may, in general, have: (i) a unique solution, (ii) no solution, or (iii) an infinite number of solutions. Suppose, for a given selected set of nonbasic variables, the reduced set of m equations in m (basic) variables has a unique solution. Then, the resulting “full” solution, with an explicit value for each of the basic and nonbasic variables, is called a *basic solution*. Note that we have not made any assumption on the sign of any of the (basic) variables. A basic solution that happens to have all nonnegative values will be called a *basic feasible solution*.

In the augmented problem above, we have four functional equality constraints and six decision variables. Hence, $m = 4$ and $n = 6$. Suppose we declare, for example, s_1 and s_4 as nonbasic variables. Then, it is easy to check that the reduced equation system,

$$\begin{array}{rcccc} 2x_1 & +3x_2 & & & = & 6 \\ -3x_1 & +2x_2 & +s_2 & & = & 3 \\ & & 2x_2 & +s_3 & = & 5 \\ 2x_1 & +x_2 & & & = & 4, \end{array}$$

has the (unique) solution $(x_1, x_2, s_2, s_3) = (3/2, 1, 11/2, 3)$. After supplementing this with two zeros for the nonbasic variables, we obtain the basic solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (3/2, 1, 0, 11/2, 3, 0)$. Moreover, since all of these values are nonnegative, what we have is a basic feasible solution as well.

In general, with m equations and n unknowns, the total number of basic solutions cannot exceed

$$\binom{n}{n-m} = \frac{n!}{(n-m)!m!}.$$

For $m = 4$ and $n = 6$, this evaluates to 15. Thus, potentially, we could have up to 15 basic solutions.