A Production Planning Problem

Suppose a production manager is responsible for scheduling the monthly production levels of a certain product for a planning horizon of twelve months. For planning purposes, the manager was given the following information:

- The total demand for the product in month j is d_j, for $j = 1, 2, \ldots, 12$. These could either be targeted values or be based on forecasts.
- The cost of producing each unit of the product in month j is c_j (dollars), for $j = 1, 2, \ldots, 12$. There is no setup/fixed cost for production.
- The inventory holding cost per unit for month j is h_j (dollars), for $j = 1, 2, \ldots, 12$. These are incurred at the end of each month.
- The production capacity for month j is m_j, for $j = 1, 2, \ldots, 12$.

The manager's task is to generate a production schedule that minimizes the total production and inventory-holding costs over this twelve-month planning horizon.

To facilitate the formulation of a linear program, the manager decides to make the following simplifying assumptions:

1. There is no initial inventory at the beginning of the first month.
2. Units scheduled for production in month j are immediately available for delivery at the beginning of that month. This means in effect that the production rate is infinite.
3. Shortage of the product is not allowed at the end of any month.

To understand things better, let us consider the first month. Suppose, for that month, the planned production level equals 100 units and the demand, d_1, equals 60 units. Then, since the initial inventory is 0 (Assumption 1), the ending inventory level for the first month would be $0 + 100 - 60 = 40$ units. Note that all 100 units are immediately available for delivery (Assumption 2); and that given $d_1 = 60$, one must produce no less than 60 units in the first month, to avoid shortage (Assumption 3). Suppose further that $c_1 = 15$ and $h_1 = 3$. Then, the total cost for the first month can be computed as: $15 \times 100 + 3 \times 40 = 1380$ dollars.

At the start of the second month, there would be 40 units of the product in inventory, and the corresponding ending inventory can be computed similarly, based on the initial inventory, the scheduled production level, and the total demand for that month. The same scheme is then repeated until the end of the entire planning horizon.

We now proceed with a linear-programming (LP) formulation of this problem.

The Decision Variables

The manager's task is to set a production level for each month. Therefore, we have twelve decision variables:

x_j = the production level for month j, $j = 1, 2, \ldots, 12$.

The Objective Function

Consider the first month again. From the discussion above, we have:

The production cost equals $c_1 x_1$.

The inventory-holding cost equals $h_1(x_1 - d_1)$, assuming that the ending inventory level, $x_1 - d_1$, is nonnegative.

Therefore, the total cost for the first month equals $c_1 x_1 + h_1(x_1 - d_1)$.

For the second month, we have:

The production cost equals $c_2 x_2$.

The inventory-holding cost equals $h_2(x_1 - d_1 + x_2 - d_2)$, assuming that the ending inventory level, $x_1 - d_1 + x_2 - d_2$, is nonnegative. This follows from the fact that the starting inventory level for this month is $x_1 - d_1$, the production level for this month is x_2, and the demand for this month is d_2.

Therefore, the total cost for the second month equals $c_2 x_2 + h_2(x_1 - d_1 + x_2 - d_2)$.

Continuation of this argument yields that:

The total production cost for the entire planning horizon equals

$$\sum_{j=1}^{12} c_j x_j \equiv c_1 x_1 + c_2 x_2 + \ldots + c_{12} x_{12} \,,$$

where we have introduced the standard summation notation ("$\equiv$" means by definition).

The total inventory-holding cost for the entire planning horizon equals

$$\begin{aligned}
\sum_{j=1}^{12} h_j \left[\sum_{k=1}^{j}(x_k - d_k)\right] &\equiv h_1 \left[\sum_{k=1}^{1}(x_k - d_k)\right] + h_2 \left[\sum_{k=1}^{2}(x_k - d_k)\right] + \ldots \\
&\quad + h_{12} \left[\sum_{k=1}^{12}(x_k - d_k)\right] \\
&= h_1 [x_1 - d_1] + h_2 [(x_1 - d_1) + (x_2 - d_2)] + \ldots \\
&\quad + h_{12} [(x_1 - d_1) + (x_2 - d_2) + \ldots + (x_{12} - d_{12})] .
\end{aligned}$$

Since our goal is to minimize the total production and inventory-holding costs, the objective function can now be stated as

$$\text{Minimize} \quad \sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j \left[\sum_{k=1}^{j}(x_k - d_k)\right] .$$

The Constraints

Since the production capacity for month j is m_j, we require

$$x_j \leq m_j$$

for $j = 1, 2, \ldots, 12$; and since shortage is not allowed (Assumption 3), we require

$$\sum_{k=1}^{j}(x_k - d_k) \geq 0$$

for $j = 1, 2, \ldots, 12$. This results in a set of 24 functional constraints. Of course, being production levels, the x_j's should be nonnegative.

LP Formulation

In summary, we have arrived at the following formulation:

$$\begin{aligned}
\text{Minimize} \quad & \sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j \left[\sum_{k=1}^{j}(x_k - d_k)\right] \\
\text{Subject to:} \quad & \\
& x_j \leq m_j \quad \text{for } j = 1, 2, \ldots, 12 \\
& \sum_{k=1}^{j}(x_k - d_k) \geq 0 \quad \text{for } j = 1, 2, \ldots, 12 \\
& x_j \geq 0 \quad \text{for } j = 1, 2, \ldots, 12 .
\end{aligned}$$

This is a linear program with 12 decision variables, 24 functional constraints, and 12 nonnegativity constraints. In an actual implementation, we need to replace the c_j's, the h_j's, the d_j's, and the m_j's with explicit numerical values.

An Alternative Formulation

In the above formulation, the expression for the total inventory-holding cost in the objective function involves a nested sum, which is rather complicated. Notice that for any given j, the inner sum in that expression, $\sum_{k=1}^{j}(x_k - d_k)$, is simply the ending inventory level for month j. This motivates the introduction of an additional set of decision variables to represent the ending inventory levels. Specifically, let

y_j = the ending inventory level for month j, $j = 1, 2, \ldots, 12$;

then, the objective function can be rewritten in the following simpler-looking form:

$$\text{Minimize} \quad \sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j y_j \,.$$

With these new variables, the no-shortage constraints also simplify to $y_j \geq 0$ for $j = 1, 2, \ldots, 12$. However, we now need to introduce a new set of constraints to "link" the x_j's and the y_j's together.

Consider the first month again. Denote the initial inventory level as y_0; then, by assumption, we have $y_0 = 0$. Since the production level is x_1 and the demand is d_1 for this month, we have $y_1 = y_0 + x_1 - d_1$. Continuation of this argument shows that for $j = 1, 2, \ldots, 12$,

$$y_j = y_{j-1} + x_j - d_j \,;$$

and these relations should appear as constraints to ensure that the y_j's indeed represent ending inventory levels. We have, therefore, arrived at the following new formulation:

$$\begin{array}{ll}
\text{Minimize} & \displaystyle\sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j y_j \\
\text{Subject to:} & \\
& \begin{array}{rcll} x_j & \leq & m_j & \text{for } j = 1, 2, \ldots, 12 \\ y_j & = & y_{j-1} + x_j - d_j & \text{for } j = 1, 2, \ldots, 12 \end{array} \\
& x_j \geq 0 \text{ for } j = 1, 2, \ldots, 12 \text{ and } y_j \geq 0 \text{ for } j = 1, 2, \ldots, 12 \,,
\end{array}$$

which is a linear program with 24 decision variables, 24 functional constraints, and 24 nonnegative variables.

Although there are twice as many decision variables in the new formulation, both formulations have the same number of functional constraints. We will show in a later section that the total amount of effort necessary to arrive at an optimal solution to a linear program depends primarily on the number of functional constraints.

In general, it is not uncommon to have several equivalent formulations of the same problem.

Discussion

If Assumption 1 is relaxed, so that the initial inventory level is not necessarily zero, we can simply set y_0 to whatever given value.

In our formulation, we assumed that there is no production delay (Assumption 2). This assumption can be easily relaxed. Suppose instead there is a production delay of one month; that is, the scheduled production for month j, x_j, is available only after a delay of one month, i.e., in month $j+1$. Then, in the alternative formulation, we can simply replace the constraint $y_j = y_{j-1} + x_j - d_j$ by $y_j = y_{j-1} + x_{j-1} - d_j$ (with $x_0 \equiv 0$), for $j = 1, 2, \ldots, 12$. Of course, for the first month, the given value of y_0 must be no less than d_1; otherwise, the resulting LP will not have any solution.

Assumption 3 can also be relaxed. If shortages are allowed, we can simply remove the nonnegativity requirements for the y_j's, and introduce a shortage penalty cost of, say, p_j per unit of shortage at the end of month j. We will discuss in a later section how to handle these "unrestricted" variables.