Experiment 8: Sinusoidal Steady-State Analysis and Power Calculations

1 Objectives

The objective of this experiment is to analyze circuits energized by sinusoidal sources of voltage or current. Particularly, the concepts of phasors, complex representation of circuit elements, and power calculations in the sinusoidal steady-state are considered here.

2 Introduction and Test Circuits

A sinusoidal voltage (or current) source produces a voltage (or current) that varies sinusoidally with time. A sinusoidal voltage source can be described mathematically as:

$$v(t) = V_m \cos(\omega t + \phi) \tag{8-1}$$

where V_m is the peak amplitude, ω is the angular frequency (in rad/s), and ϕ is the phase angle.

2.1 Sinusoidal steady-state analysis

When a sinusoidal source energizes a circuit, all the signals of interest in the circuit have the same frequency. Therefore, it is sufficient to represent those signals in terms of their amplitude and phase angle. This leads to the *phasor* representation of sinusoidal signals:

$$\mathbf{V} = V_m e^{j\phi} = V_m \angle \phi = V_m \cos\phi + jV_m \sin\phi = \mathcal{P}[V_m \cos(\omega t + \phi)] \tag{8-2}$$

where $\mathcal{P}$ is the phasor transform (from the time domain to the frequency domain), whose inverse (from the frequency domain to the time domain) yields:

$$\mathcal{P}^{-1}\{V_m e^{j\phi}\} = \Re\{V_m e^{j\phi} e^{j\omega t}\} = v(t) \tag{8-3}$$

In a resistor, the sinusoidal voltage and current are in phase. At the terminals of an inductor the voltage leads the current by 90°, and at the terminals of a capacitor the current leads the voltage by 90°. The relationship between phasor current and phasor voltage for resistors, inductors, and capacitors is

$$\mathbf{V} = Z\mathbf{I} \tag{8-4}$$

where Z is the *impedance* (in Ω) of the element. The reciprocal of impedance is *admittance* (Y, in *mhos* $\mho$), which relates the phasor voltage and phasor current of an element:

$$\mathbf{V} = \mathbf{I}/Y \tag{8-5}$$

In general, impedance and admittance are complex numbers. The imaginary part of impedance is called *reactance*, and the imaginary part of admittance *susceptance*. Table 8-1 summarizes the values of impedance, reactance, admittance and susceptance for resistors, inductors, and capacitors.

All the techniques used in DC circuit analysis (Experiment 2 and 3) may be used to analyze a frequency domain circuit using phasors. Phasor diagrams are commonly used to visualize the relationship among the

Element	Impedance	Reactance	Admittance	Susceptance
Resistor	R (resistance)	—	G (conductance)	—
Capacitor	$1/j\omega C$	$-1/\omega C$	$j\omega C$	ωC
Inductor	$j\omega L$	ωL	$1/j\omega L$	$-1/\omega L$

Table 8-1: Impedance and related values of common elements.

different signals in the frequency-domain. A simple test circuit operating in the frequency domain is shown in Fig. 8-1. Here, eleven different phasors can be identified. Namely, $\mathbf{V_s}$, $\mathbf{V_{in}}$, $\mathbf{I_{in}}$, and the voltage phasor and current phasor for L, C, R_L, and R_C. Notice that phasor $\mathbf{V_s}$ remains constant at different frequencies, however, it cannot be used as the reference phasor in a real circuit as it is internal to the voltage source. Instead, $\mathbf{V_{in}}$ is used as the phasor with the reference phase of 0° and a normalized magnitude of 1 to describe the other voltage phasors. The current phasors are described in terms of the voltage phasors by using equations (8-4) or (8-5).

2.2 Sinusoidal steady-state power calculations

When dealing with sinusoidal sources and the signals generated by them in a circuit it is often desirable to determine the power delivered to and dissipated by the circuit elements. The importance of discussing such calculations roots in that the voltage and current phasors may have different phase angles, hence, determining power is not limited to the product of the magnitudes of voltages and currents. In fact, there are several types of power that arise in a circuit operating in the sinusoidal steady-state.

- **Instantaneous power**
  The instantaneous power (in *watts* W) delivered to a load is a function of time defined as the product of the instantaneous terminal voltage and current:

  $$p(t) = v(t)i(t) \tag{8-6}$$

  The instantaneous power can be positive or negative, depending on whether the power is being dissipated by the load or is being delivered by the load. The frequency of the instantaneous power is twice the frequency of the voltage (or current).

- **Average power**
  For many purposes, an average value of the power over time is more useful than instantaneous power. It is the power that is converted from electrical to another form of energy (e.g. heat), for this reason it is often referred to as real power. Its units are also watts. The average value of the instantaneous power over one period is expressed as

  $$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = V_{rms} I_{rms} \cos(\theta_v - \theta_i) \tag{8-7}$$

- **Reactive power**
  Reactive power is the electric power exchanged between an inductor (or a capacitor) and the source that drives it. Reactive power is never converted to nonelectric power. Its units are *var* (volt amp reactive, or VAR), and is expressed as

  $$Q = \frac{1}{2}V_m I_m \sin(\theta_v - \theta_i) = V_{rms} I_{rms} \sin(\theta_v - \theta_i) \tag{8-8}$$

- **Complex power**
  The complex sum of the real and reactive powers is the complex power. Its units are *volt-amp* (VA). Complex power is expressed as

  $$S = P + jQ = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \mathbf{V}_{rms}\mathbf{I}^*_{rms} = I^2_{rms}Z = \frac{V^2_{rms}}{Z^*} \tag{8-9}$$

- **Apparent power**
  The magnitude (in VA) of the complex power is referred to as apparent power:

$$|S| = \sqrt{P^2 + Q^2} \tag{8-10}$$

Associated with reactive loads is the parameter known as *power factor*, which is the cosine of the phase angle between the voltage and the current:

$$\text{pf} = \cos(\theta_v - \theta_i) \tag{8-11}$$

Similarly, the *reactive factor* is the sine of the phase angle between the voltage and the current:

$$\text{rf} = \sin(\theta_v - \theta_i) \tag{8-12}$$

Generally, it is preferred to have circuits (systems) with a power factor close to unity such that the real power is maximized. In circuits operating in the sinusoidal steady-state the *maximum power transfer* occurs when the load impedance is the complex conjugate of the Thévenin impedance as viewed from the terminals of the load impedance.

3 Preparation

In preparation for the lab compute the required quantities.

1. Consider the circuit of Fig. 8-1 and the component values of Table 8-2. Consider $\mathbf{V_{in}}$ as the reference phasor, i.e. assume $\mathbf{V_{in}}$ has magnitude 1 and phase zero at all frequencies disregarding $\mathbf{V_s}$. Complete the entries of Table 8-3 corresponding to the theoretical values. Include the calculations in a separate sheet.

2. Consider the circuit of Fig. 8-2 when the capacitor is connected as indicated by the dashed lines. Derive an expression for the value of the input frequency (in Hertz) as a function of L, C and R such that the impedance of the load is real (i.e. the imaginary part is zero).

3. Consider the circuit of Fig. 8-3 and the component values of Table 8-6. Write an expression for the average power P delivered to the load in terms of $\mathbf{V_s}$, R_s and the load resistance R_i. Complete the entries of Table 8-6 corresponding to the theoretical values. Include the calculations in a separate sheet.

4 Procedure

This part of the experiment requires assembling the circuits presented in the previous section and measuring data from all of them. Refer to the appendices regarding the use of the equipment. Pay special attention to the procedure necessary to measure the phase angle between two signals and the *peak* voltage value of a signal.

1. Assemble the circuit of Fig. 8-1 using the component values of Table 8-2. Observe $\mathbf{V_{in}}$ (the reference phasor) with channel 2 of the oscilloscope at all times, use channel 1 to observe the other signals. Take measurements to complete the entries of Table 8-3 corresponding to the experimental values. Notice that the magnitude of $\mathbf{V_{in}}$ depends on the input frequency and must be measured for both frequencies. Pay attention to the sign of the phase angle.

2. Assemble the circuit of Fig. 8-2 using the component values of Table 8-4. Measure the actual values of the components and calculate the frequency at which the load impedance is real using the expression derived in the Preparation section. Set the frequency of the signal generator to this value. Take measurements, with and without the capacitor connected to the circuit, to complete the entries of Table 8-4 corresponding to $|\mathbf{V}|$ (using channel 2), $|\mathbf{I}|$ (using channel 1 across R_1), and the phase between the two of them $\theta_V - \theta_{V_{R_1}}$. Notice that the reading from channel 1 is the scaled value of $|\mathbf{I}|$ by R_1, *therefore it must be divided by the actual value of* R_1. The experimental values of pf, P, Q, and S in Table 8-4 will be calculated from these measurements in the Analysis section.

3. Assemble the circuit of Fig. 8-3 using the component values of Table 8-6. Use a signal frequency of 1 kHz. For each value of R_i measure the RMS voltage across the load and calculate the experimental values of Table 8-6.

5 Analysis

This section is intended for the analysis and comparison of the experimental and theoretical results. Answer all the questions.

1. Normalize the experimental values of the voltage phasors recorded in Table 8-2 such that $\mathbf{V_{in}}$ appears with magnitude 1 and compare these results with the theoretical values. Find the current phasors $\mathbf{I}_{in}$, $\mathbf{I}_1$, and $\mathbf{I}_2$ for the frequency of 1.8 kHz. Represent $\mathbf{I}_{in}$ as the sum of $\mathbf{I}_1$ and $\mathbf{I}_2$ in a phasor diagram.
2. Make the necessary calculations to complete the entries of Table 8-5 corresponding to pf, P, Q, and S. Explain the effect of the capacitor on the power factor of the circuit, and its effect on the reactive power.
3. From the results of Table 8-6, determine which of the three resistor values gives the maximum power transfer. Compare this resistor value with the source resistance R_s.

Param	Unit	Theor	Exper	Param	Unit	Theor	Exper
R_s	Ω	50	◇	V_s	V	5	◇
R_L		100		L	mH	1	
R_C		100		C	μF	1	

Table 8-2: Phasor test circuit component values.

Frequency (kHz)	Phasor	Theoretical		Experimental	
		Mag (V)	Phase (°)	Mag (V)	Phase (°)
1.8	$\mathbf{V_{in}}$	1	0		0
	$\mathbf{V_1}$				
	$\mathbf{V_2}$				
18	$\mathbf{V_{in}}$	1	0		0
	$\mathbf{V_1}$				
	$\mathbf{V_2}$				

Table 8-3: Phasor table for test circuit.

Param	V_s	R_s	R	R_1	L	C
Units	V	Ω			mH	μF
Theor	10	50	150	5.1	1	0.01
Exper	◇	◇				

Table 8-4: Power calculations test circuit component values.

Param	Unit	With C	Without C
$\|\mathbf{V}\|$	V		
$\|\mathbf{I}\|$	mA		
$\theta_V - \theta_{V_R}$	°		
pf	—		
P	W		
S	VAR		
Q	VA		

Table 8-5: Power calculations.

Param	Unit	Theor	Exper	Param	Unit	Theor	Exper
R_s	Ω	50	◇	V_s	V	2	◇
R_1		24		P_1	mW		
R_2		51		P_2			
R_3		100		P_3			

Table 8-6: Maximum power transfer calculations.

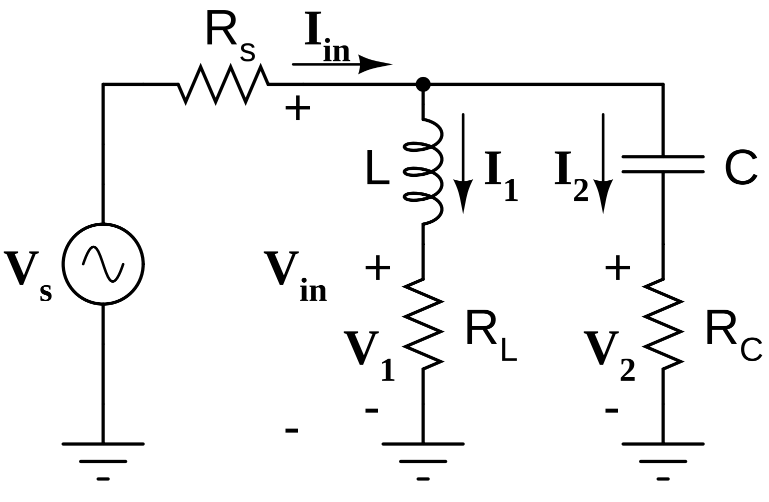

Figure 8-1: Phasors test circuit.

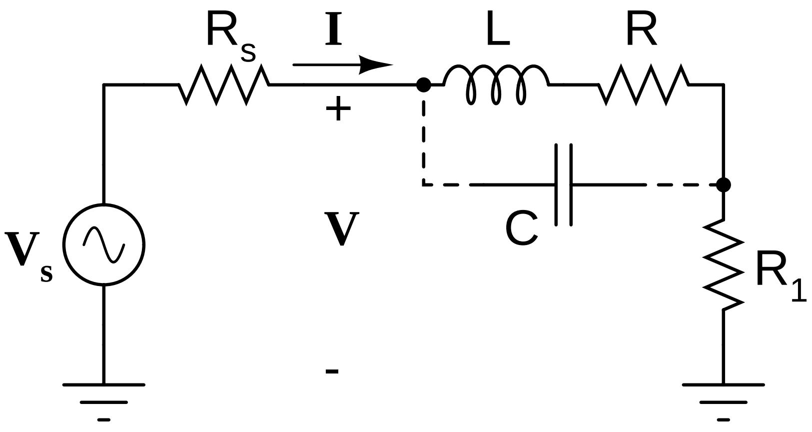

Figure 8-2: Power calculations test circuit.

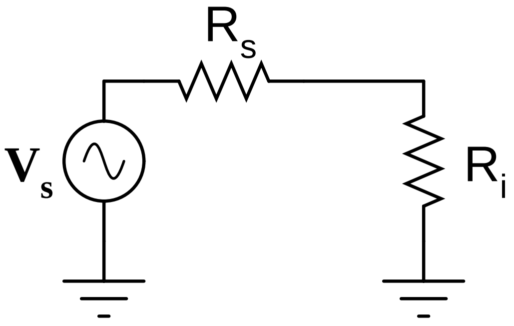

Figure 8-3: Maximum power transfer test circuit.