EXAMINATION II

Show your work! No credit otherwise!!
No books, no notes, no calculators, no nothing!

1. [15] \(X\) is a RV with PDF \(f_X(x) = |x|\) for \(-1 \leq x \leq +1\) and zero otherwise. [Note: You should know that \(|x| = -x\) for \(x \leq 0\) and \(|x| = x\) for \(x \geq 0\)]. Find the standard deviation of \(X\).

\[
\begin{align*}
E[X] &= \int_{-\infty}^{100} x f_X(x) \, dx = \int_{-1}^0 x(-x) \, dx + \int_0^1 x(x) \, dx = -\frac{x^2}{2}\bigg|_0^1 + \frac{x^3}{3}\bigg|_0^1 = -\frac{1}{2} + \frac{1}{3} = 0 \\
E[X^2] &= \int_{-\infty}^{100} x^2 f_X(x) \, dx = \int_{-1}^0 x^2(-x) \, dx + \int_0^1 x^2(x) \, dx = -\frac{x^3}{3}\bigg|_0^1 + \frac{x^4}{4}\bigg|_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{1}{2} \\
\text{Var}(X) &= \frac{1}{2} - 0^2 = \frac{1}{2} \\
\sigma &= \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}
\end{align*}
\]

2. [15] \(X\) is a Gaussian RV with \(E[X] = 0\) and \(P[|X| \leq 44] = 0.34\). Find the standard deviation of \(X\). [See tables]

\[
P[|X| \leq 44] = P[-44 \leq X \leq 44] = P[X \leq 44] - P[X \leq -44] = 2P[X \leq 44] - 1
\]

\[
= 2 \Phi\left(\frac{44 - 0}{\sigma}\right) - 1 = 0.34
\]

\[
\Phi\left(\frac{44}{\sigma}\right) = \frac{13}{2} = 0.67
\]

\[
\frac{44}{\sigma} = 0.44 \quad \text{from table}
\]

\[
\sigma = \frac{44}{0.44} = 100
\]

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3. [15] \( X \) is a RV with CDF \( F_X(x) = 0 \) for \( x < -1 \); \( = (x+2)/4 \) for \( -1 \leq x < +1; = 1 \) for \( x \geq 1 \). Draw the PDF \( f_X(x) \) and find the standard deviation of \( X \).

\[
\begin{align*}
E[X] &= \frac{1}{4}(-1) + \int_{-1}^{1} x \left( \frac{4}{4} \right) \, dx + \frac{1}{4}(+1) = -\frac{1}{4} + \frac{1}{4} x^2 \bigg|_{-1}^{1} = \frac{1}{4} = 0 & \text{obvious from symmetry} \\
E[X^2] &= \frac{1}{4}(-1)^2 + \int_{-1}^{1} x^2 \left( \frac{4}{4} \right) \, dx + \frac{1}{4}(+1)^2 = \frac{1}{4} + \frac{1}{12} x^3 \bigg|_{-1}^{1} + \frac{1}{4} = \frac{1}{4} + \frac{1}{6} + \frac{1}{4} = \frac{2}{3} \\
\text{Var}(X) &= \frac{2}{3} - 0^2 \\
\sigma &= \sqrt{\frac{2}{3}}
\end{align*}
\]

4. [15] Your friendly computer software vendor has supplied you with a random number generator supposedly yielding values uniformly distributed between 0 and 1. In order to simulate a system you need a RV \( X \) with PDF \( f_X(x) = 3x^2 \) for \( 0 \leq x \leq 1 \) and zero otherwise. What transformation do you apply to the random numbers given?

\[
\begin{align*}
F_X(x) &= \int_{-\infty}^{x} f_X(t) \, dt = \int_{0}^{x} 3x^2 \, dt = x^3 \bigg|_{0}^{x} = x^3 \quad 0 \leq x \leq 1 \\
F_Y(u) &= u \quad 0 \leq u \leq 1 \\
u &= x^3 \\
x &= \sqrt[3]{u}
\end{align*}
\]

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5. [20] RVs \( X \) and \( Y \) have joint PDF \( f_{X,Y}(x,y) = 2 \) for \( 0 \leq y \leq x \leq 1 \) and zero otherwise. Define the RV \( W = Y/X \) which clearly takes on nonzero values only in \( 0 \leq w \leq 1 \). Find the CDF of \( W \), the PDF of \( W \), and \( E[W] \).

\[
F_W(w) = P[W \leq w] = P\left[\frac{Y}{X} \leq w\right] = P\left[Y \leq wX\right] = \int_0^w \int_0^{\frac{w}{x}} f_{X,Y}(x,y) \, dy \, dx
\]

\[
= 2 \int_0^w w \, dx = 2w \int_0^w \frac{x^2}{2} \, dx = w - \frac{w^3}{2}, \quad 0 \leq w \leq 1
\]

\[
f_W(w) = \left\{ \begin{array}{ll}
1 & 0 \leq w \leq 1 \\
0 & \text{otherwise}
\end{array} \right.
\]

\[
E[W] = \frac{1}{2}
\]

6. [20] RVs \( X \) and \( Y \) have joint PDF \( f_{X,Y}(x,y) = 2 \) for \( 0 \leq y \leq x \leq 1 \) and zero otherwise (just as in the previous problem). It is not very hard to find the marginal PDFs so I’ll give them to you along with their means and variances:

\[
f_X(x) = \left\{ \begin{array}{ll}
2x & 0 \leq x \leq 1 \\
0 & \text{otherwise}
\end{array} \right. \quad E[X] = \frac{2}{3} \quad \text{Var}(X) = \frac{1}{18}
\]

\[
f_Y(y) = \left\{ \begin{array}{ll}
2(1-y) & 0 \leq y \leq 1 \\
0 & \text{otherwise}
\end{array} \right. \quad E[Y] = \frac{1}{3} \quad \text{Var}(Y) = \frac{1}{18}
\]

Find the correlation coefficient \( \rho_{X,Y} \).

\[
E[XY] = \int_0^1 \int_0^x xy \cdot 2 \, dy \, dx = \int_0^1 x \int_0^x y^2 \, dy \, dx = \int_0^1 x \left[ \frac{y^3}{3} \right]_0^x \, dx = \int_0^1 \frac{x^4}{4} \, dx = \frac{1}{4}
\]

\[
\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{4} - \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}
\]

\[
\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{\frac{1}{36}}{\frac{1}{18} \cdot \frac{1}{18}} = \frac{\frac{1}{36}}{\frac{1}{324}} = \frac{1}{\frac{1}{2}} = \frac{1}{2}
\]