EXAMINATION 2.5

Show your work! No credit otherwise!!
No books, no notes, no calculators, no nothing!

1. [20] The mean weight of 1000 students at a certain college is exactly 151 lbs. with a standard deviation of exactly 15 lbs. assuming a Gaussian distribution. At what weight \( W \) would we expect to find exactly 1 student with more than that weight? [Do calculations by hand using the table]

\[
1 - \Phi \left( \frac{W-151}{15} \right) = 1 - \frac{1}{1000} = 0.999 = 0.999 \\
\text{Use \( \Phi \) table:} \quad \Phi \left( \frac{W-151}{15} \right) = 0.001 = 1 \times 10^{-3} \\
\frac{W-151}{15} = 3.09 \\
W = 151 + 15(3.09) \approx 197.35
\]

2. [20] \( T \) is a RV with CDF \( F_T(t) = 0 \) for \( t < 0 \); \( = t^3/32 \) for \( 0 \leq t < 2 \); \( = 1 \) for \( t \geq 2 \). Draw the PDF \( f_T(t) \) and find the mean (expected) value of \( T \).

\[
f_T(t) = \begin{cases} 
0 & t < 0, t > 2 \\
\frac{t^3}{8} & 0 \leq t < 2 \\
\frac{1}{8} & t = 2 
\end{cases}
\]

\[
E[T] = \int_0^2 t \left( \frac{t^3}{8} \right) dt = 2 \left( \frac{1}{2} \right) \\
= \frac{2 \cdot 3^2}{40} + 1 = \frac{3^2}{40} + 1 = \frac{9}{5}
\]
3. [20] Your friendly computer software vendor has supplied you with a random number generator supposedly yielding values uniformly distributed between 0 and 1. In order to simulate a system you need a RV $X$ with PDF $f_X(x) = (3/2) x^2 + (1/2) \delta(x-1)$ for $0 \leq x \leq 1$ and zero otherwise. What transformation do you apply to the random numbers given? [HINT: Note that the values of $x$ must end up in the range $0 \leq x \leq 1$ with lots of them right at 1]

$$
F_X(x) = \int_0^x \left[ \frac{3}{2} t^2 + \frac{1}{2} \delta(t-1) \right] dt \quad 0 \leq x \leq 1,
$$

$$
= \frac{1}{2} x^3 + \frac{1}{2} \delta(x-1) x
$$

$$
U = \frac{1}{2} x^3 \Rightarrow x = 3 \sqrt{2U}
$$

but if $U > \frac{1}{2}$, $2U > 1$, $x > 1$ not wanted.

OK. Just set $x = 1$ for all $\frac{1}{2} < U < 1$.

$$
X = \min \left( 1, 3 \sqrt{2U} \right) \text{ for example}
$$

4. [20] RVs $X$ and $Y$ have joint PDF $f_{X,Y}(x,y) = 24xy$ for $0 \leq x, y \leq 1$ and zero otherwise. Define the RV $W = X+Y$ which clearly takes on nonzero values only in $0 \leq w \leq 1$. Find the CDF of $W$.

$$
F_W(w) = P[X+Y \leq w] = P[Y \leq w-X] = \int_0^w \int_0^{w-x} 24xy \, dy \, dx = \int_0^w 12x^2y \, dy \, dx
$$

$$
= 12 \int_0^w \left( x^2 w - 2 x^2 w + x^3 \right) \, dx = 12 \left[ \frac{x^3 w^2}{3} - \frac{2x^4 w}{3} + \frac{x^4}{4} \right]_0^w
$$

$$
= 12 \left( \frac{w^4}{3} - \frac{2w^4}{3} + \frac{w^4}{4} \right) = w^4(6-8+3) = w^4
$$

$$
: F_W(w) = \begin{cases} 0 & w < 0 \\ w^4 & 0 \leq w \leq 1 \\ 1 & w > 1 
\end{cases}
$$
5. [20] RVs $X$ and $Y$ have joint PDF $f_{X,Y}(x,y) = 2$ for $0 \leq y \leq x \leq 1$ and zero. It is not very hard to find the marginal PDFs so I'll give them to you along with their means and variances:

- $f_X(x) = 2x$ for $0 \leq x \leq 1$ and zero elsewhere  
  $E[X] = 2/3$  
  $Var(X) = 1/18$

- $f_Y(y) = 2(1-y)$ for $0 \leq y \leq 1$ and zero elsewhere  
  $E[Y] = 1/3$  
  $Var(Y) = 1/18$

Find the correlation coefficient $\rho_{X,Y}$.

$$E[XY] = \int_0^1 \int_0^x xy \cdot 2 \, dy \, dx$$

$$= \int_0^1 x \cdot \left[ \frac{y^2}{2} \right]_0^x = \int_0^1 x^3 \, dx = \frac{1}{4} \left[ \frac{x^4}{4} \right]_0^1 = \frac{1}{4}$$

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = \frac{1}{4} - \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$$

$$\rho_{X,Y} = \frac{\frac{1}{36}}{\sqrt{\frac{1}{18}} \cdot \sqrt{\frac{1}{18}}} = \frac{\frac{1}{36}}{\frac{1}{18}} = \frac{1}{2}$$