\[ f_X(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \]

1. \[ E[X] = \int_0^2 x \cdot \frac{x}{2} \, dx = \left[ \frac{x^3}{6} \right]_0^2 = 2 - \frac{2}{6} = \frac{5}{3} \]

2. \[ E[Y] = \int_0^2 y \, dy = \left[ \frac{y^2}{2} \right]_0^2 = \frac{4}{2} = \frac{2}{3} \]

3. \[ E[Z] = E[X] + E[Y] = \frac{5}{3} + \frac{2}{3} = 2 \]

4. \[ E[X^2] = \int_0^2 x^2 \cdot \frac{x}{2} \, dx = \left[ \frac{x^4}{8} \right]_0^2 = \frac{16}{8} - \frac{16}{8} = \frac{2}{3} \quad \text{VAR}_X = \frac{2}{3} - \left( \frac{5}{3} \right)^2 = \frac{4}{9} \]

5. \[ E[Y^2] = \int_0^2 y^2 \, dy = \left[ \frac{y^3}{3} \right]_0^2 = 2 \quad \text{VAR}_Y = 2 - \left( \frac{2}{3} \right) = \frac{4}{9} \quad \text{VAR}_Z = \frac{4}{9} + \frac{4}{9} = \frac{8}{9} \]

6. **Convolution**

   \[ f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \, dx \]

   \[ \int_0^2 \left( \frac{x}{2} \right) (z-x) \, dx = \frac{1}{4} \int_0^2 \left( 2z - 2x^2 - x^3 + \frac{1}{3} x^3 \right) \, dx = \frac{1}{4} \left( 2z - \frac{2}{3} x^2 + \frac{1}{3} x^3 \right) \]

   If \( z < 0 \): \( f_Z(z) = 0 \)

   If \( 0 < z < 2 \):

   \[ f_Z(z) = \int_0^2 \left( \frac{x}{2} \right) (z-x) \, dx = \frac{1}{4} \left( 2z - \frac{2}{3} x^2 + \frac{1}{3} x^3 \right) \]

   = \frac{1}{4} \left( 2z - \frac{2}{3} \cdot \frac{4}{3} + \frac{1}{3} \cdot \frac{8}{3} \right) = \frac{1}{4} \left( z - \frac{8}{6} \right) \]

   If \( z > 2 \):

   \[ f_Z(z) = \int_{z-2}^2 \left( \frac{x}{2} \right) (z-x) \, dx = \frac{1}{4} \left( 2z - \frac{2}{3} x^2 + \frac{1}{3} x^3 \right) \]

   \[ = \frac{1}{4} \left[ \left( z - 2 \right) (z-2)^2 - \left( z^2 - 2z \right) \left( z-2 \right) + \frac{1}{3} (z-2)^3 \right] \]

   \[ = \cdots \]

   = \frac{1}{4} \left( -2z + \frac{1}{3} \right)
18. The Convolution. Consider a linear system in which the effect at the present time $t$ of a stimulus $f(r)\, dr$ at any past time $r$ is proportional to the stimulus. On physical grounds we assume that the proportionality constant depends only on the elapsed time $t - r$ and hence has the form $g(t - r)$. The effect at the present time $t$ is therefore

$$f(r)g(t - r)\, dr.$$  

Since the system is linear the response to the whole past history can be obtained by adding these separate effects, and we are led to the integral

$$h(t) = \int_0^t f(r)g(t - r)\, dr. \quad (18-1)$$

The lower limit is 0 because it is assumed that the process started at time $t = 0$; in other words, $f(r) = 0$ for $r < 0$.

The expression (18-1) is called the convolution of $f$ and $g$. It gives the response at the present time $t$ as a weighted superposition over the inputs at the times $r \leq t$. The weighting factor $g(t - r)$ characterizes the system, and $f(r)$ characterizes the past history of the input. Because of this physical interpretation, the convolution is sometimes called the superposition integral.

The importance of the convolution can hardly be overestimated. It plays a prominent role in the study of heat conduction, wave motion, plastic flow and creep, and other areas of mathematical physics (cf. Chap. 7). The convolution is also encountered in several branches of mathematics; for example, a large part of the theory of distributions can be based on (18-1) rather than on the concept of a linear functional.\(^1\)

The function $h$ in (18-1) is often denoted by $f \ast g$, so that $(f \ast g)(t) = h(t)$ is the value of $f \ast g$ at the argument $t$. A similar notation applies in other cases; for example,

$$(a \ast b)(t) = \int_0^t a(r)b(t - r)\, dr. \quad (18-2)$$

The $\ast$ product behaves like ordinary multiplication, in that

$$a \ast (b + c) = a \ast b + a \ast c \quad a \ast b = b \ast a \quad (a \ast b) \ast c = a \ast (b \ast c). \quad (18-3)$$

(See Prob. 6.) We now show that convolution in the time domain actually does correspond to multiplication in the frequency domain. This is the content of the following:

**CONVOLUTION THEOREM.** If $f$ and $g$ are admissible then $L(f \ast g) = (L_f)(L_g)$.

To simplify the proof, define $f(t) = 0$ for all negative $t$. With a similar convention for $g(t)$ the Laplace transforms can be written\(^2\)

$$L_f = \int_{-\infty}^{\infty} e^{-s}f(t)\, dt \quad L_g = \int_{-\infty}^{\infty} e^{-s}g(t)\, dt \quad (18-4)$$

and the convolution $h(t)$ is

$$h(t) = \int_{-\infty}^{\infty} f(r)g(t - r)\, dr. \quad (18-5)$$

Indeed, the lower limit $-\infty$ in (18-5) can be replaced by 0 because $f(r) = 0$ when $r < 0$, and the upper limit can be replaced by $t$ because $g(t - r) = 0$ when $t - r < 0$. The convolution theorem now follows by a change in order of integration, as indicated next.


\(^2\) Transforms of the type (18-4) are called bilateral, in contrast to the unilateral transform (12-1). An account of the bilateral Laplace transform may be found in B. Van der Pol and H. Bremmer, "Operational Calculus," Cambridge University Press, London, 1959.