



Principle Component Analysis

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- λ is an **eigenvalue** of a matrix $A \in \mathbb{R}^{n \times n}$ if the linear system $Ax = \lambda x$ has at least one non-zero solution
 - If $Ax = \lambda x$ we say that λ is an eigenvalue of A with corresponding **eigenvector** x
 - Could be multiple eigenvectors for the same λ

Eigenvalues of Symmetric Matrices



- If $A \in \mathbb{R}^{n \times n}$ is symmetric, then it has n linearly independent eigenvectors v_1, \dots, v_n corresponding to n real eigenvalues
 - Moreover, it has n linearly independent **orthonormal** eigenvectors
 - $v_i^T v_j = 0$ for all $i \neq j$
 - $v_i^T v_i = 1$ for all i

Eigenvalues of Symmetric Matrices



- If $A \in \mathbb{R}^{n \times n}$ is symmetric, then it has n linearly independent eigenvectors v_1, \dots, v_n corresponding to n real eigenvalues
- A symmetric matrix A is **positive definite** if and only if all of its eigenvalues are positive or equivalently $x^T A x > 0$ for all vectors $x \neq 0$
 - The orthonormal eigenvectors form a **basis** of \mathbb{R}^n (similar to the standard coordinate axes)

Examples



- The 2x2 identity matrix has all of its eigenvalues equal to 1 (it is positive definite) with orthonormal eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues 0 and 2 with orthonormal eigenvectors $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$
- The matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvalues 1 and 3 with orthonormal eigenvectors $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

- Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric
- Any $x \in \mathbb{R}^n$ can be written as $x = \sum_{i=1}^n c_i v_i$ where v_1, \dots, v_n are the eigenvectors of A
 - $Ax = \sum_{i=1}^n \lambda_i c_i v_i$
 - $A^2 x = \sum_{i=1}^n \lambda_i^2 c_i v_i$
 - \vdots
 - $A^t x = \sum_{i=1}^n \lambda_i^t c_i v_i$

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- Any $x \in \mathbb{R}^n$ can be written as $x = \sum_{i=1}^n c_i v_i$ where v_1, \dots, v_n are the eigenvectors of A
 - $c_i = v_i^T x$, this is the projection of x along the line given by v_i (assuming that v_i is a unit vector)

Eigenvalues of Symmetric Matrices



- Let $Q \in \mathbb{R}^{n \times n}$ be the matrix whose i^{th} column is v_i and $D \in \mathbb{R}^{n \times n}$ be the diagonal matrix such that $D_{ii} = \lambda_i$
 - $Ax = QDQ^T x$
 - Can throw away some eigenvectors to approximate this quantity
 - For example, let Q_k be the matrix formed by keeping only the top k eigenvectors and D_k be the diagonal matrix whose diagonal consists of the top k eigenvalues

- The Frobenius norm is a matrix norm given by

$$\|A\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |A_{ij}|^2}$$

- $Q_k D_k Q_k^T$ is the best **rank** k approximation of the symmetric matrix A with respect to the Frobenius norm

$$Q_k D_k Q_k^T = \underset{B \in \mathbb{R}^{n \times n} \text{ s.t. } \text{rank}(B)=k}{\text{argmin}} \|A - B\|_F$$

Principal Component Analysis



- Principle component analysis
 - Can be used to reduce the dimensionality of the data while still maintaining a good approximation of the sample mean and variance
 - Can also be used for selecting good features that are combinations of the input features
 - Unsupervised – just finds a good representation of the data in terms of combinations of the input features

Principal Component Analysis



- Input a collection of data points sampled from some distribution $x_1, \dots, x_p \in \mathbb{R}^n$

- Construct the matrix $W \in \mathbb{R}^{n \times p}$ whose i^{th} column is

$$x_i - \frac{\sum_j x_j}{p}$$

- The matrix WW^T is the sample covariance matrix
 - WW^T is symmetric and positive semidefinite

Principal Component Analysis



- PCA finds a set of orthogonal vectors that best explain the variance of the sample covariance matrix
 - From our previous discussion, these are exactly the eigenvectors of WW^T
 - We can discard the eigenvectors corresponding to small magnitude eigenvalues to yield an approximation
 - Simple algorithm to describe, MATLAB and other programming languages have built in support for eigenvector computation

- Forming the matrix WW^T can require a lot of memory (especially if $n \gg p$)
 - Need a faster way to compute this without forming the matrix explicitly
 - Typical approach: use the singular value decomposition

Singular Value Decomposition (SVD)



- Every matrix $B \in \mathbb{R}^{n \times p}$ admits a decomposition of the form

$$B = U\Sigma V^T$$

- where $U \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, $\Sigma \in \mathbb{R}^{n \times p}$ is non-negative diagonal matrix, and $V \in \mathbb{R}^{p \times p}$ is an orthogonal matrix
- A matrix $C \in \mathbb{R}^{m \times m}$ is **orthogonal** if $C^T = C^{-1}$. Equivalently, the rows and columns of C are orthonormal vectors

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Diagonal elements of Σ called
singular values

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- Returning to PCA
 - Let $W = U\Sigma V^T$ be the SVD of W
 - $WW^T = U\Sigma V^T V \Sigma^T U^T = U\Sigma\Sigma^T U^T$
 - If we can compute the SVD of W , then we don't need to form the matrix WW^T

- For any matrix A , AA^T is symmetric and positive semidefinite
 - Let $A = U\Sigma V^T$ be the SVD of A
 - $AA^T = U\Sigma V^T V \Sigma^T U^T = U\Sigma\Sigma^T U^T$
 - U must be a matrix of eigenvectors of AA^T
 - The eigenvalues of AA^T are all non-negative because $\Sigma\Sigma^T = \Sigma^2$ which are the square of the singular values of A

An Example: “Eigenfaces”



- Let's suppose that our data is a collection of images of the faces of individuals



An Example: “Eigenfaces”



- Let's suppose that our data is a collection of images of the faces of individuals
 - The goal is, given the "training data", to correctly match new images to the training data
 - Let's suppose that each image is an $s \times s$ array of pixels: $x_i \in \mathbb{R}^n, n = s^2$
 - As before, construct the matrix $W \in \mathbb{R}^{n \times p}$ whose i^{th} column is $x_i - \sum_j \frac{x_j}{p}$

An Example: “Eigenfaces”



- Forming the matrix WW^T requires a lot of memory
 - $s = 256$ means WW^T is 65536×65536
 - Need a faster way to compute this without forming the matrix explicitly
 - Could use the singular value decomposition

An Example: “Eigenfaces”



- A different approach when $p \ll n$
 - Compute the eigenvectors of $A^T A$ (this is an $p \times p$ matrix)
 - Let v be an eigenvector of $A^T A$ with eigenvalue λ
 - $AA^T Av = \lambda Av$
 - This means that Av is an eigenvector of AA^T with eigenvalue λ (or 0)
 - Save the top k eigenvectors - called eigenfaces in this example

An Example: “Eigenfaces”



- The data in the matrix is “training data”
 - Given a new image, we’d like to determine which, if any, member of the data set that it is most similar to
- Step 1: Compute the projection of the recentered, new image onto each of the k eigenvectors
 - This gives us a vector of weights c_1, \dots, c_k

An Example: “Eigenfaces”



- The data in the matrix is “training data”
 - Given a new image, we’d like to determine which, if any, member of the data set that it is most similar to
- Step 2: Determine if the input image is close to one of the faces in the data set
 - If the distance between the input and it's approximation is too large, then the input is likely not a face

An Example: “Eigenfaces”



- The data in the matrix is “training data”
 - Given a new image, we’d like to determine which, if any, member of the data set that it is most similar to
- Step 3: Find the person in the training data that is closest to the new input
 - Replace each group of training images by its average
 - Compute the distance to the i^{th} average $\|c - a^i\|$ where a^i are the coefficients of the average face for person i