Convergent and Correct Message Passing Algorithms

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Overview

- A simple derivation of a new family of message passing algorithms
- Conditions under which the parameters guarantee correctness upon convergence
- An asynchronous algorithm that generalizes the notion of "bound minimizing" algorithms [Meltzer, Globerson, Weiss 2009]
- A simple choice of parameters that guarantees both convergence and correctness of the asynchronous algorithm

Previous Work

- Convergent message passing algorithms:
  - Serial TRMP [Kolmogorov 2006]
  - MPLP [Globerson and Jaakkola 2007]
  - Max-sum diffusion [Werner 2007]*
  - Norm-product BP [Hazan and Shashua 2008]
- Upon convergence, any assignment that simultaneously minimizes all of the beliefs must minimize the objective function

Outline

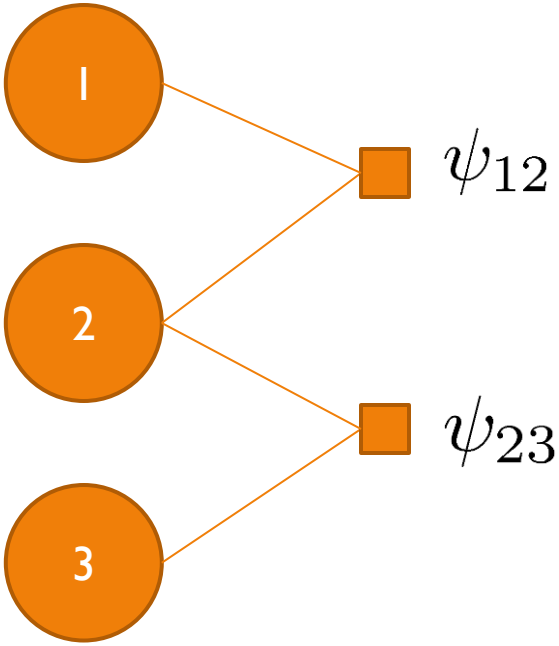
- **Review of the min-sum algorithm**
- The splitting algorithm
  - Correctness
  - Convergence
- Conclusion

Min-Sum

$$f(x_1, ..., x_n) = \sum_i \phi_i(x_i) + \sum_\alpha \psi_\alpha(x_\alpha)$$

- Minimize an objective function that factorizes as a sum of potentials
- $\alpha \in \mathcal{A}$ (some multiset whose elements are subsets of the variables)

Corresponding Graph



$$f(x_1, x_2, x_3) = \phi_1 + \phi_2 + \phi_3 + \psi_{23} + \psi_{12}$$

Min-Sum

- Messages at time t:

$$m^t_{i\to\alpha}(x_i) = \kappa + \phi_i(x_i) + \sum_{\beta\in\partial i\setminus\alpha} m^{t-1}_{\beta\to i}(x_i)$$

$$m^t_{\alpha\to i}(x_i) = \kappa + \min_{x_{\alpha\setminus i}} \left[\psi_\alpha(x_\alpha) + \sum_{k\in\alpha\setminus i} m^{t-1}_{k\to\alpha}(x_k)\right]$$

- Normalization factor is arbitrary for each message

- Initial messages must be finite

Computing Beliefs

- At each step, construct a set of beliefs:

$$
\begin{aligned}
b_i^t(x_i) &= \kappa + \phi_i(x_i) + \sum_{\alpha \in \partial i} m_{\alpha \to i}^t(x_i) \\
b_\alpha^t(x_\alpha) &= \kappa + \psi_\alpha(x_\alpha) + \sum_{i \in \alpha} m_{i \to \alpha}^t(x_i)
\end{aligned}
$$

- Estimate the optimal assignment as

$$
x_i^t = \arg\min_{x_i} b_i^t(x_i)
$$

- Converges if we obtain a fixed point of the message updates

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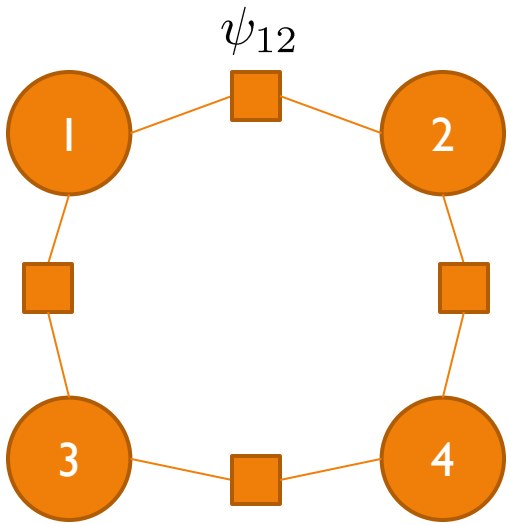
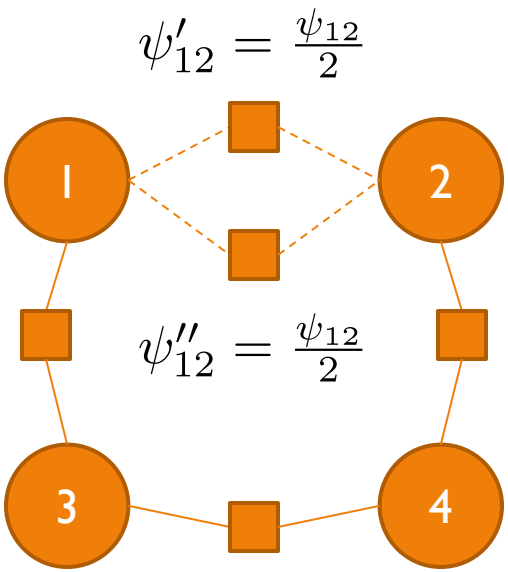
“Splitting” Heuristic



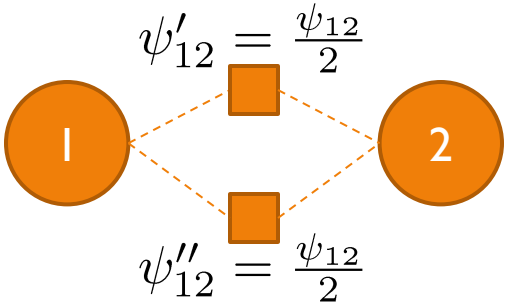
Original Graph



Split the (1,2) factor

Reduce the potentials on each edge by 1/2

Min-Sum with Splitting



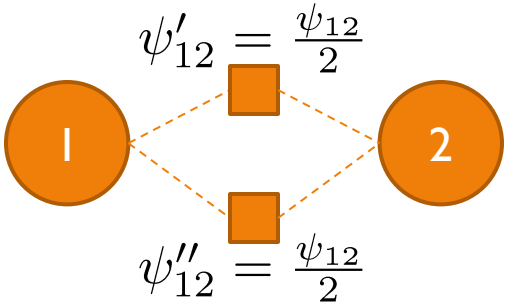
- Consider the message passed across one of the split edges:

$$m_{1\to(1,2)'}(x_1) = \phi_1(x_1) + m_{(3,1)\to 1}(x_1) + m_{(1,2)''\to 1}(x_1)$$

- If the initial messages are the same for the copied edges, then, at each time step, the messages passed across the duplicated edges are the same:

$$m^t_{1\to(1,2)'}(x_1) = m^t_{1\to(1,2)''}(x_1)$$

Min-Sum with Splitting



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Min-Sum with Splitting

- We can split any of the factors c_α times and divide the potentials by a factor of c_α

$$m^t_{i\to\alpha}(x_i) = \kappa + \phi_i(x_i) + (c_\alpha - 1)m^{t-1}_{\alpha\to i}(x_i) + \sum_{\beta\in\partial i\setminus\alpha} c_\beta m^{t-1}_{\beta\to i}(x_i)$$

$$m^t_{\alpha\to i}(x_i) = \kappa + \min_{x_{\alpha\setminus i}}\left[\frac{\psi_\alpha(x_\alpha)}{c_\alpha} + \sum_{k\in\alpha\setminus i} m^{t-1}_{k\to\alpha}(x_k)\right]$$

- Messages are passed on the original factor graph

- Update equation is similar to TRMP

General Splitting Algorithm

- Splitting the variable nodes and the factor nodes:

$$m^t_{i\to\alpha}(x_i) = \kappa + \frac{\phi_i(x_i)}{c_i} + (c_\alpha - 1)m^{t-1}_{\alpha\to i}(x_i) + \sum_{\beta\in\partial i\setminus\alpha} c_\beta m^{t-1}_{\beta\to i}(x_i)$$

$$m^t_{\alpha\to i}(x_i) = \kappa + \min_{x_{\alpha\setminus i}}\left[\frac{\psi_\alpha(x_\alpha)}{c_\alpha} + (c_i - 1)m^{t-1}_{i\to\alpha}(x_i) + \sum_{k\in\alpha\setminus i} c_k m^{t-1}_{k\to\alpha}(x_k)\right]$$

- Recover min-sum if all parameters are chosen to be equal to one

Beliefs

- Corresponding beliefs:

$$
\begin{aligned}
b_i^t(x_i) &= \kappa + \frac{\phi_i(x_i)}{c_i} + \sum_{\alpha \in \partial i} c_\alpha m_{\alpha \to i}^t(x_i) \\
b_\alpha^t(x_\alpha) &= \kappa + \frac{\psi_\alpha(x_\alpha)}{c_\alpha} + \sum_{k \in \alpha} c_k \Big[\frac{\phi_k(x_k)}{c_k} + (c_\alpha - 1) m_{\alpha \to k}^t(x_k) \\
&\quad + \sum_{\beta \in \partial k \setminus \alpha} c_\beta m_{\beta \to k}^t(x_k) \Big]
\end{aligned}
$$

- Not the same as before

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Admissibility and Min-consistency

- A vector of beliefs, b, is **admissible** for a function f if

$$f(x) = \kappa + \sum_i c_i b_i(x_i) + \sum_\alpha c_\alpha \left[b_\alpha(x_\alpha) - \sum_{k \in \alpha} c_k b_k(x_k) \right]$$

  - Makes sense for any non-zero, real-valued parameters

- A vector of beliefs, b, is **min-consistent** if for all α and all i$\in\alpha$:

$$\min_{x_{\alpha \setminus i}} b_\alpha(x_\alpha) = \kappa + b_i(x_i)$$

Conical Combinations

- If there is a unique x^* that simultaneously minimizes each belief, does it minimize f?

$$f(x) = \kappa + \sum_i c_i b_i(x_i) + \sum_\alpha c_\alpha \left[b_\alpha(x_\alpha) - \sum_{k \in \alpha} c_k b_k(x_k) \right]$$

- f can be written as a **conical combination** of the beliefs if there exists a vector, d, of non-negative reals such that

$$f(x) = \sum_{i,\alpha : i \in \alpha} d_{i\alpha}(b_\alpha(x_\alpha) - b_i(x_i)) + \sum_\alpha d_{\alpha\alpha} b_\alpha(x_\alpha) + \sum_i d_{ii} b_i(x_i)$$

Global Optimality

Theorem: Given an objective function, *f*, suppose that

- *b* is a vector of admissible and min-consistent beliefs for the function *f*
- *c* is chosen such that *f* can be written as a conical combination of the beliefs
- There exists an x^* that simultaneously minimizes all of the beliefs

then x^* minimizes the objective function

Global Optimality

- Examples of globally optimal parameters:
  - TRMP: $c_i = 1$ and c_α is the "edge appearance probability"
  - MPLP: $c_\alpha = 1$ and $c_i = 1/2$
  - Others: $c_\alpha > 0$ and $c_i(1 - \sum_{\alpha \in \partial i} c_\alpha) \geq 0$
- Weaker conditions on c guarantee local optimality (wrt the Hamming distance)

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Notion of Convergence

- Let c be chosen as: $c_i = 1$

$$c_\alpha = 1/\max_i |\partial i|$$

- Consider the lower bound:

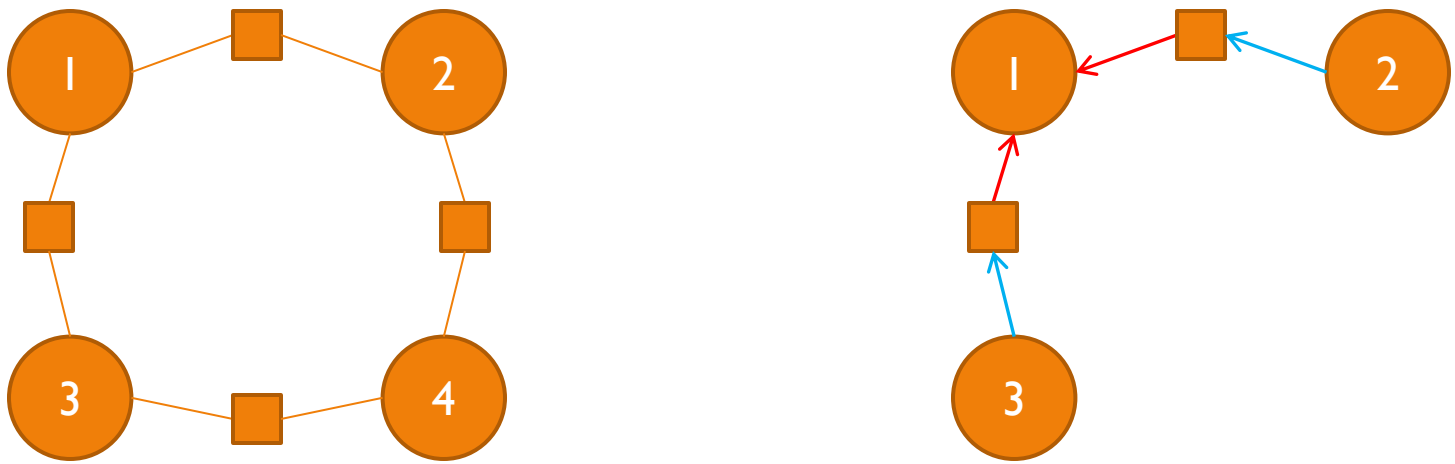
$$\begin{aligned} \min_x f(x) &\geq \sum_i c_i (1 - \sum_{\alpha \in \partial i} c_\alpha) \min_{x_i} b_i(x_i) + \sum_\alpha c_\alpha \min_{x_\alpha} b_\alpha(x_\alpha) \\ &\equiv LB(m) \end{aligned}$$

- We will say that the algorithm has converged if this lower bound cannot be improved by further iteration

Convergence

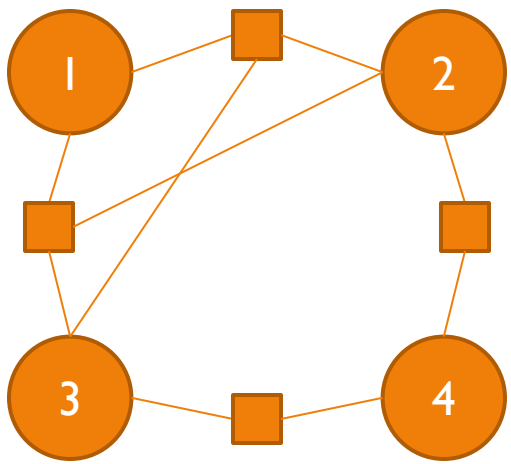
- For certain c, there is an asynchronous message passing schedule that is guaranteed not to decrease this bound
- All "bound minimizing" algorithms have similar guarantees [Meltzer, Globerson, Weiss 2009]
  - Ensure min-consistency over one subtree of the graph at a time

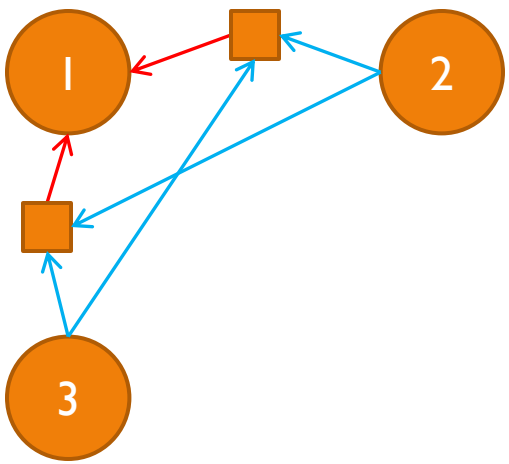
Asynchronous Algorithm



- Update the blue edges then the red edges
- After this update beliefs are min-consistent with respect to the variable x_1

Asynchronous Algorithm





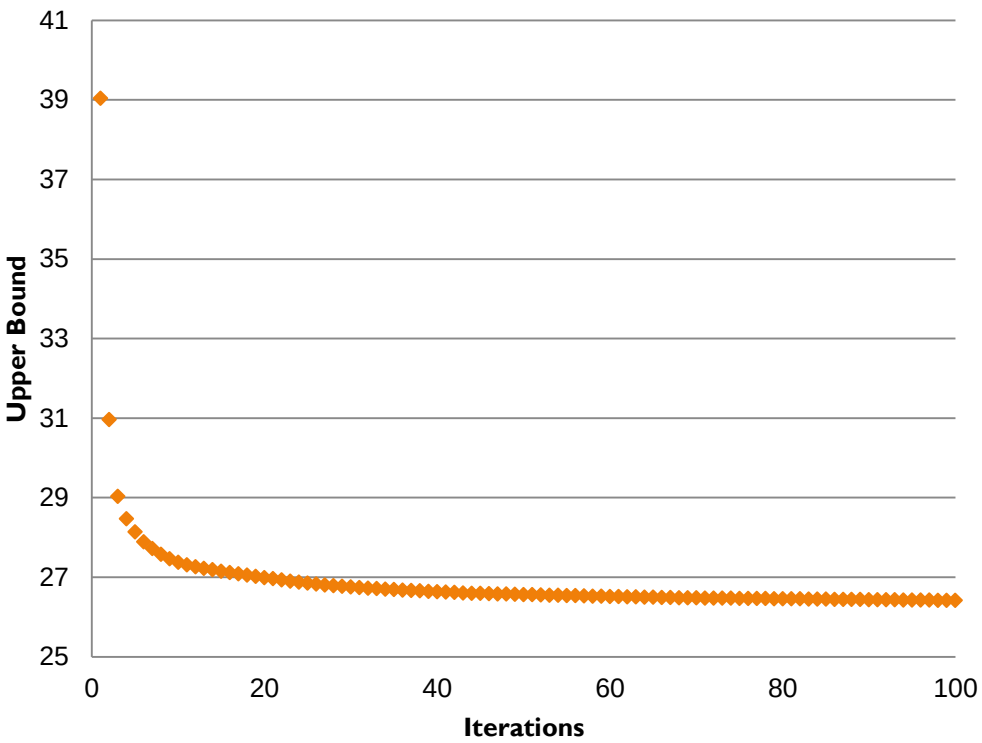
- Update the blue edges then the red edges
- After this update beliefs are min-consistent with respect to the variable x_1

Asynchronous Convergence

- Order the variables and perform the one step update for a different variable at each step of the algorithm
  - This cannot decrease the lower bound
  - This update can be thought of as a coordinate ascent algorithm in an appropriate dual space

An Example: MWIS

- 100x100 grid graph
- Weights randomly chosen between 0 and 1
- Convergence is monotone as predicted



Conclusion

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- An asynchronous algorithm that generalizes the notion of "bound minimizing" algorithms [Meltzer, Globerson, Weiss 2009]
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Questions?

Preprint available online at: http://arxiv.org/abs/1002.3239

Other Results

- We can extend the computation tree story to this new algorithm
- Can use graph covers to understand when the algorithm can converge to a collection of beliefs that admit a unique estimate
- The lower bound is always tight for pairwise binary graphical models
  - New proof that uses graph covers instead of duality