SCN : 3. Multistage Inventory Systems

1 Reformulation of Single Stage EOQ Model

With all the EOQ assumptions

$$z = \min_{Q \geq 0} \frac{\lambda K}{Q} + \frac{hQ}{2}$$

With a change of variable $T = Q/\lambda$

$$z = \min_{T \geq 0} \frac{K}{T} + \frac{h\lambda T}{2} = \min_{T \geq 0} \frac{K}{T} + gT$$

where $g = \lambda h/2$. The optimal solution is $T^* = \sqrt{K/g}$ and $z^* = 2\sqrt{Kg}$.

Consider the power of two policies. Let T_L be base planning period, we assume that $T_L \leq T^*$. Consider the minimization of inventory costs when ordering period is a power of two multiple of T_L:

$$\min_{T \geq 0} \{f(T) = \frac{K}{T} + gT\}$$

$$T = 2^k T_L$$

$$k \in Z_+$$

Now k is the decision variable and it is nonnegative integer so look for the first difference. Find the smallest k such that

$$f(2^{k+1}T_L) \geq f(2^k T_L) \quad \Rightarrow \quad \frac{K}{2^{k+1}T_L} + g2^{k+1}T_L \geq \frac{K}{2^k T_L} + g2^k T_L \Rightarrow 2^{2k} \geq \frac{K}{2T_L^2 g}$$

$$2^k \geq \frac{1}{\sqrt{2}T_L}\sqrt{\frac{K}{g}} = \frac{1}{\sqrt{2}T_L}T^* \Rightarrow \frac{1}{\sqrt{2}}T^* \leq 2^k T_L < \sqrt{2}T^*$$

Thus, the optimal power of two cycle time is close to T^*. Let us check the value of the objective function with the power of two policy. The objective value is between $f(T^*/\sqrt{2})$ and $f(\sqrt{2}T^*)$:

$$f(T^*/\sqrt{2}) = f(\sqrt{2}T^*) = \frac{K}{\sqrt{2}\sqrt{K/g}} + g\sqrt{2}\sqrt{K/g} = (\sqrt{2} + \frac{1}{\sqrt{2}})\sqrt{Kg} = (\sqrt{2} + \frac{1}{\sqrt{2}})z^*/2 = 1.06z^*$$

2 Multistage Inventory System

Now we consider N stages. The holding costs are h'_i for each stage but we define echelon holding costs as $h_i = h'_i - h'_{i+1}$ for $i < N$ and $h_N = h'_N$. The echelon inventory at stage i is defined as the sum of inventory at stage i and all the other stages j for $j < i$. In a sense, this is the total inventory between stage i and the customer. Using the echelon inventory holding costs and echelon inventories and equivalent inventory holding cost accounting is possible; see Figure 1. This accounting scheme makes the inventory graphs saw-tooth type at each stage so that we can continue to use EOQ type costs.

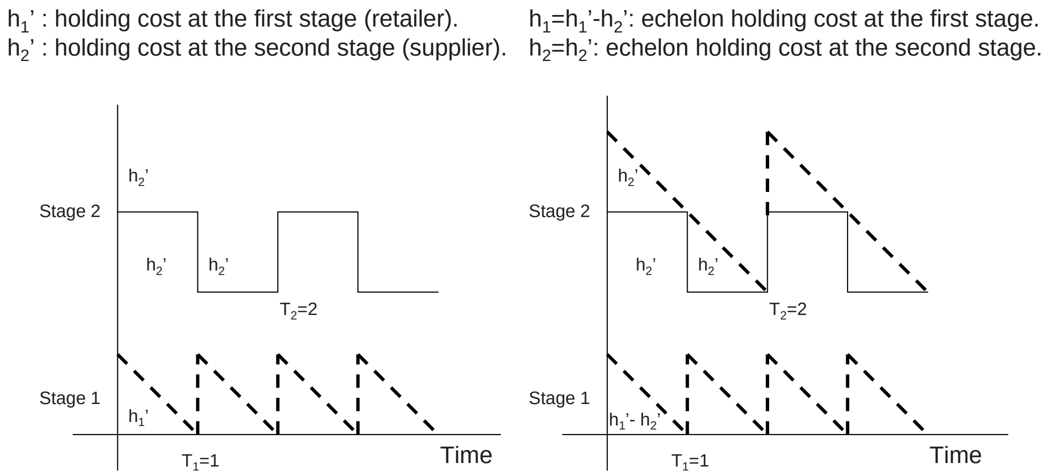


Figure 1: Two alternative but equivalent ways of holding cost accounting.

The N stage problem is presented below. The cost parameter g_i is based on echelon holding costs.

$$\min \sum_{i=1}^{N} \left(\frac{K_i}{T_i} + g_i T_i \right)$$

$$\begin{aligned} \text{s.t.} \quad T_i &\geq T_{i-1} \geq 0 \\ T_i &= 2^l T_L \\ T_0 &= 0 \end{aligned}$$

Ignore $T_i = 2^l T_L$ and rewrite the above problem:

$$\min \sum_{i=1}^{N} f_i(T_i)$$

s.t.

$$\begin{aligned} T_{i-1} - Ti \leq 0 & \qquad i = 1, \cdots, N \\ f_i(T_i) = \frac{K_i}{T_i} + g_i T_i & \qquad i = 1, \cdots, N \\ T_0 = 0 & \end{aligned}$$

This is a convex nonlinear minimization problem. Use Kuhn-Tucker conditions to solve it.

$$L(T, \lambda) = \sum_{i=1}^{N} f_i(T_i) + \sum_{i=1}^{N} \lambda_{i-1}(T_{i-1} - T_i)$$

KKT:

$$\begin{array}{lll} (1) & f_i'(T_i) - \lambda_{i-1} + \lambda_i = 0 & i = 1, \cdots, N \\ (2) & T_{i-1} - T_i \leq 0 & i = 1, \cdots, N \\ (3) & (T_{i-1} - T_i)\lambda_{i-1} = 0 & i = 1, \cdots, N \\ (4) & \lambda_{i-1} \geq 0 & i = 1, \cdots, N \\ & T_0 = 0 & \end{array}$$

Note that the index of λ is from 0 to $N - 1$.

Define:
Echelon cluster $C = \{i, i+1, \cdots, j-1, j\}$ such that $T_i = T_{i+1} = \cdots = T_j$
$\max(C) = \max(\{i, \cdots, j\}) = j$
$\min(C) = \min(\{i, \cdots, j\}) = i$

$$T_i = \arg\min\{\sum_{j \in C} f_j(T) : i \in C\}$$

Let

$$\begin{cases} \lambda_i = 0 & \text{if } i = \max(C) \\ \lambda_i = \sum_{n=i+1}^{\max(C)} f_n'(T_n) & \text{otherwise} \end{cases}$$

Note that $\lambda_{\max(C)}$ can also be written as

$$\lambda_{\max(c)} = \sum_{n=\max(C)+1}^{\text{max of the next cluster}} f_n'(T_n) = 0$$

We can check that constraints (1) and (3) hold.

Constraint (1): $\lambda_{i-1} - \lambda_i = f_i'(T_i)$

- $i - 1, i \in C$

$$\lambda_{i-1} - \lambda_i = \sum_{n=i}^{\max(C)} f_n'(T_n) - \sum_{n=i+1}^{\max(C)} f_n'(T_n) = f_i'(T_i)$$

- $i - 1 = \text{max of a cluster}, i$ is a min of cluster C

$$\lambda_{i-1} - \lambda_i = 0 - \sum_{n=\min(C)+1}^{\max(C)} f_n'(T_n) = -(\sum_{n=\min(C)}^{\max(C)} f_n'(T_n) - f_{\min(C)}'(T) = f_i'(T_i)$$

Constraint (3): $\lambda_{i-1} - \lambda_i = f_i'(T_i)$

- $i - 1 = \text{max of a cluster}$: $\lambda_{i-1} = 0$

For n=1..N do
    $C := \{n\}$, UniteCompulete:=false
    While UniteComplete = false
        $k = \min(C) - 1$
        Compute $t_C := \arg\min\{\sum_{j\in C} f_j(T)\}$
        Remember T_k from memory
        If $T_k > T_C$
            Unite: $C = C \cup$(The cluster containing k)
        else UniteComplete = true
    EndWhile
    Update T_n for $n \in C$ in the memory
EndFor

Table 1: Cluster Algorithm

- $i-1$ is the only element of a cluster: $\lambda_{i-1} = 0$
- $i-1$ is one of the element of a cluster: $T_{i-1} - T_i = 0$

We use the Cluster Algorithm of Table 1 to solve the N stage problem. A numerical example is presented next.

Example: $f_1(T) = (T-8)^2, f_2(T) = (T-2)^2 = f_3(T), f_4(T) = (T-8)^2$

$n = 1$
    $C = \{1\}$ UniteComplete $= 0$
    while
        $k = 0$
        Compute $T_1 = 8, T_0 = 0$,UniteComplete $= 1$
    Remember $T_1 = 8, C = \{1\}$

$n = 2$
    $C = \{2\}$ UniteComplete $= 0$
    while
        $k = 1, T_1 = 8$
        $T_C = 2$
        $T_k = 8 > T_C$, then $C = \{1,2\}, T_C = 5$

        $k = 0, T_k = 0$,UniteComplete $= 1$
        $T_1 = T_2 = 5, C = \{1,2\}$

$n = 3$
    $C = \{3\}$ UniteComplete $= 0$
    while
        $k = 2, T_2 = 5$
        $T_C = 2$

$T_k = 5 > T_C$, then $C = \{1, 2, 3\}, T_C = 4$

$k = 0, T_k = 0$,UniteComplete $= 1$
$T_1 = T_2 = T_3 = 4, C = \{1, 2, 3\}$

$n = 4$
$C = \{4\}$ UniteComplete $= 0$
while
$k = 3, T_3 = 4$
$T_C = 8$
$T_k = 4 < T_C$, then UniteComplete $= 1$

So there are two clusters - $\{1, 2, 3\}$ and $\{4\}$ with $T_1 = T_2 = T_3 = 4, T_4 = 8$.

Note that Constraint (2) always holds for this cluster algorithm.

Constraint (4): $\lambda_{i-1} \geq 0$

- $i - 1 = \max(C)$ of a cluster: $C : \lambda_{i-1} = 0$

  The constraint is satisfied.

- $i - 1$ is one of the element of a cluster C:

$$\lambda_{i-1} = \sum_{n=i}^{\max(C)} f'_n(T_C)$$

  Define $A = \{\min(C), \cdots, i - 1\}$

$$T_A = \arg\min \sum_{j \in A} f_j(T) > \arg\min f_i(T)$$

  According to the algorithm.

  Since f_i's are convex functions, it can be shown that

$$\lambda_{i-1} = \sum_{n=i}^{\max(C)} f'_n(T_C) \geq 0$$

We solve

$$Z = \min \sum_{n=1}^{N} f_n(T) = \min \sum_{n=1}^{N} \left(\frac{K_n}{T_n} + g_n T_n \right)$$
$$\text{s.t.} \quad T_{n-1} \leq T_n$$

$$C = \{1\} \qquad T_1 = \sqrt{\frac{K_1}{g_1}} \qquad \Leftarrow \quad \min(\frac{K_1}{T_1} + g_1 T_1)$$
$$C = \{1,2\} \qquad T_1 = T_2 = \sqrt{\frac{K_1 + K_2}{g_1 + g_2}} \quad \Leftarrow \quad \min(\frac{K_1 + K_2}{T} + (g_1 + g_2)T)$$

We get

$$T_j = \sqrt{\frac{\sum_{i \in C} K_i}{\sum_{i \in C} g_i}} \qquad j \in C$$

Notice that we have solved a problem without the constraint $T_n = 2^l T_L$. Now add this constraint.

For every j, find l_j such that

$$2^{l_j} \geq \frac{T_j}{\sqrt{2} T_L} > 2^{j_j - 1}$$

and set

$$T_j^* = 2^{l_j} T_L$$

It can be shown that the constraint $T_{n-1} \leq T_n$ will not be violated and clusters will not be changed with this new T_j^*. In addition, costs only increase at most 6% by rounding T_j to power of two. That is

$$1.06Z \geq \text{The costs of the power of two policy}$$

A compact presentation of these arguments can be found in Chapter 2 of [1] for details see the references at the end of that chapter.

3 Exercises

1. Redraw Figure 1 with $T_1 = 2$ and $T_2 = 3$. Remember that order quantities at each stage must be constant. Do you get the saw-tooth echelon inventory pattern with this power of 1.5 policy, explain?

2. Consider the power of three policies. Let T_L be base planning period and consider

$$\min_{T \geq 0} \{f(T) = \frac{K}{T} + gT\}$$
$$T = 3^k T_L$$
$$k \in Z_+$$

   How far is the optimal solution to power of three policy to the optimal solution of $\min_{T \geq 0}\{f(T) = \frac{K}{T} + gT\}$?

3. Let f_1 and f_2 be two differentiable and convex functions which are not constant on any interval. Let

$$T_i^* = \arg\min_{T \geq 0} f_i(T) \quad \text{for } i \in \{1,2\} \quad \text{and} \quad T_{1,2}^* = \arg\min_{T \geq 0} f_1(T) + f_2(T)$$

   Show that $\min\{T_1^*, T_2^*\} \leq T_{1,2}^* \leq \max\{T_1^*, T_2^*\}$. Do you think differentiability assumption is critical for your argument? Note that this property is used when clusters are united in the Cluster Algorithm.

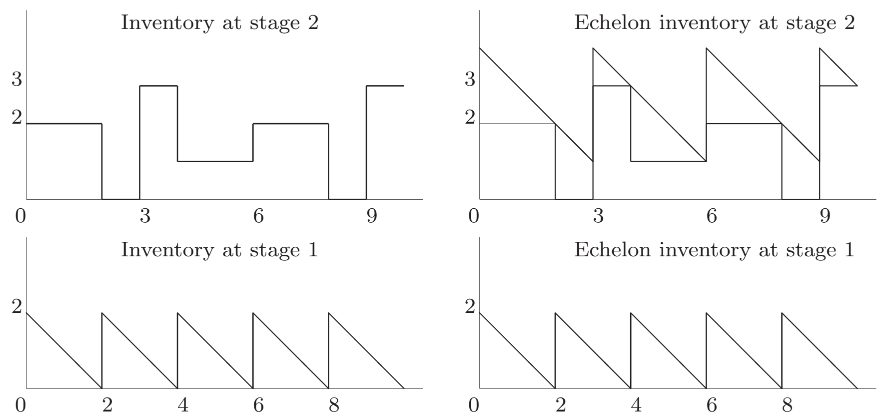
4. What is the running time of the Cluster Algorithm in terms of N, is it $O(N)$ or $O(N^2)$ or something else?

References

[1] Logistics of Production and Inventory (1993). Edited by S.C. Graves, A.H.G. Rinnooy Kan and P.H. Zipkin. Volume 4 of Handbooks in ORMS. Publishied by Elsevier, Amsterdam.

4 Solutions

1. See the following figure.



Both stage 1 and 2 have saw-tooth echelon inventory pattern. However, the difference with respect to Figure 1 is that the echelon inventory for stage 2 never hits zero. Since the holding cost for 1 unit of the product is always charged in stage 2, power of 1.5 policy is not very efficient.

2. The problem is

$$\min_{T\geq 0}\left\{f(T)=\frac{K}{T}+gT\right\}$$
$$s.t.\quad T=3^kT_L$$
$$k\in Z_+.$$

Find the smallest k such that

$$f(3^{k+1}T_L)\geq f(3^kT_L)\quad\Rightarrow\quad\frac{K}{3^{k+1}T_L}+g3^{k+1}T_L\geq\frac{K}{3^kT_L}+g3^kT_L\Rightarrow 3^{2k}\geq\frac{K}{3T_L^2g}$$

$$3^k\geq\frac{1}{\sqrt{3}T_L}\sqrt{\frac{K}{g}}=\frac{1}{\sqrt{3}T_L}T^*\Rightarrow\frac{1}{\sqrt{3}}T^*\leq 3^kT_L<\sqrt{3}T^*$$

Thus, the optimal power of two cycle time is close to T^*. Let us check the value of the objective function with the power of two policy. The objective value is between $f(T^*/\sqrt{3})$ and $f(\sqrt{3}T^*)$:

$$f(T^*/\sqrt{3})=f(\sqrt{3}T^*)=\frac{K}{\sqrt{3}\sqrt{K/g}}+g\sqrt{3}\sqrt{K/g}=(\sqrt{3}+\frac{1}{\sqrt{3}})\sqrt{Kg}=(\sqrt{3}+\frac{1}{\sqrt{3}})z^*/2=1.1547z^*$$

3. By the minimality of T_1^* and $T_{1,2}$ we have

$$f_1(T_1^*) \leq f_1(T_{1,2}^*) \tag{1}$$
$$f_1(T_1^*) + f_2(T_1^*) \geq f_1(T_{1,2}^*) + f_2(T_{1,2}^*) \tag{2}$$

From (1) and (2) we have $f_2(T_1^*) \geq f_2(T_{1,2}^*)$, together with the minimality of T_2^* we have

$$f_2(T_1^*) \geq f_2(T_{1,2}^*) \geq f_2(T_2^*).$$

Since f_2 is convex, we deduce that either $T_1^* \leq T_{1,2}^* \leq T_2^*$ or $T_2^* \leq T_{1,2}^* \leq T_1^*$. Clearly, this result has nothing to do with differentiability.