OPRE 6377. DemReMan : Capacity Constraint

1 Solved Exercises

1. [**Marginal Opportunity Cost of Capacity for Demand with Constant Elasticity**] We suppose that $d(p) = Cp^{-\varepsilon}$ for $\varepsilon \geq 1$. Let the profit under capacity b be $\Pi(b) = \max_p\{(p-c)d(p) : d(p) \leq b\}$. We want to obtain the marginal opportunity cost of the capacity, i.e.,

$$\frac{d}{db}\Pi(b).$$

ANSWER From our basic price optimization discussion, we know that the unconstrained optimal price is

$$p_0 = c\frac{\varepsilon}{\varepsilon - 1}.$$

This price generates a demand of $d(p_0) = C(c\varepsilon/(\varepsilon-1))^{-\varepsilon}$. If the capacity is more than this price, its marginal opportunity costs is zero:

$$\frac{d}{db}\Pi(b) = 0 \quad \text{for} \quad b \geq C\left(c\frac{\varepsilon}{\varepsilon-1}\right)^{-\varepsilon}.$$

We now suppose that $b < C(c\varepsilon/(\varepsilon-1))^{-\varepsilon}$ which implies $C/b > (c\varepsilon/(\varepsilon-1))^{\varepsilon}$. On the other hand, the inverse demand function is found from setting $d(p) = Cp^{-\varepsilon} = b$ and solving p in terms of b:

$$d^{-1}(b) = p = \left(\frac{C}{b}\right)^{1/\varepsilon}.$$

Then

$$\begin{aligned}
\frac{d}{db}\Pi(b) &= \frac{d}{db}b\left(d^{-1}(b) - c\right) \\
&= \frac{d}{db}b\left(\left(\frac{C}{b}\right)^{1/\varepsilon} - c\right) \\
&= \left(\frac{C}{b}\right)^{1/\varepsilon} - c + b\left(-\frac{1}{\varepsilon}\left(\frac{b}{C}\right)^{-1/\varepsilon-1}\frac{1}{C}\right) \\
&= \left(\frac{C}{b}\right)^{1/\varepsilon}\left(1 - \frac{1}{\varepsilon}\right) - c
\end{aligned}$$

This marginal opportunity cost is nonnegative because $C/b > (c\varepsilon/(\varepsilon-1))^{\varepsilon}$. It decreases in the capacity b. It starts with infinity when $b = 0$ and becomes zero when $b = C(c\varepsilon/(\varepsilon-1))^{-\varepsilon}$. We can check the last assertion:

$$\left.\frac{d}{db}\Pi(b)\right|_{b=C(c\varepsilon/(\varepsilon-1))^{-\varepsilon}} = \left((c\varepsilon/(\varepsilon-1))^{\varepsilon}\right)^{1/\varepsilon}\left(1 - \frac{1}{\varepsilon}\right) - c = 0.$$

2. Consider selling tickets for UTDallas and UTAustin ladies volleyball game to be held at a stadium with C seating capacity. Demands for general public and students are given by

$$d_g(p_g) = a_g - b_g p_g \quad \text{and} \quad d_s(p_s) = a_s - b_s p_s.$$

These lead to the following pricing problem

$$\begin{aligned} \max_{p_g, p_s} p_g(a_g - b_g p_g) + p_s(a_s - b_s p_s) \\ a_g - b_g p_g + a_s - b_s p_s \leq C \\ p_g \leq a_g/b_g \\ p_s \leq a_s/b_s. \end{aligned}$$

Find the optimal prices from the optimization problem above.

ANSWER The maximization objective above is equivalent to the following objectives

$$\begin{aligned} &\equiv \max_{p_g, p_s} p_g(a_g - b_g p_g) + p_s(a_s - b_s p_s) \\ &\equiv \min_{p_g, p_s} b_g p_g^2 - a_g p_g + b_s p_s^2 - a_s p_s \\ &\equiv \min_{p_g, p_s} b_g(p_g - a_g/(2b_g))^2 + b_s(p_s - a_s/(2b_s))^2 \end{aligned}$$

Thus, the equivalent optimization problem is

$$\begin{aligned} \min_{p_g, p_s} b_g(p_g - a_g/(2b_g))^2 + b_s(p_s - a_s/(2b_s))^2 \\ b_g p_g + b_s p_s \geq a_g + a_s - C \\ p_g \leq a_g/b_g \\ p_s \leq a_s/b_s. \end{aligned}$$

The unconstrained optimal price is $(a_g/(2b_g), a_s/(2b_s))$ which satisfies both constraints $p_g \leq a_g/b_g$ and $p_s \leq a_s/b_s$. Then there are two possibilities: Either $a_g + a_s \leq 2C$ or $a_g + a_s > 2C$. These cases can be found from the problem data wgich specify a_g, a_s, C.

If $a_g + a_s \leq 2C$, we have

$$b_g a_g/(2b_g) + b_s a_s/(2b_s) = a_g/2 + a_s/2 \geq a_g + a_s - C,$$

so the capacity constraint is satisfied at the unconstrained price. In summary, the unconstrained price satifies all theree constraints and it is the constrained optimal as well.

If $a_g + a_s > 2C$, we have

$$b_g a_g/(2b_g) + b_s a_s/(2b_s) = a_g/2 + a_s/2 < a_g + a_s - C,$$

so the capacity constraint is not satisfied at the unconstrained price. The optimal constrained price will use up all the available capacity:

$$b_g p_g + b_s p_s = a_g + a_s - C.$$

In addition to the last equality, we obtain another equality by considering the gradient of the objective $\Pi(p_g, p_s) = b_g(p_g - a_g/(2b_g))^2 + b_s(p_s - a_s/(2b_s))^2$:

$$\nabla \Pi = \left[\frac{\partial \Pi}{\partial p_g}, \frac{\partial \Pi}{\partial p_s}\right] = [2b_g((p_g - a_g/(2b_g)), \; 2b_s((p_s - a_s/(2b_s))].$$

At the optimal price, the gardient must be perpendicular to the slope of the capacity constraint which can be represented by either $[b_s; -b_g]$ or $[-b_s; b_g]$ in coordinate system where x-axis is p_g and $y - axis$ is p_s. Note that while computing the gradient, we have treated p_g as the first variable of the function Π. The consistency of the first and second variables is important.

The perpendicular vectors must give zero when they are multiplied:

$$\begin{aligned} [2b_g(p_g - a_g/(2b_g)),\ 2b_s(p_s - a_s/(2b_s))][b_s; -b_g] &= 2b_g b_s(p_g - a_g/(2b_g)) - 2b_s b_g(p_s - a_s/(2b_s)) \\ &= [2b_g b_s][p_g - a_g/(2b_g) - p_s + a_s/(2b_s)] \\ &= 0. \end{aligned}$$

This implies the optimality equation

$$p_g - p_s = a_g/(2b_g) - a_s/(2b_s).$$

When we intersect this equation with the constraint $b_g p_g + b_s p_s = a_g + a_s - C$, we obtain the optimal prices

$$\begin{aligned} p_g &= \frac{1}{b_g + b_s}\left\{a_g\left(1 + \frac{b_s}{2b_g}\right) + \frac{a_s}{2} - C\right\} \\ p_s &= \frac{1}{b_g + b_s}\left\{a_s\left(1 + \frac{b_g}{2b_s}\right) + \frac{a_g}{2} - C\right\} \end{aligned}$$

We summarize the results:

$$\begin{aligned} a_g + a_s \le 2C &\implies \left\{\begin{array}{l} p_g = a_g/(2b_g) \\ p_s = a_s/(2b_s) \end{array}\right\}, \\ a_g + a_s > 2C &\implies \left\{\begin{array}{l} p_g = \frac{1}{b_g+b_s}\left\{a_g\left(1 + \frac{b_s}{2b_g}\right) + \frac{a_s}{2} - C\right\} \\ p_s = \frac{1}{b_g+b_s}\left\{a_s\left(1 + \frac{b_g}{2b_s}\right) + \frac{a_g}{2} - C\right\} \end{array}\right\}. \end{aligned}$$

2 Homework Questions

1. The cost of an ordinary haircut is $c = \$4$. On the other hand, the demand can vary a lot with the price. If the price is more than \$10, only the regular customers consider having a haircut and their demand function is $d_r(p) = 150 - 5p$ for $0 \le p \le 30$. If the price drops below \$10, opportunistic customers also consider having a haircut and their demand function is $d_o(p) = 750 - 75p$ for $0 \le p \le 10$.
   a) What is the total demand function $d(p)$ for $0 \le p \le 30$?
   b) Find the optimal price for the haircut.
   c) Compute the demand corresponding to the optimal price and check that it is above the capacity of $b = 60$. Find a new price that maximizes the profit and yields a demand that can be dealt with the capacity of $b = 60$.
   d) How does the price change if we expand the capacity to $b = 80$.
   e) How does the price change if we expand the capacity to $b = 120$.

2. For the haircut demand function $d(p)$ defined in the last exercise, obtain the marginal opportunity cost for $0 \le b \le 400$. Draw this cost as function of capacity b for $0 \le b \le 400$.

3. Consider selling tickets for a concert at UTD Theater which has a seating capacity of 400. Demands for general public and students are given by

$$d_g(p_g) = 400 - 10p_g \quad \text{and} \quad d_s(p_s) = 200 - 20p_s.$$

   What should be the optimal ticket prices for general public and students?

4. Consider selling tickets for a concert at UTD Theater which has a seating capacity of 200. Demands for general public and students are given by

$$d_g(p_g) = 400 - 10p_g \quad \text{and} \quad d_s(p_s) = 200 - 20p_s.$$

What should be the optimal ticket prices for general public and students?

5. Consider the 3-period variable pricing formulation for a system constrained by C:
max $p_1x_1 + p_2x_2 + p_3x_3$;
st. $x_i \le D_i - m_ip_i + s(p_j - p_i) + s(p_k - p_i)$ for $i \ne j \ne k$ and $i, j, k \in \{1, 2, 3\}$,
$x_i \le C$ for $i \in \{1, 2, 3\}$,
$p_i, x_i \ge 0$.
a) Besides the formulation above, we are presented with another one:
max $p_1x_1 + p_2x_2 + p_3x_3$;
st. $x_1 \le D_1 - m_1p_1 - s(p_1 - p_2) - s(p_1 - p_3)$,
$x_2 \le D_2 - m_2p_2 + s(p_1 - p_2) - s(p_2 - p_3)$,
$x_3 \le D_3 - m_3p_3 + s(p_1 - p_3) + s(p_2 - p_3)$,
$x_1, x_2, x_3 \le C$,
$p_i, x_i \ge 0$.
Determine if this formulation is written under the condition $p_3 \le p_2 \le p_1$ or $p_2 \le p_1 \le p_3$ or $p_1 \le p_2 \le p_3$ or no condition on prices, explain.
b) The formulation in a) is modified as:
max $p_1x_1 + p_2x_2 + p_3x_3$;
st. $x_1 \le D_1 - m_1p_1 - s_1(p_1 - p_2) - s_2(p_1 - p_3)$,
$x_2 \le D_2 - m_2p_2 + s_1(p_1 - p_2) - s_2(p_2 - p_3)$,
$x_3 \le D_3 - m_3p_3 + s_2(p_1 - p_3) + s_2(p_2 - p_3)$,
$x_1, x_2, x_3 \le C$,
$p_i, x_i \ge 0$.
Express parameters s_1 and s_2 in English.

6. The art museum in your town is closed on Mondays and charges three different entrance fees. The fee p_1 on Tuesdays and Wednesdays is the lowest. The fee on p_3 on weekends is the highest. The fee on p_2 charged on Thursdays and Fridays satisfies $p_1 < p_2 < p_3$. Recall that s in general represents the number of customers switching between periods which have a price differential of one monetary unit. The same switching behavior from period j to period k is expected from museum patrons and can be modelled with
   - s_{jk} that depends on the period j and period k and $s_{jk} \ne s_{kj}$;
   - s_{jk} that depends on the period j and period k and $s_{jk} = s_{kj}$;
   - s_m that depends on the number m of periods between the periods j and k;
   - s that is a constant independent of periods.

   A permanent collection remains in the museum for many weeks unlike a temporary one (a collection that tours museums) that may be there for just one week. Discuss which one of the switching models is more appropriate for studying permanent collection patrons and temporary collection patrons.

7. Suppose that the switching behavior from period j to period k is modelled with s_{jk} that depends on the period j and period k and $s_{jk} \ne s_{kj}$. Modifying the 3-period variable pricing formulation, we obtain.
max $p_1x_1 + p_2x_2 + p_3x_3$;
st. $x_1 \le D_1 - m_1p_1 + s_{21}(p_2 - p_1)^+ + s_{31}(p_3 - p_1)^+ - s_{12}(p_1 - p_2)^+ - s_{13}(p_1 - p_3)^+$,
$x_2 \le D_2 - m_2p_2 - s_{21}(p_2 - p_1)^+ + s_{12}(p_1 - p_2)^+ + s_{32}(p_3 - p_2)^+ - s_{23}(p_2 - p_3)^+$,
$x_3 \le D_3 - m_3p_3 + ???$,
$x_1, x_2, x_3 \le C$,

$p_i, x_i \geq 0.$

a) Is the constraint for the second period correct?

b) Complete the right-hand side of the constraint for the third period.

8. If you are a PhD student, you may attempt to prove that the optimal solutions with and without constraint $x_i \geq 0$ are the same. In other words, the optimal solution cannot have negative x_i even without $x_i \geq 0$ constraint.