Overbooking

Outline

- **Overbooking**
- **Policies**
- **Extensions**

Based on Phillips (2005) Chapter 9

Overbooking in Practice

Industry	Importance	Consequence
Passenger airlines	Very high	Compensate & reaccommodate at other flights
Business hotels	High	Reaccommodate at other hotels
Rental cars	High	Delay (backorder) customer or reaccommodate at other companies
Air cargo	High	Reaccommodate at a later flight
Health clinics	Medium	Reaccommodate with overtime
Cruise lines	Low	Typically do not overbook or reaccommodate at a better room
Resort hotels	Low	Typically do not overbook or reaccommodate at a better room
Events (sport, music)	Low	Typically do not overbook. Tickets are nonrefundable but transferable

- Overbooking is applicable when
  - Capacity is fixed and perishable
  - Customers cancel tickets (reservations, orders) or do not simply show
  - Consequence of denying service to overbooked customers is low

Overbooking Terminology

- Bumping a passenger = Denied boarding
  - Voluntary denied boarding
    - An airline offers vouchers and reaccommodation (at another flight) to passengers of overbooked flights
    - Some passengers volunteer to fly on an alternate flight to receive these vouchers
  - Involuntary denied boarding
    - Even after some passengers voluntarily give up their seats on an overbooked flight, there may not be enough seats for all the passengers that show up at the time of departure
    - The airline may increase the compensation level once or twice to motivate more passengers to give up their seats.
    - After all these, if there are more passengers than the seats, the airline involuntarily bumps the passengers

- Your rights according to Department of Transportation (circa 2004) when you are involuntarily bumped:
  - If < 1 hour delay to arrive your final destination with alternative flights, compensation=0.
  - Else <2 hours delay to arrive your final destination, compensation=min{one-way fare, 200}
  - Else, compensation=2*min{one-way fare, 200}
  - These threshold times are for domestic flights, multiply the threshold times by 2 for international flights
- You are entitled to the compensation above in addition to reaccommodation on an alternative flight

Cost of Service Denial

- Direct cost or compensation
  - Voucher for future travel or hotel stays
- Provision cost
  - Meals and lodging for bumped passenger
- Reaccommodation cost
  - Airline: Flying on another flight of the airline or of another airline
  - Hotel: Alternative accommodation in another , higher class room of the hotel or of another hotel
  - Rental car: Alternative possibly higher-class car of the rental car company or another rental car company
- Ill will cost
  - Cost of upsetting the customer
  - Loss of future demand
  - This cost is elusive

- Sum all these costs to obtain *D*, the cost of denial of service per customer

Booking Process

- Booking for fixed capacity of C.
- Supplier decides on the overbooking limit b in advance.

- During booking process,
  - Accept bookings until the total bookings are less than the limit b.
  - Reject bookings afterwards.
  - The total demand received during the booking process is d. Number of accepted bookings is $n(b)=\min\{d,b\}$.

- At the time of service (departure time, check-in time),
  - Only a portion ρ of accepted bookings show up. Customers showing up are called shows and their number is denoted by s. The other customers are called no-shows and their number is denoted by x.
  - Each showing up customer pays price p.
  - If $s \leq C$, every customer is served
  - If $s > C$ and $(s-C)$ customers are denied service. Each such denial costs D.

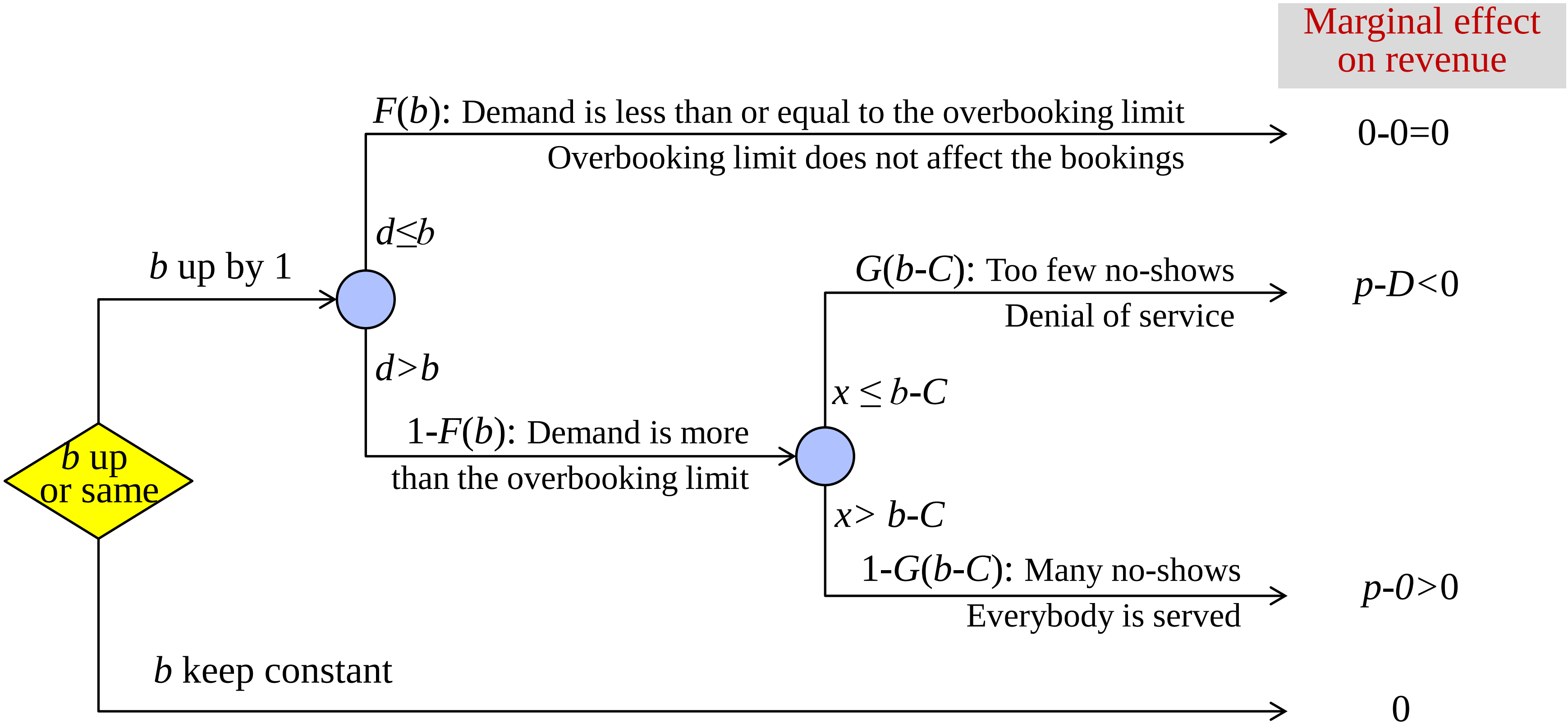
- Fixing b high
  - increases shows from which we make revenue but also
  - increases denial of services from which we lose revenue.

Finding Overbooking Limit

- Deterministic Heuristic: Inflate the capacity by $1/\rho$ to find the overbooking limit, $b = C/\rho$.
- Risk-Based Policies
  - Trade off the price p against the cost D of denial of service in a profit objective
  - Take the distribution of shows into account
    - Independent distribution of no-shows from the number of bookings
    - Dependent distribution of no-shows on the number of bookings
- Service-Level policies which limit
  - the probability of denial of service
  - the percentage of customers who are denied of service
- Hybrid Policies: Mixture of Risk-Based and Service-Level policies

- We focus on Risk-Based policies

Overbooking Limit
G: Independent Distribution of No-Shows



Marginal effect on revenue

b up or same

b up by 1

$F(b)$: Demand is less than or equal to the overbooking limit

Overbooking limit does not affect the bookings

$0-0=0$

$d \le b$

$d > b$

$1-F(b)$: Demand is more than the overbooking limit

$x \le b-C$

$G(b-C)$: Too few no-shows

Denial of service

$p-D<0$

$x > b-C$

$1-G(b-C)$: Many no-shows

Everybody is served

$p-0>0$

b keep constant

0

Expected marginal effect on revenue

$$= [1-F(b)]\left[G(b-C)(p-D)-[1-G(b-C)]p\right]$$
$$= [1-F(b)]\left[p-G(b-C)D\right]$$

Graphical Depiction
Independent Distribution of No-Shows

The critical term is $p-G(b-C)D$

When b=C, $p-G(b-C)D=p>0$ so increase b.

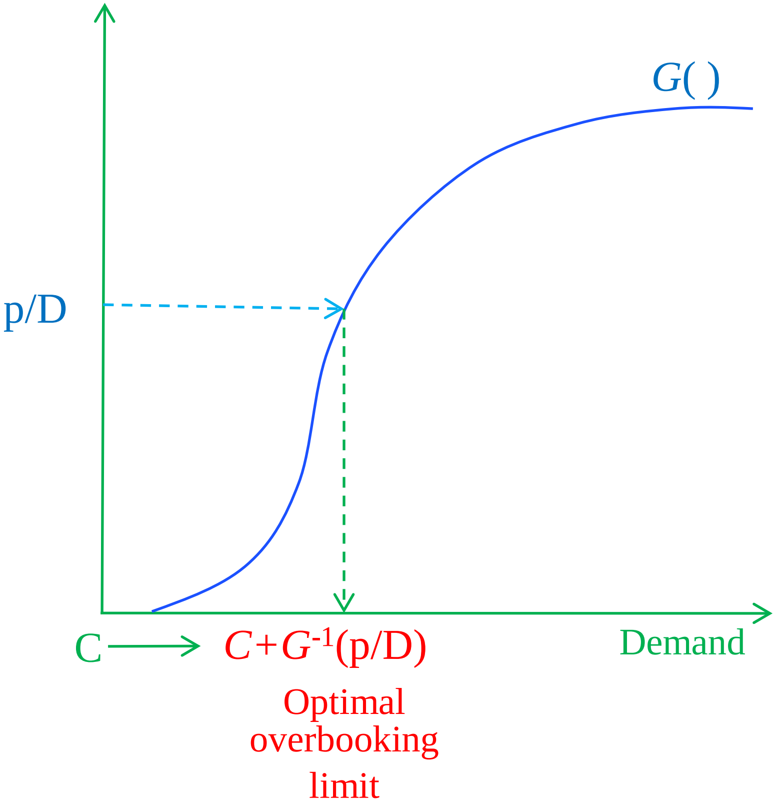
$G(b-C)$ increases as b increases because G is cumulative density and increases in its argument.

For any b, increase b as long as $p/D>G(b-C)$.

For a continuous demand distribution,

b^*-C= $G^{-1}(p/D)$ and $b^* \geq C$.

$$\boxed{b^* = C + G^{-1}(p/D)}$$



Example
Independent Distribution of No-Shows

A MD-80 Delta flight from ATL to PUJ (Punta Cana, Dominican Republic) has a capacity of 120 passengers and fills up quickly for the Thanksgiving holiday. Delta does not want any seat to be wasted on this flight due to no-shows. The ticket costs \$600. In the past, no-shows have ranged uniformly from 6 to 14.

If Delta cannot accommodate a passenger on this flight, the passenger will be sent to Miami on an Delta flight and then to San Juan, PR on an American Airlines flight. The passenger will spend the night at a San Juan airport hotel. The next morning, the passenger will be flown by American to Punta Cana. The cost of these flights to Delta is \$1000. The hotel and food vouchers cost \$100.

According to Department of Transportation (DOT), how much does Delta pay to involuntary denied boarding? Find the cost of denial of service and the optimal overbooking limit.

Delta pays \$400 per ticket according to DOT rules.

Ignoring the ill-will cost, we find D=400+1000+100=1500.

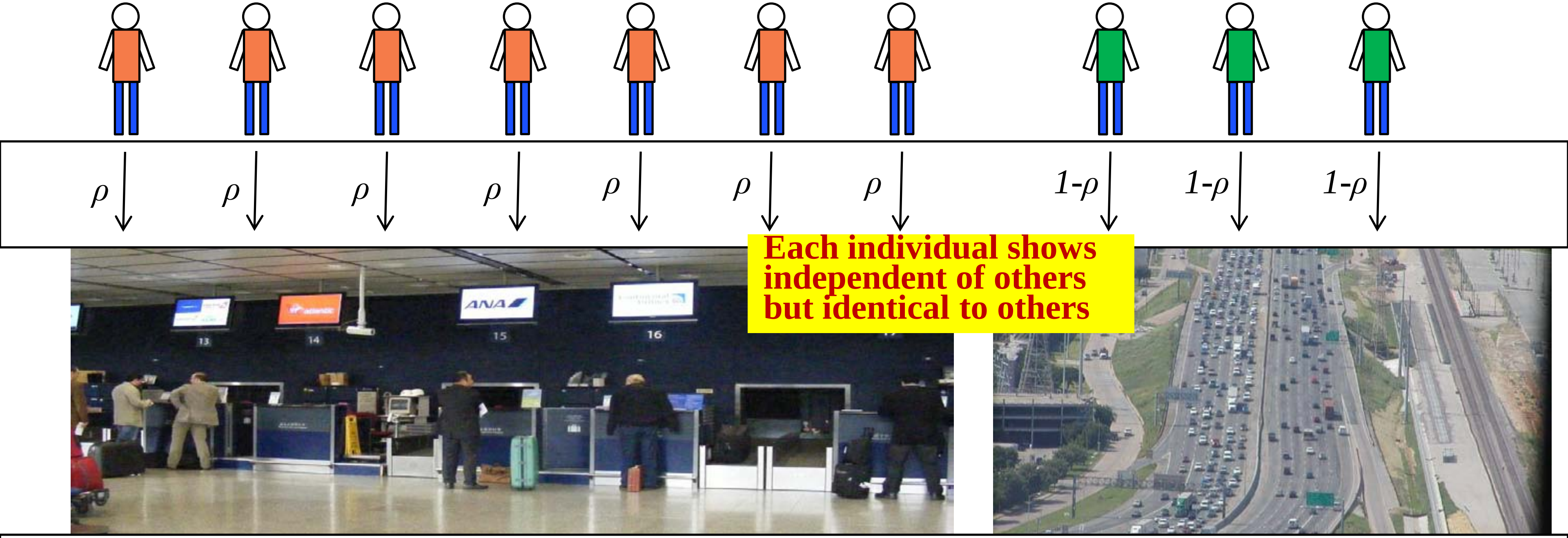
No-show distribution is uniform from 6 to 14, so it is G(x)=(x-6)/8 over this range.

Solving for G(b-C)=(b-C-6)/8=600/1500, we have b-C-6=16/5=3.2.

Rounding 3.2 down to 3, we obtain b=129.

Distribution of Shows and No-Shows

- Number of accepted bookings is $n(b)=\min\{d,b\}$.
- Assume: These people show up independent of each other, each with probability ρ.
- Both the numbers of shows s and no-shows x are Binomially distributed, consider $n(b)=10$:



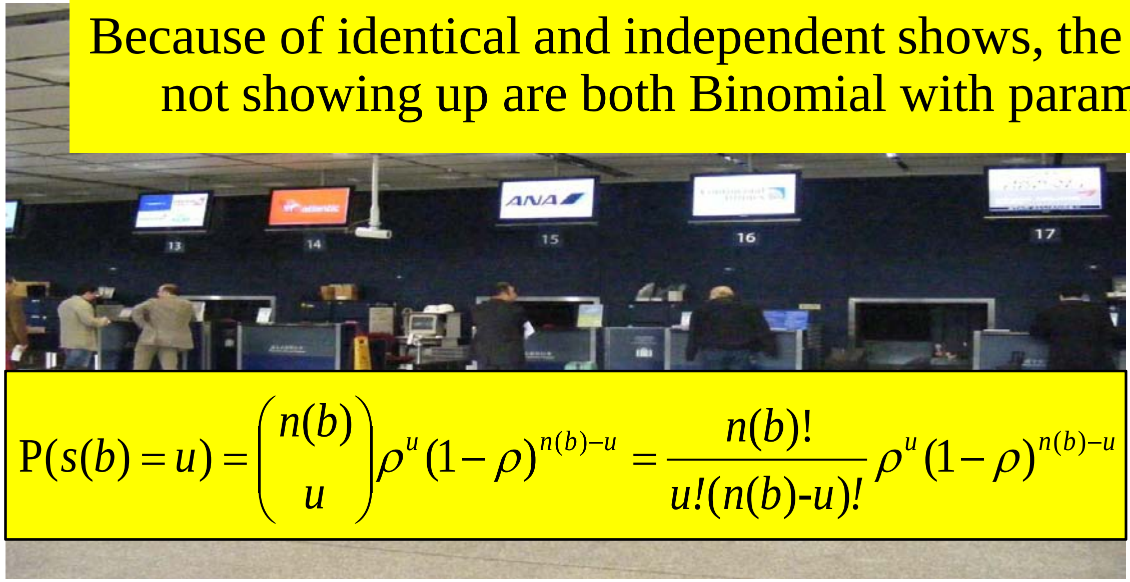
Each individual shows independent of others but identical to others

$s=7$ shows and $x=3$ no-shows, $s+x= n(b)=10$.

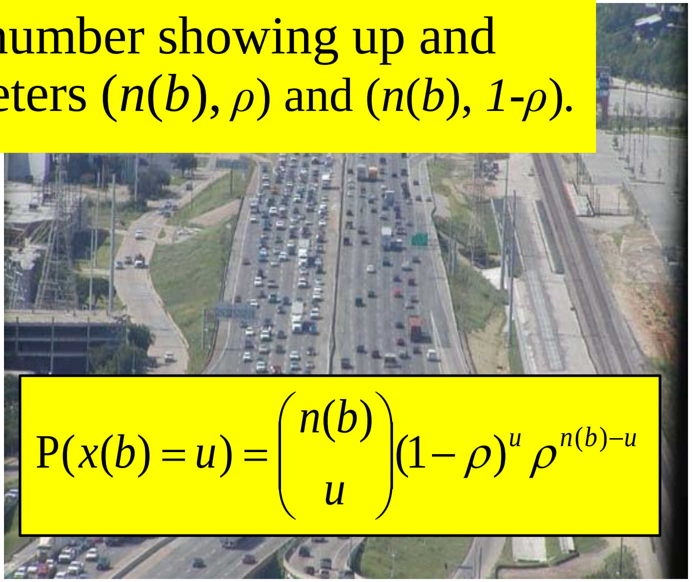
$$\mathrm{P}(s=7)=\binom{10}{7}\rho^{7}(1-\rho)^{3}=\frac{10!}{7!3!}\rho^{7}(1-\rho)^{3} \quad \text{and} \quad \mathrm{P}(x=3)=\binom{10}{3}(1-\rho)^{3}\rho^{7}=\frac{10!}{3!7!}(1-\rho)^{3}\rho^{7}$$

Binomial Distribution of Shows and No-Shows

Because of identical and independent shows, the number showing up and not showing up are both Binomial with parameters ($n(b)$, ρ) and ($n(b)$, $1-\rho$).



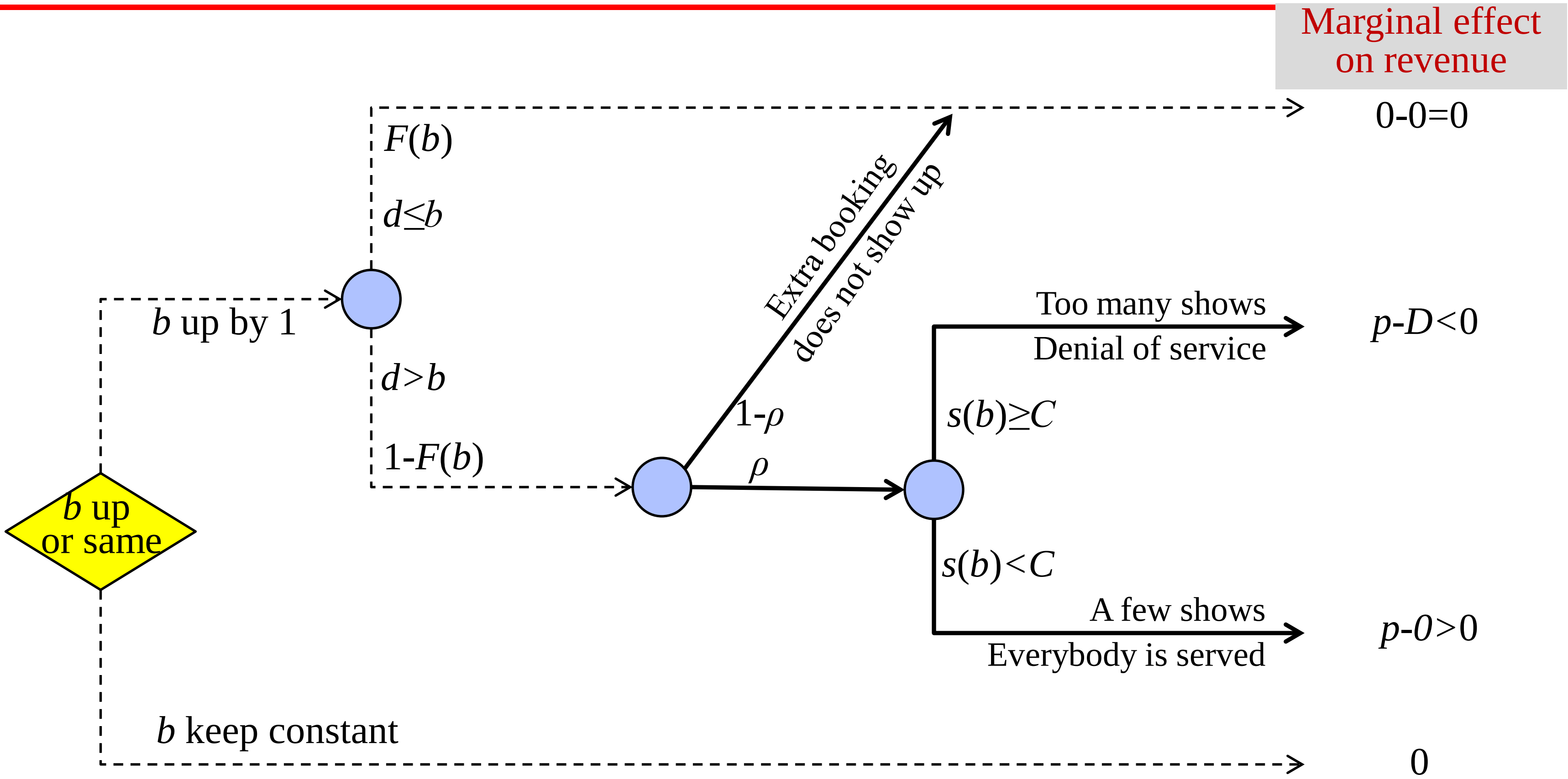
$$P(s(b)=u)=\binom{n(b)}{u}\rho^{u}(1-\rho)^{n(b)-u}=\frac{n(b)!}{u!(n(b)-u)!}\rho^{u}(1-\rho)^{n(b)-u}$$



$$P(x(b)=u)=\binom{n(b)}{u}(1-\rho)^{u}\rho^{n(b)-u}$$

- Both shows s and no-shows x depend on b, so we write $s(b)$ and $x(b)$.
- Since $s(b)+x(b)=n(b)=\min\{b,d\}$, the two events below are the same:
  - $[s(b)\geq C]$ and $[x(b)\leq n(b)-C]$
- When no-shows were independent of bookings, the event $[x(b)\leq n(b)-C]$ became $[x\leq b-C]$.
  - You can see the decision tree branch that corresponds to the event $[x\leq b-C]$.
  - The same branch now will correspond to the event $[s(b)\geq C]$.

Overbooking Limit
Shows Depend on Bookings



Expected marginal effect on revenue

$$=[1-F(b)]\,\rho\left[\mathrm{P}(s(b)\geq C)(p-D)-[1-\mathrm{P}(s(b)\geq C)]p\right]$$

$$=[1-F(b)]\,\rho\left[p-D\mathrm{P}(s(b)\geq C)\right]$$

Computing the Probability of Selling the Entire Capacity $P(s(b) \geq C)$

The critical term is $p\text{-}DP(s(b) \geq C)$. We need to compute $P(s(b) \geq C)$.

For $b=C$, $[s(C) \geq C]$ if there are more than C-1 passengers and all bookings $n(b)=b=C$ show up. Thus,

$$P(s(C) \geq C)=[1- F(C-1)]\, \rho^{C}$$

This can be rewritten as

$$P(s(C) \geq C) = P(s(C-1) \geq C) + [1-F(C-1)]\rho \binom{C-1}{C-1} \rho^{C-1}(1-\rho)^{(C-1)-(C-1)}$$

because $P(s(C-1) \geq C) = 0$ and $\binom{C-1}{C-1} = 1.$

For general $b \geq C$, $[s(b+1) \geq C]$ can be obtained by generalizing the above formula:

$$\begin{aligned} P(s(b+1) \geq C) &= P(s(b) \geq C) + [1-F(b)]\,\rho\, P(s(b) = C-1) \\ &= P(s(b) \geq C) + [1-F(b)]\rho \binom{b}{C-1} \rho^{C-1}(1-\rho)^{b-(C-1)}. \end{aligned}$$

See the aside after summary for details of obtaining this formula.

Computing the Overbooking Limit

The critical term is $p-DP(s(b) \ge C)$. The probability $P(s(b) \ge C)$ increases as we increase b. We can start with $b=C$ and increase b by one when $p/D > P(s(b) \ge C)$.

Since $$P(s(b+1) \ge C) - P(s(b) \ge C) = [1-F(b)]\rho \binom{b}{C-1} \rho^{C-1}(1-\rho)^{b-(C-1)} \le [1-F(C)]\rho$$

$P(s(b) \ge C)$ increases at most by $[1-F(C)]\,\rho$ when we increase b. For example, when the demand is low [1-F(C)] will be low. Similarly, when the show up probability is low, ρ will be low. In these cases, we may never achieve $p/D \le P(s(b) \ge C)$. So we may limit the overbooking limit by a given number (upperlimit), which tends to be twice to thrice of C.

These observations yield the algorithm below.

```
Set $b=C$.
If $b<$upperlimit and $P(s(b) \ge C) < p/D$,
    $b \leftarrow b+1$
```

<See the implementation of this algorithm in overbooking.xlsx>

Service Level and Hybrid Policies

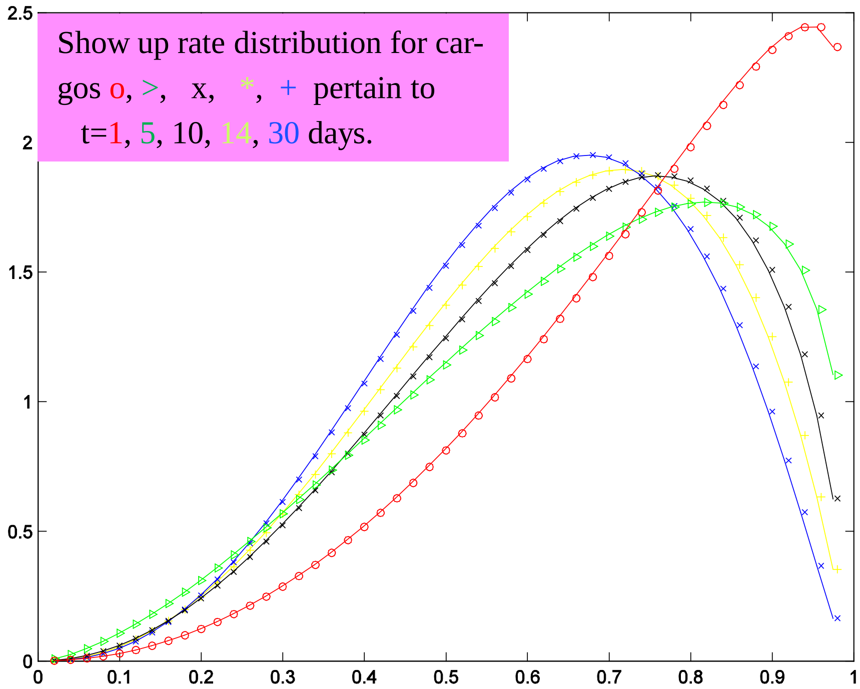
- Service-Level policies can be appropriate especially when D is elusive
  - Implemented at American Airlines and National Car Rental
  - Limit the probability of denial of service
    - Find the largest b such that $P(s(b) \geq C+1)<30\%$.
  - Limit the percentage of customers who are denied of service
    - Find the largest b such that $E[(s(b)-C)+]/E[s(b)]<10\%$.

- Hybrid Policies: Mixture of Risk-Based and Service-Level policies
  - Example: Find the largest b such that

    $P(s(b) \geq C)<p/D$ and $P(s(b) \geq C+1)<30\%$.

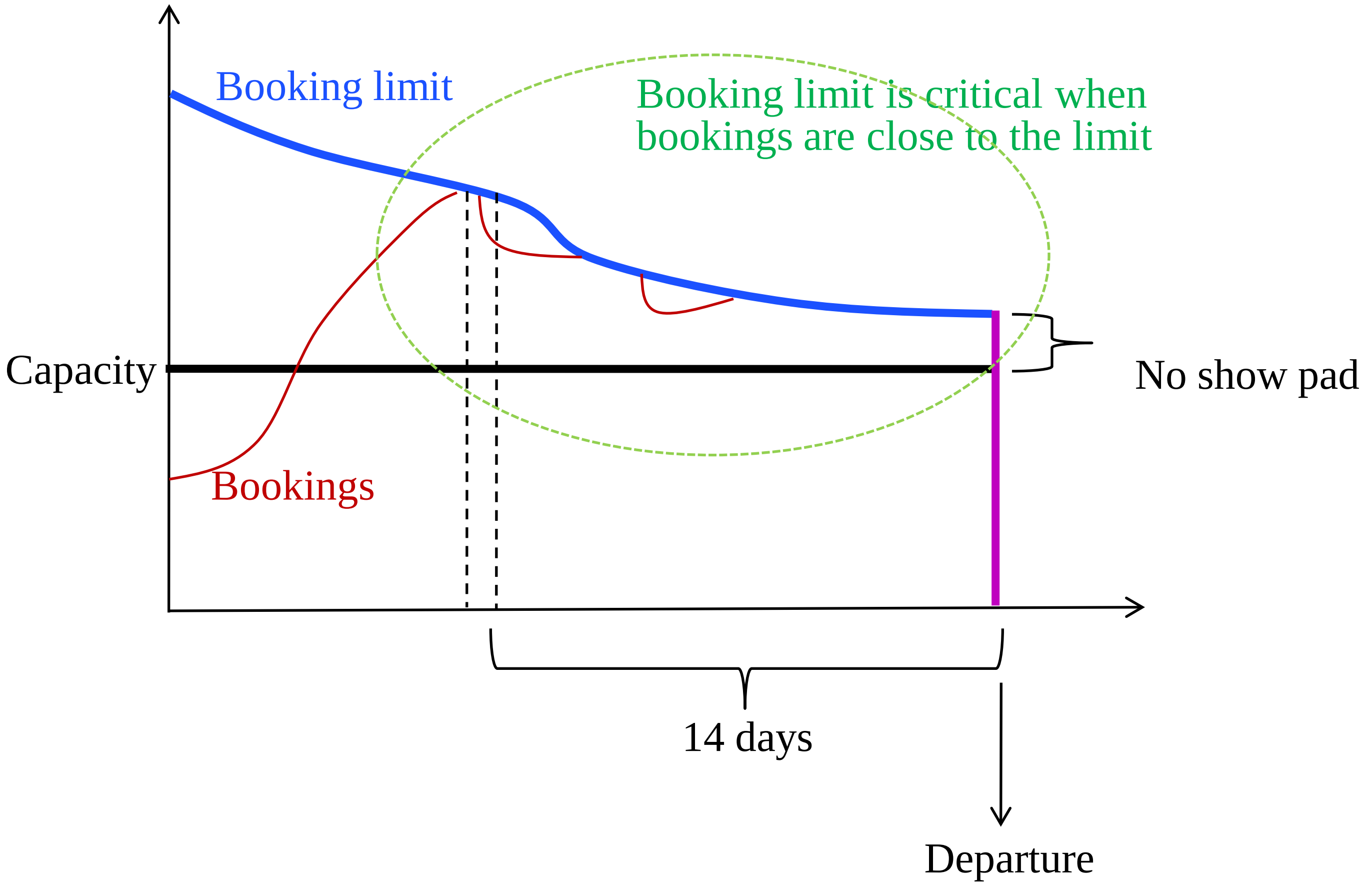
Extension 1: Cancellation

- Some customers cancel their reservation in advance of service delivery (departure or check-in)
  - According to AA 35% of all bookings are cancelled while 15% does not show up
  - More cancellation than no-shows but cancellations are not as critical as no-shows
  - Cancelled reservation can be sold to customers arriving after cancellation but before the service delivery
- Probability of cancellation decreases in the time *t* remaining until the service delivery
  - Not necessarily continuous because expiration of specific can-cancel time windows
  - Some discount and group bookings cannot be cancelled when t<14 days
  - Expect heavy drops in bookings from 15th to 14th day before departure.
  - Try to book a flight on day t=14 if earlier attempts failed.



Show up rate distribution for cargos o, >, x, *, + pertain to t=1, 5, 10, 14, 30 days.

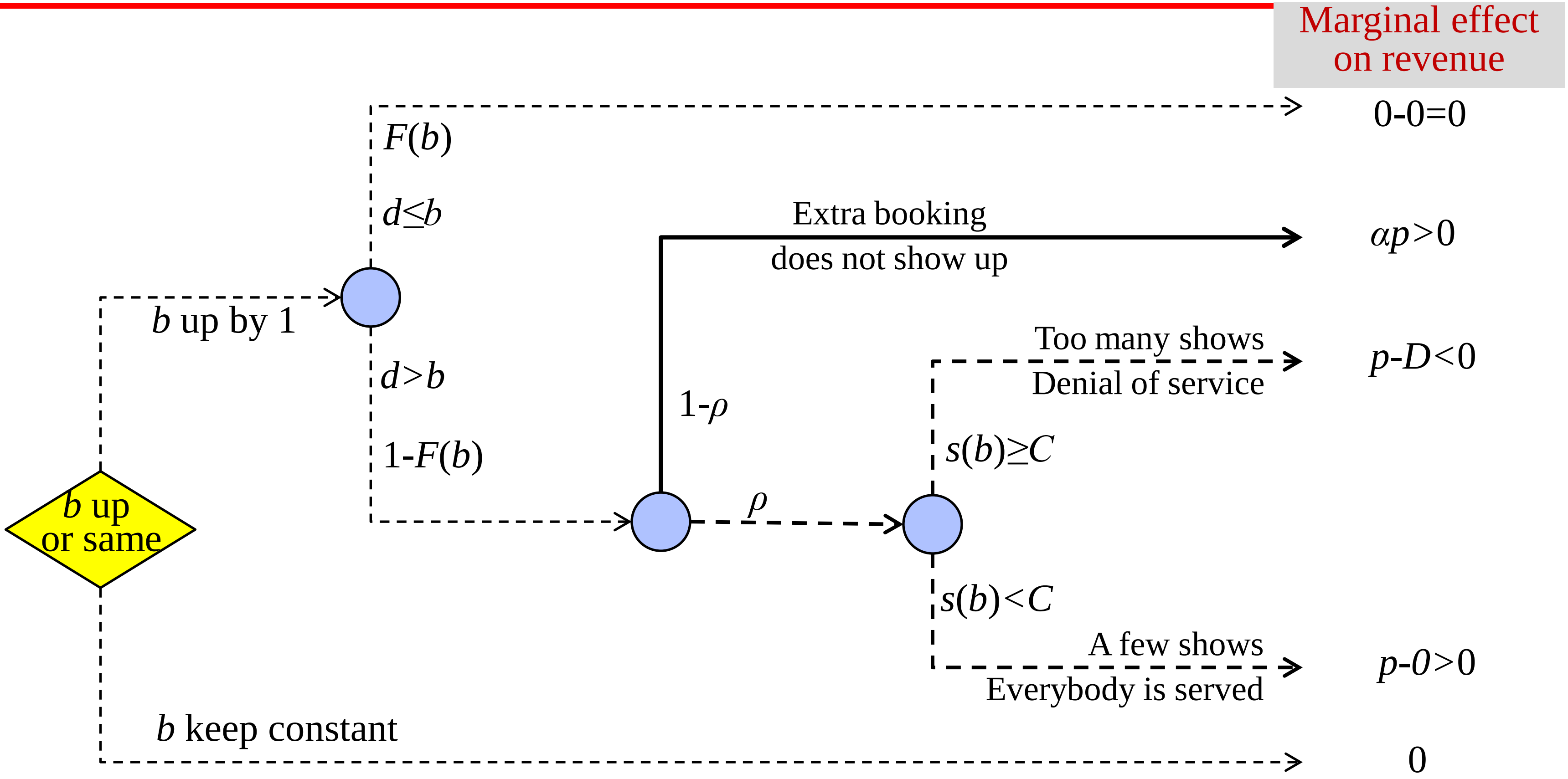
Extension 1: Booking Limit Incorporating Show up rate and Cancellation



Extension 2: Multiple Fare Classes

- In this case, fare classes booking limits must be combined with overbooking limit
- This is quite a challenging optimization problem
- Instead, we separate them and use a heuristic
  - 1. Solve overbooking problem first to find the overbooking limit
    - Aggregate all classes into a single class
  - 2. Solve for protection levels by ignoring the overbooking limit (first step)
  - 3. Solve for the booking limit for each class by setting it equal to the overbooking limit (first step) minus the protection level (second step)

Extension 3: No Show Penalty



Expected marginal effect on revenue	?

Summary

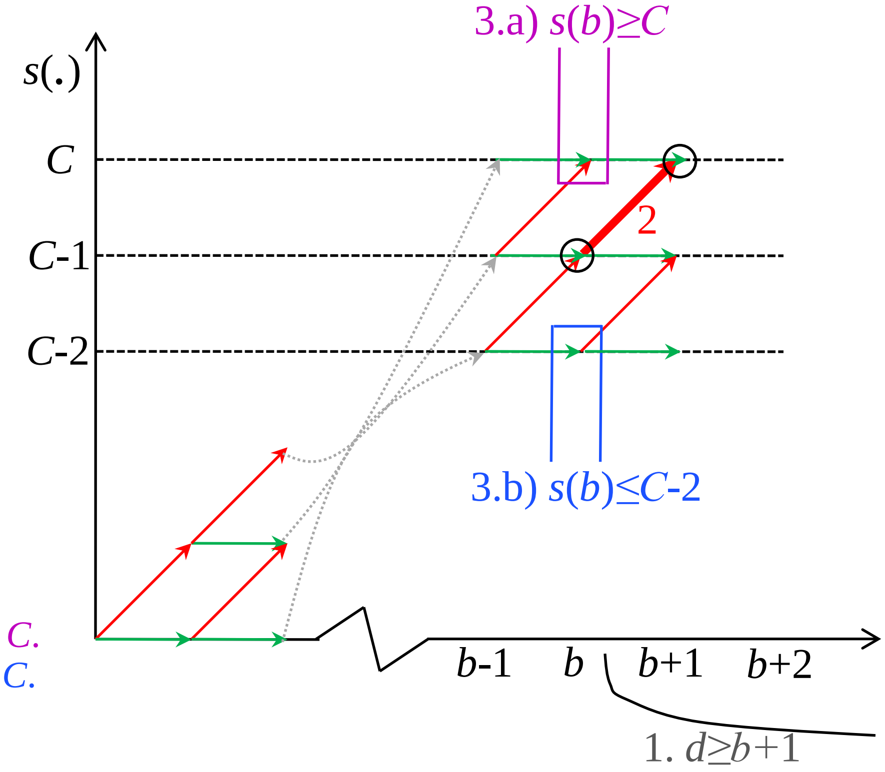
- Overbooking
- Policies
- Extensions

Aside: Computing $P(s(b+1) \geq C)$-$P(s(b) \geq C)$

1. If $d<b+1$, the same number of people book under overbooking limits b and $b+1$. Then the same number of people show up: $s(b)=s(b+1)$. This makes

$$P(s(b+1) \geq C|d<b+1)\text{- } P(s(b) \geq C|d<b+1)=0.$$

We consider only $d \geq b+1$ which happens with probability $1-F(b)$.



2. If $d \geq b+1$, there are b and b+1 bookings made with $s(b)$ and $s(b+1)$. Since each of these bookings show up iid (independent and identically distributed) with probability ρ. Both $s(b)$ and $s(b+1)$ are binomial random variables with parameters (b, ρ) and $(b+1, \rho)$. Moreover,

$$s(b+1) = s(b) + \begin{Bmatrix} 1 \text{ with prob } & \rho \\ 0 \text{ with prob } & 1-\rho \end{Bmatrix}.$$

3. Only when $s(b)=C$-1, $P(s(b+1) \geq C)$- $P(s(b) \geq C)>0$; Otherwise difference is 0:

   3.a) If $s(b) \geq C$, then $s(b+1) \geq C$. So, both $s(b)$ and $s(b+1) > C$.
   3.b) If $s(b) \leq C-2$, then $s(b+1) \leq C-1$. So, both $s(b)$ and $s(b+1) < C$.

Then $P(s(b+1) \geq C)$- $P(s(b) \geq C) = \rho\, P(s(b)=C-1)$.

In summary, only when $d \geq b+1$ and $s(b)=C$-1, and $b+1^{st}$ customer shows up with ρ; then

$$\begin{aligned} P(s(b+1)=C) - P(s(b)=C) &= P(\; d \geq b+1, \quad s(b)=C-1, \quad s(b+1)\text{-}s(b) = 1 \;) \\ &= \quad [1-F(b)] \quad \binom{b}{C-1}\rho^{C-1}(1-\rho)^{b-(C-1)} \quad \rho \end{aligned}$$