Pricing with Constrained Supply

Outline

- **Pricing with a Supply Constraint**
- **Opportunity Cost**
- **Variable Pricing**
- **Variable Prices in Practice**

Based on Phillips (2005) Chapter 5

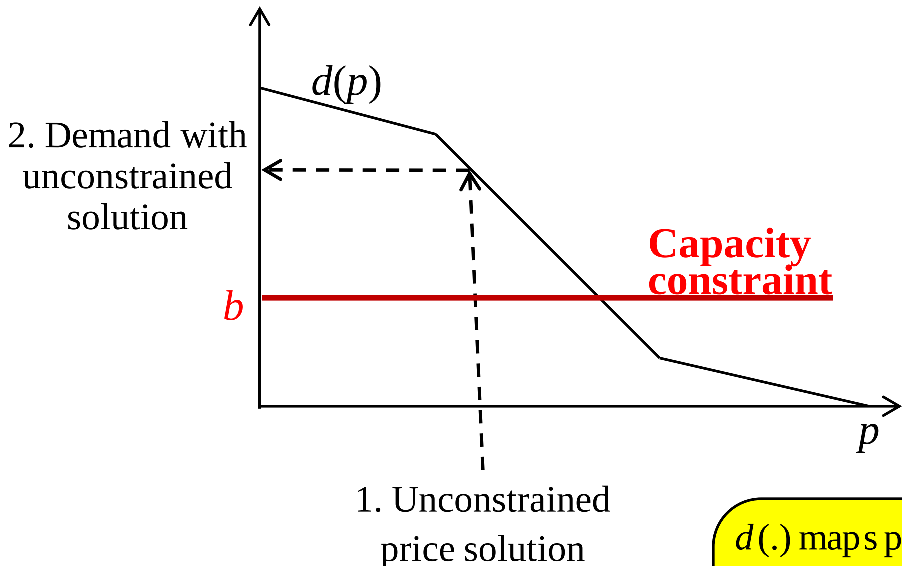
Supply Constraints

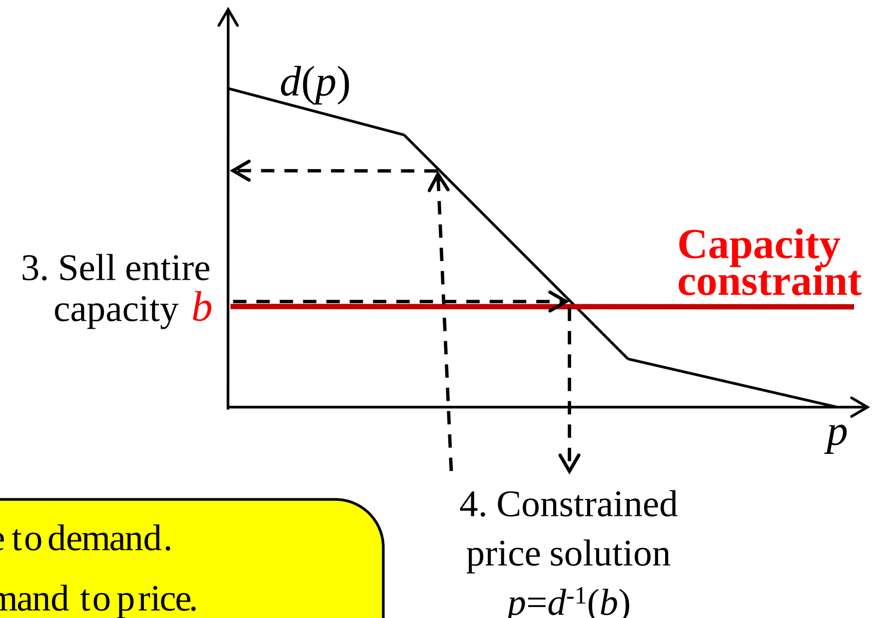
- UTD's Cohort (full-time) MBA has 50 spots
- Eismann Center performance hall has 1563 seats
- Dallas Cowboys Stadium can accommodate up to 111,000 people
- Royal Caribbean's cruise ship Voyager of the Seas departs from Galveston.
  - Passengers: 3114 passengers
  - Inside Cabins: 618
  - Outside Cabins: 939
  - Balconycabins: 757
  - Suites: 119
- Airbus A380 can seat 550 passengers
- During your favorite sit-com (situation comedy), there are 3 advertisement breaks approximately about 6th, 16th and 26th minutes of the show. An average sit-com lasts for 22 minutes without advertisements and 29.5 minutes with advertisements; there is a 30 second long filler for every 30 minutes.
  - 7.5 minutes are available for commercials during a 30 minute sitcom
  - Equally long 3 advertisement pods are used, each pod lasts 150 seconds (2.5 minutes)
  - During 150 seconds 8-12 commercials can be shown
- Yahoo.com has three vertical panels: "My favorites", "Today-news", "Advertisements"
  - Advertisements panel can accommodate 1 big and 1 small advertisement
  - Total advertisement area: 10 cm wide and 10 cm long; it can be split in to smaller pieces

Supply Constraint: *b*

- Constrained pricing problem is

$$\begin{aligned} \max \quad & \Pi(p) = d(p)(p-c) \\ st \quad & d(p) \le b \\ & p \ge 0 \end{aligned}$$





$d(.)$ maps price to demand.

$d^{-1}(.)$ maps demand to price.

If $d(p) = D - mp$ then $d^{-1}(b) = \dfrac{D-b}{m}$.

What is $d(d^{-1}(b))$?

Example: Constrained Supply

- Example: Suppose that $d(p) = 200 - 10p$ if $0 \le p \le 20$
  Find total margin maximizing price when the cost is c=10 and the capacity constraint is b=20.

For the derivative of $(p-c)d'(p)+d(p)=0$, we have $-20p+300=0$.

The root of this equation is 15. The derivative of the total margin is positive from 0 to 15 and negative from 15 to 20. The price p=15 solves the optimality equation but yields the demand of $50>b=20$.

If we sell all of the capacity, we price so that $20=200-10p$. This yields p=18.

Aside: To compute $d^{-1}(b)$, set $b=200-10p$ and solve for p.

$p=20-b/10$, so $d^{-1}(b)=20-b/10$.

$$(p-c)d'(p)+d(p)=0$$

Opportunity Cost: Going from 20 to 40

- With constraint *b*, we find

$$\Pi(b) = \max\{d(p)(p-c):\ d(p) \le b,\ p \ge 0\}$$

- The benefit of more capacity, say by going from b=20 to b=40, is

$$\Pi(b=40) - \Pi(b=20)$$

- Under b=20 in the last example, we obtain a profit of (18-10)(20)=160.
- Under b=40, we still have that the demand of 50 with p=15 violates the capacity.
  - We solve for $p=d^{-1}(b=40)=20-b/10=16$.
  - The profit then is (16-10)(40)=240.
- Opportunity cost of having the capacity of 20 as opposed to 40 is 80:

$$80 = 240 - 160 = \Pi(b=40) - \Pi(b=20).$$

Opportunity Cost: Going from 40 to 50

- Under b=40 in the last example, we obtain a profit of (16-10)(40)=240.
- Under b=50, the demand of 50 with p=15 is within the capacity so the profit is (15-10)(50)=250.
- Opportunity cost of having the capacity of 60 as opposed to 40 is 10:

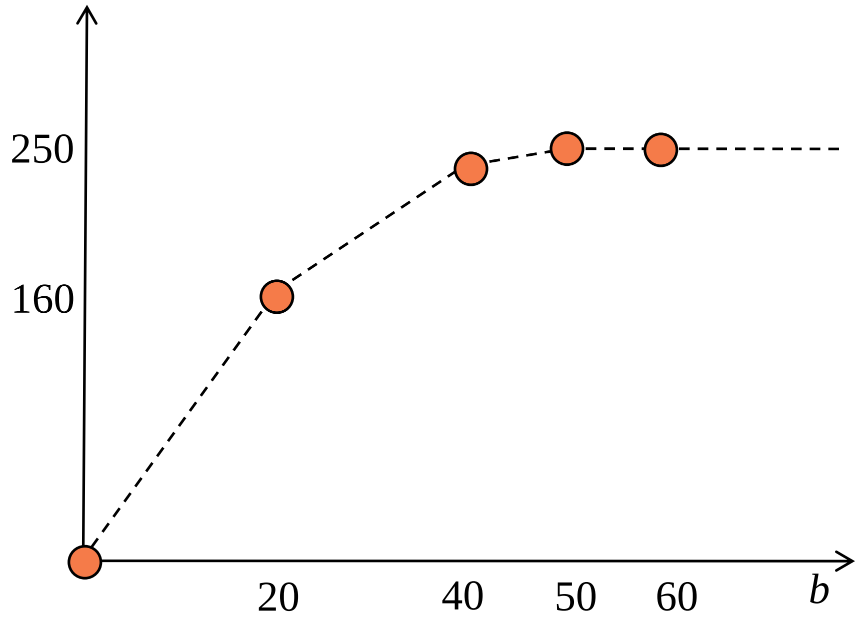
$$\Pi(b=50)-\Pi(b=40)=250-240$$

- Under b=60, the demand of 50 with p=15 is within the capacity so the profit is (15-10)(50)=250.
- Opportunity cost of having the capacity of 60 as opposed to 50 is 0:

$$\Pi(b=60)-\Pi(b=50)=250-250$$

Putting Various Capacities Together
Opportunity Cost

$$\Pi(b=0)=0;\ \Pi(b=20)=160;\ \Pi(b=40)=240;\ \Pi(b=50)=250);\ \Pi(b\geq 60)=250.$$



- The profit curve between the pairs of (capacity, profit) that we have evaluated above is not linear. So the profit is drawn in dots.
- Because of nonlinearity, marginal opportunity cost is not constant.

Linear Demand Curve Marginal Opportunity Cost

- Suppose that the demand curve is linear: $d(p)=D-mp$.
- The unconstrained price p_0 that maximizes revenue is found from the derivative of $\Pi(p)=(p-c)(D-mp)$. The derivative is

$$D+mc-2mp=0 \text{ which yields}$$

$$p_0=\frac{D+mc}{2m}.$$

The demand under this price is

$$d(p_0)=D-m\left(\frac{D+mc}{2m}\right)=\frac{D-mc}{2}.$$

- Marginal opportunity cost of the capacity is zero when the capacity is more than the demand under the price p_0.

$$\frac{\mathrm{d}}{\mathrm{d}b}\Pi(b)=0 \quad \text{for } b\geq d(p_0)$$

- See the opportunity cost in the last example for capacity higher than 50.

Linear Demand Curve

Marginal Opportunity Cost with Insufficient Capacity

- Suppose that the capacity is insufficient so $b<d(p_0)$.
- Set the price equal to $d^{-1}(b)$ to sell the entire capacity.
- The marginal opportunity cost is

$$\frac{\mathrm{d}}{\mathrm{d}b}\Pi(b) = \frac{\mathrm{d}}{\mathrm{d}b} b(d^{-1}(b) - c) \text{ for linear demand curve } d^{-1}(b) = (D-b)/m \text{ so}$$

$$= \frac{\mathrm{d}}{\mathrm{d}b}\left\{ \mathrm{b}\left(\frac{D}{m} - \frac{b}{m} - c\right)\right\}$$

$$= \frac{D}{m} - c - \frac{2b}{m}$$

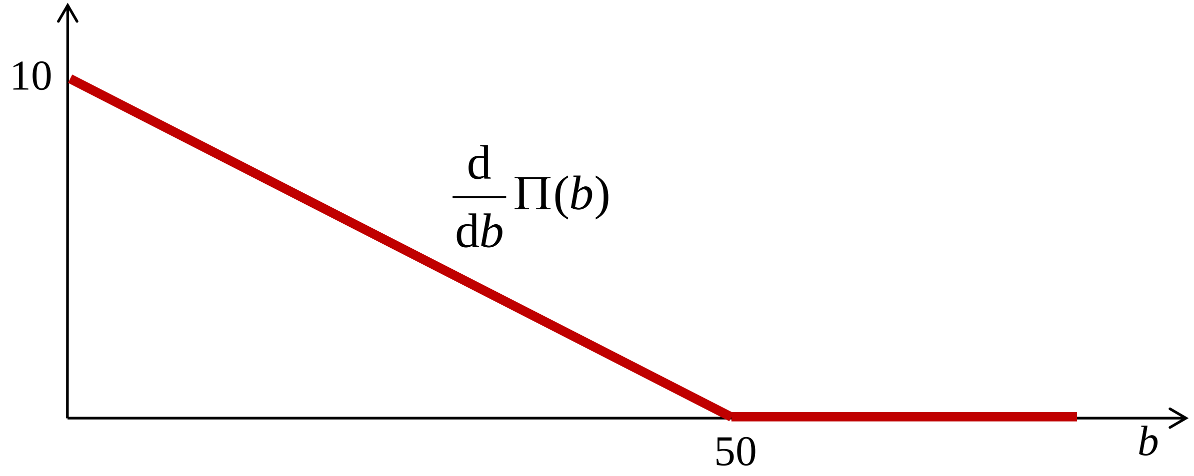
As an exercise check $\left.\frac{\mathrm{d}}{\mathrm{d}b}\Pi(b)\right|_{b=d(p_0)} = 0$, so the marginal cost is continuous.

Example: Linear Demand Curve Marginal Opportunity Cost

- Suppose that the demand is linear $d(p)=D\text{-}mp=200\text{-}10p$ and $c=10$.
- We already computed $p_0=15$ and $d(p_0)=50$.
- Then the marginal opportunity cost is

$$\frac{\mathrm{d}}{\mathrm{d}b}\Pi(b)=\frac{D}{m}-c-\frac{2b}{m}=\frac{200}{10}-10-\frac{2b}{10}=10-\frac{b}{5} \quad \text{for } \mathrm{b}\le 50.$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}b}\Pi(b)\right|_{b=50}=0.$$



Market Segmentation with a Capacity Constraint

- Consider accepting MBA students with and without scholarships where the capacity is 50 for full-time MBA.
  - Scholarships segment the market.
- Consider selling tickets for Eismann Center performances to senior citizens and non-seniors. Seniority segments the market for the performance hall whose capacity is 1563 seats
- Consider selling tickets for Berkeley and Stanford football game to be held at a stadium with 60000 seating capacity. Demand curves for general public and students are given by

$$d_g(p_g) = (120{,}000 - 3{,}000\,p)^+ \quad , \quad d_s(p_s) = (20{,}000 - 1{,}250\,p)^+ .$$

Market Segmentation with a Capacity Constraint
Single Price

- When a single price is charged to both general and student attendees, we have

$$R(p) = p(120{,}000 - 3{,}000p)^+ + p(20{,}000 - 1{,}250p)^+, \text{ whose derivative is}$$

$$R'(p) = \begin{cases} 0 & \text{if } p > 40 \\ 120000 - 6000p & \text{if } 16 < p \le 40 \\ 120000 - 6000p + 20000 - 2500p & \text{if } p \le 16 \end{cases}$$

The only feasible prices that can be solutions are $p = 20$ and $p = 16$.

- Since the stadium is sold out at the price of \$20, the optimal price is \$20.
- The optimal revenue under single-price is 20(60,000)=1,200,000.

Market Segmentation with a Capacity Constraint Formulation with Multiple Prices

- When separate prices are charged to general and student attendees, we have

$$\max \; p_g(120{,}000 - 3{,}000 p_g) + p_s(20{,}000 - 1{,}250 p_s)$$

st

$$120{,}000 - 3{,}000 p_g + 20{,}000 - 1{,}250 p_s \leq 60{,}000$$

$$\text{(or equivalently } 3 p_g + 1.25 p_s \geq 80\text{)}$$

$$p_g \leq 40, \qquad p_s \leq 16$$

- This objective defines a distorted circle in terms of prices

Market Segmentation with a Capacity Constraint Graphical Depiction with Multiple Prices

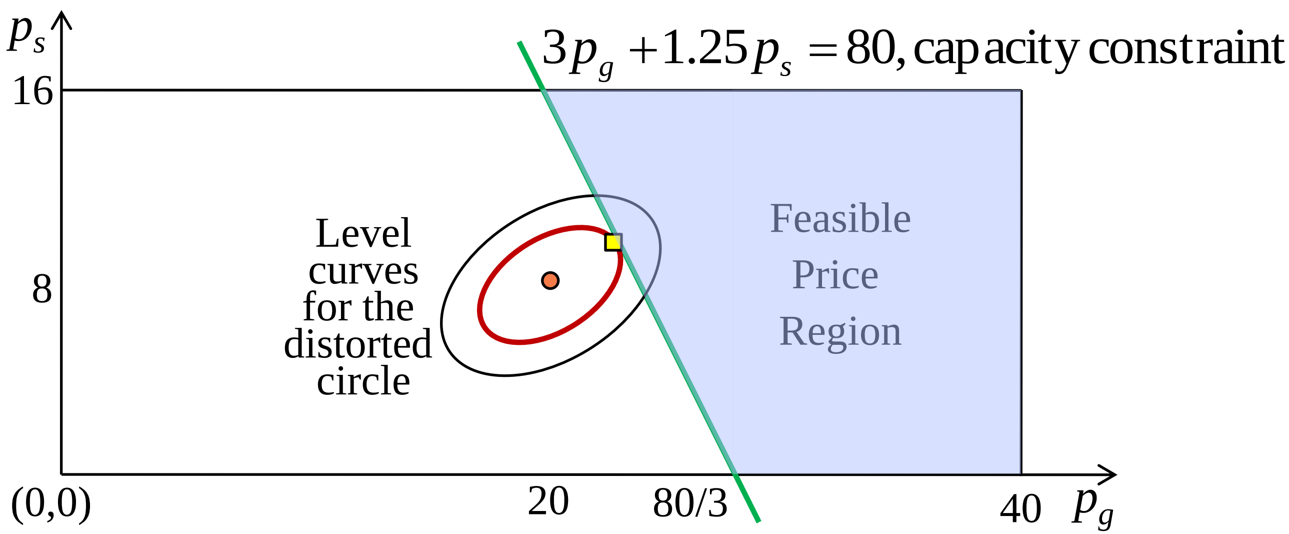
- The maximization objective above is equivalent to the following objectives

$$\equiv \max 120{,}000\, p_g - 3{,}000\, p_g^2 + 20{,}000\, p_s - 1{,}250\, p_s^2$$

$$\equiv \min 3{,}000\, p_g^2 - 120{,}000\, p_g + 1{,}250\, p_s^2 - 20{,}000\, p_s$$

$$\equiv \min 3(p_g - 20)^2 - 3(400) + 1.25(p_s - 8)^2 - 1.25(64)$$

$$\equiv \min 3(p_g - 20)^2 + 1.25(p_s - 8)^2$$



- At the optimal prices, the capacity constraint is tangent to the distorted circle.

Market Segmentation with a Capacity Constraint Solution for an Instance with Multiple Prices

The solution of

$$\begin{aligned} \max\ & p_g(a_g - b_g p_g) + p_s(a_s - b_s p_s) \\ \text{st}\ & a_g - b_g p_g + a_s - b_s p_s \le C \\ & p_g \le a_g/b_g, \quad p_s \le a_s/b_s \end{aligned}$$

If $a_g + a_s > C$, i.e., capacity is binding, then

$$p_g^* = \frac{\{2a_g + a_s - 2C + a_g b_s / b_g\}}{2(b_g + b_s)} \qquad p_s^* = \frac{\{2a_s + a_g - 2C + a_s b_g / b_s\}}{2(b_s + b_g)}$$

If $a_g + a_s \le C$, i.e., capacity is non-binding, then

$$p_g^* = \frac{a_g}{2b_g} \qquad p_s^* = \frac{a_s}{2b_s}$$

For details see constrainedEx.pdf

Market Segmentation with a Capacity Constraint
Algebraic Solution with Multiple Prices

The solution of

$$\max \; p_g(120{,}000 - 3{,}000\,p_g) + p_s(20{,}000 - 1{,}250\,p_s)$$

$$\text{st} \quad 120{,}000 - 3{,}000\,p_g + 20{,}000 - 1{,}250\,p_s \le 60{,}000$$

$$p_g \le 40, \qquad p_s \le 16$$

Since $a_g + a_s = 120 + 20 > 60 = C$, i.e., capacity is binding, so

$$p_g^* = \frac{\{2*120 + 20 - 2*60 + 120*1.25/3\}}{2(3+1.25)} = 22.35$$

$$p_s^* = \frac{\{2*20 + 120 - 2*60 + 20*3/1.25\}}{2(1.25+3)} = 10.35$$

Variable Pricing

- The prices for
  - Theme park tickets vary over a week
  - Movie theater rickets vary over a week
  - Sport events vary over a season
  - Airline tickets vary over a season
  - Electricity vary over time of the day
  - Phone calls vary over time of the day

Variable Pricing
Single Price

- Index periods (time, day, week etc.) by i
- Demand in period i is $D_i - m_i p$
- We apply the same price p everyday
- The only decision variable is the price p
- As a consequence of this variable, we obtain demand for period i, which is
  - » $x_i = \text{Min}\{D_i - m_i p, C\}$

where C is the capacity.

$$\max \quad p \sum_i x_i$$

$$\text{st}$$

$$x_i \leq D_i - mp \quad \text{for period } i$$

$$x_i \leq \text{C} \quad \text{for period } i$$

$$p \geq 0$$

Variable Pricing
Multiple Prices

- Price charged in period *i* is denoted by p_i
- If $p_i < p_j$, some customers switch from period *j* to period *i*
- How many customers switch?
  - First suppose that the number of switching customers does not depend on either *i* or *j*
  - Suppose that *s* customers switch for each dollar of price difference
  - This number *s* is difficult to find out
- Given a price for each period, the demand for period becomes

$$d_i \leq D_i - mp_i + s\sum_j (p_j - p_i) \quad \text{for period } i$$

- Then the price optimization problem is

$$\max \sum_i p_i x_i$$

$$\text{st}$$

$$x_i \leq D_i - mp_i + s\sum_j (p_j - p_i) \quad \text{for period } i$$

$$x_i \leq C \quad \text{for period } i$$

$$p_i \geq 0$$

Variable Pricing
Revenue of Multiple Prices > Revenue of Single Price

- The multiple price formulation can be reduced to the single price formulation by inserting the following set of constraints

$$p_i = p \text{ for period } i$$

- Inserting this constraint makes the objective value worse.
- In other words, revenue can be made higher with multiple prices.

Sporting Events
Variable Prices in Practice

- Soccer leagues
  - Big four in English Premier League: Liverpool (shipyard workers), Chelsea (bourgeoisie), Arsenal (alternative), Manchester United (middle-class, higher Brits)
  - Big four in Turkish Super League: Trabzon (seamen, fishermen), Fener (bourgeoisie), Besiktas (alternative; read "*The View from the Stands*" by E. Batuman, The New Yorker, March 7, 2011), Galata (middle-class, higher Turks)
  - Derbies are games among big four and their ticket prices are substantially higher
  - Generally, the championship is won by one of the four teams.

- National Basketball League (NBA)
  - What are the premier teams: LA Lakers, Boston Celtics?
  - Premier teams are not easy to identify
  - Not much of a derby game concept
    Except for Lakers – Celtics rivalry.
- Baseball leagues
  - Outdoor games in the spring (cold), summer (nice), fall (cold)
  - Charge higher for summer games
  - No dynamic pricing until 2010 season.
- Football league
  - Dallas Cowboys new arena – no variable prices
    Cheapest ticket $75 vs Atlanta Falcons on Oct 25; vs Washington Redskins on Nov 22; vs. Philly Eagles on Jan 3
- Variable pricing increase revenue per game

Baseball League - Update in 2010
Variable Prices at San Francisco Giants

- San Francisco Giants experienced with dynamic pricing in 2009 season
- They implemented it in 2010 season; revenue up by 6% with similar attendance figures
- Ticket price depends on
  - Data from the secondary ticket market (ticket agencies)
  - Status of the pennant race
  - Success of Giants on the field
  - Opposing team, higher prices with
    - Historic rivalry: Giants versus Dodgers
    - Pennant (e.g., National League West title) contenders
    - Rarely seen inter-league teams (such as Yankees)
  - Pitching match-ups
  - Day of the week
  - Weather forecast

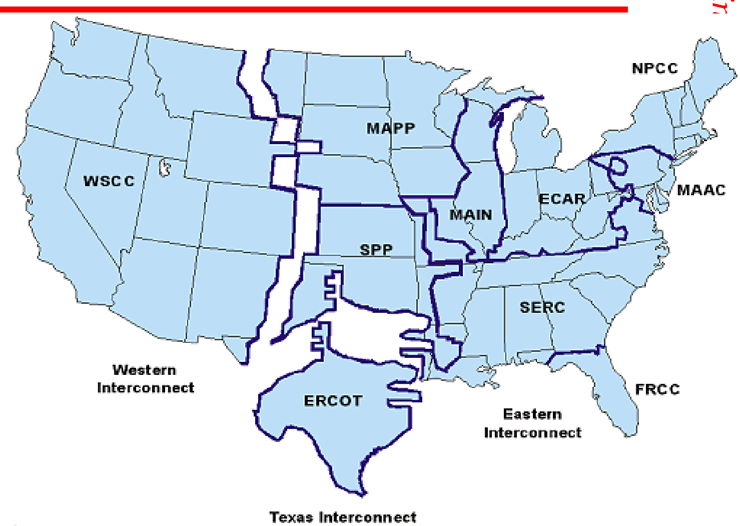
- Example ticket prices for Giants vs. San Diego Padres game on Oct 1
  - Left-field upper deck stand: $5 at the start of the season; $5.75 on Aug 1, 2010; $20 after Giants and Padres become contenders of NL West title
  - Field Club behind home plate: $68 at the start of the season; $92 on Aug 1, 2010; $121 on Aug 9, 2010; $145 on Sep 4, 2010; **$175** before the game.
  - Prices went up as Giants were competing to advance to playoffs first time since 2003.
    - See Page 4 of *ORMS Today* , October 2010 Issue, Vol.37, No.5.

Passenger Airlines
Variable Prices in Practice

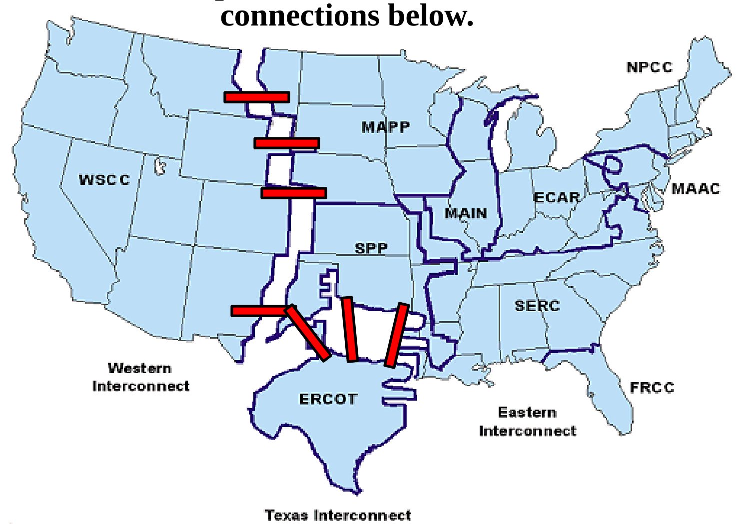
- Southwest (discount) airlines
  - Limited customer segmentation
  - If demand for the next weekend's morning flight to Chicago-Midway is high, increase price.
- Other airlines (full-service) airlines
  - Customer segmentation
  - Take coach class and split into fare classes
  - Some classes are more expensive than others
  - If demand is high, close low-fare classes

Electric Power
Variable Prices in Practice

- Peak electricity demand at 6 pm
- Off-peak electricity demand at 4 am
- Power generation from coal, nuclear, etc.
  - Big generators are on throughout the day, smaller ones are turned on during peak periods
  - Smaller generators are not efficient so their electricity is more costly
  - A 5% reduction in electricity production during peak period reduces marginal cost by 55%
- Reduce the demand during the peak period by increasing the price
- Adjusting price with demand is not fully implemented
  - Historical reasons: Electric utilities were monopolies in local markets
  - Political reasons: Power companies generated little electricity to keep prices high in California. Then governor Davis was recalled and Schwarzenegger replaced him.
  - US electricity grid is not integrated – east , west and Texas. ERCOT: Electric Reliability Council of Texas manages the flow of electric to 22 million Texas customers - representing 85 percent of the state's electric load and 75 percent of the land area.
- Isolated applications in US: Recallable capacity in Wisconsin and Texas.



Grid in 2010 above; Proposal with AC-DC-AC connections below.



Television Advertising
Variable Prices in Practice

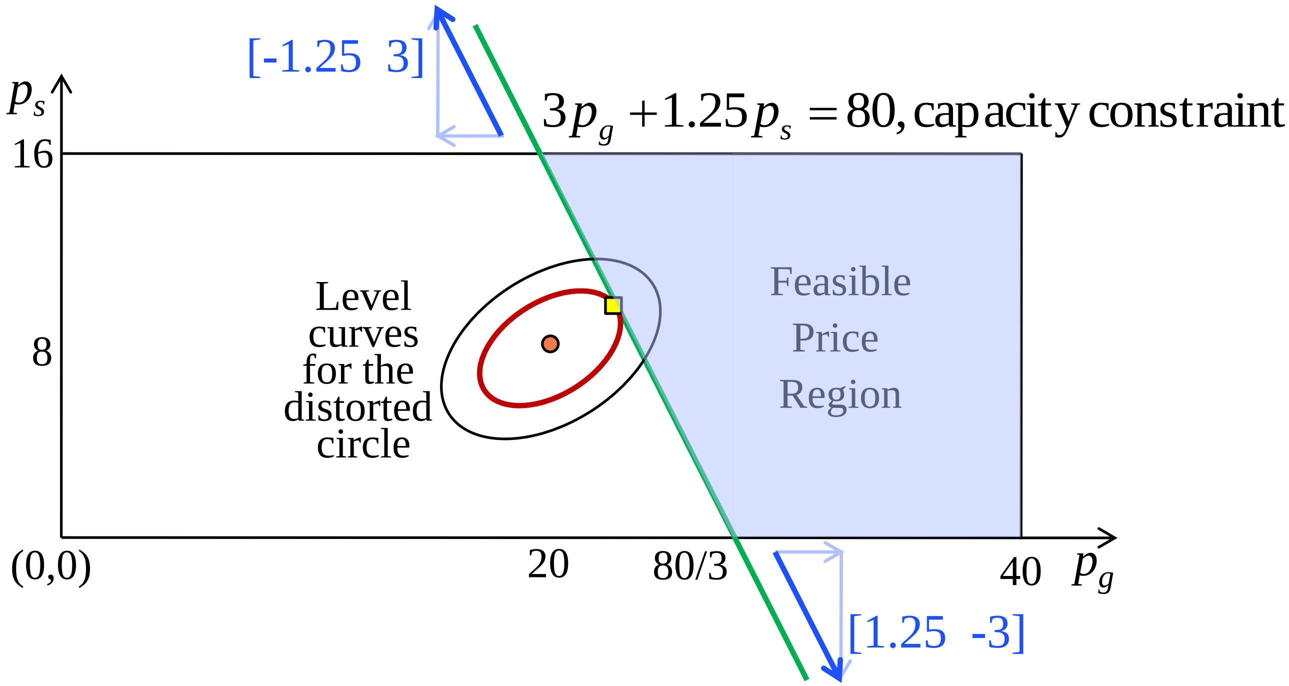
- During a 30 min program, 3 pods of advertisements are shown. A pod lasts 150 secs.
- Pods during prime time are more demanded than those during insomnia time
- Advertisement buyers require a certain number of viewers for pods
  - Program ratings are important
- Upfront market
  - For Fall, it lasts 1-2 weeks in May or June after the announcement of the Fall schedule
  - Broadcaster guarantee of a number of viewers
  - To fulfill the guarantee, broadcasters run additional advertisements (makeups)
  - Broadcaster do not charge extra when there are more viewers than guaranteed
  - How to fulfill the guarantee without overshooting it?
    - Issue: Number of viewers is not known in advance
- Scatter market
  - Scatter market for Fall happens in Fall
  - Scatter sales are cheaper and have no guarantees for the number of viewers
- Constraints: No competing advertisements in the same pod; Not more than a certain number of advertisements for the same product in a single day.
- Broadcasters have advertisement rate sheets (catalog prices) but give big discounts (pocket price is much less).

Pricing with Constrained Supply

Summary

- **Pricing with a Supply Constraint**
- **Opportunity Cost**
  - **With and without market segmentation**
- **Variable Pricing**
- **Variable Prices in Practice**
  - **Sport events; Airlines; Electricity Markets; Advertising**

Market Segmentation with a Capacity Constraint Graphical Depiction with Multiple Prices Solution Method



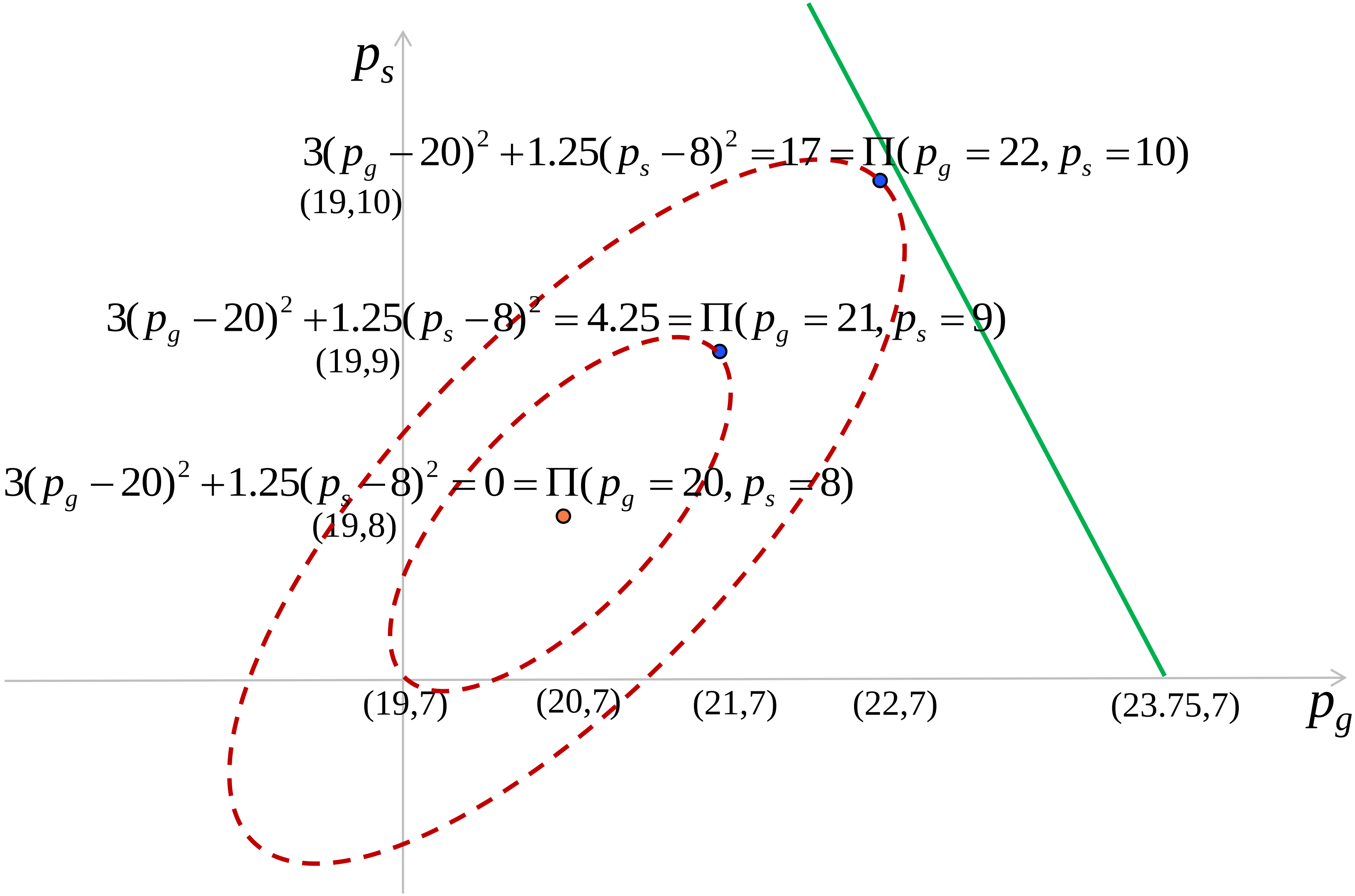
- At the optimal prices, the capacity constraint is tangent to the distorted circle.
- If we treat student price as y-variable and the general price as x-variable, the slope of the capacity constraint is

$$\frac{dp_s}{dp_g} = -\frac{3}{1.25}$$

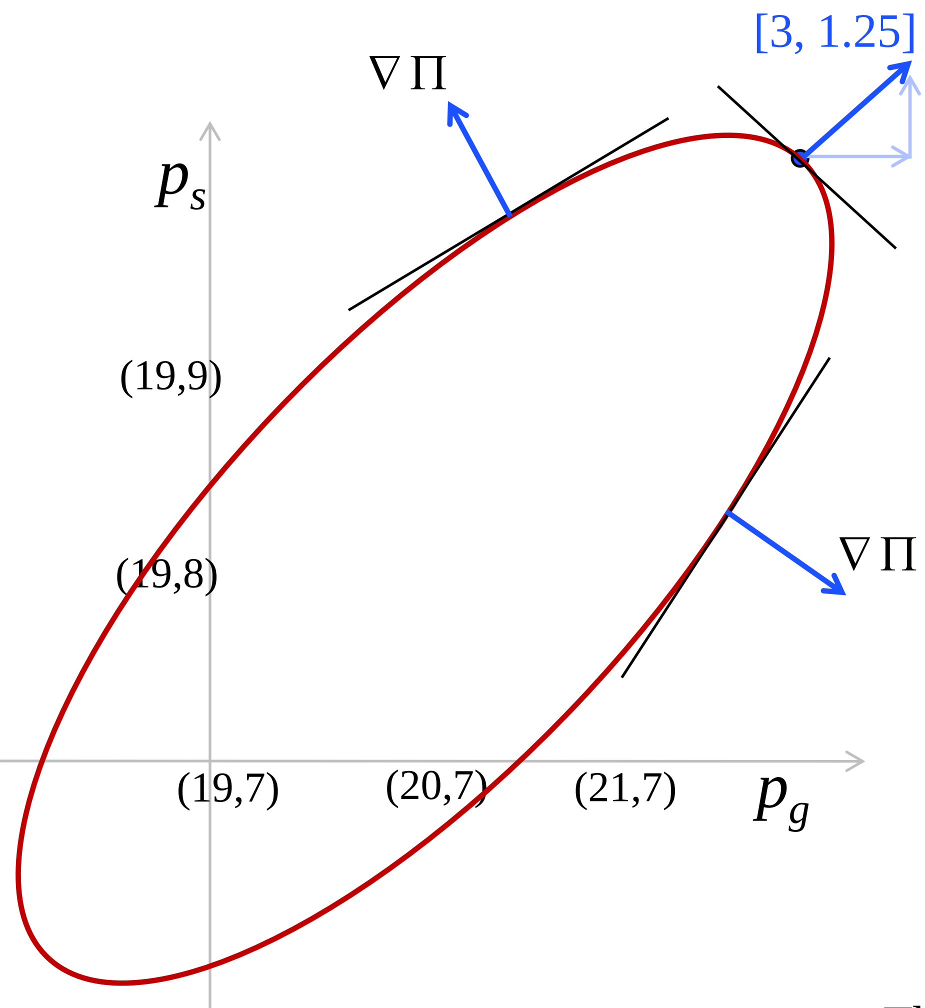
- This slope can be represented by either one of the following vectors:

[-1.25 3] or [1.25 -3]

Level Curves



Normal Vectors to Level Curves



$$3(p_g-20)^2+1.25(p_s-8)^2=17$$

Gradient vector of a curve is normal (perpendicular) to the curve. Given a curve

$$f(p_1, p_2, \ldots, p_n) = \text{Constant},$$

the associated gradient is

$$\nabla f = \left[\frac{\partial f}{\partial p_1}, \frac{\partial f}{\partial p_2}, \ldots, \frac{\partial f}{\partial p_n}\right]$$

Given the curve for the distorted circle

$$\Pi(p_g, p_s) = 3(p_g-20)^2 + 1.25(p_s-8)^2 = 17$$

the associated gradient is

$$\nabla\Pi = [3(2)(p_g-20),\ 1.25(2)(p_s-8)]$$

For example, at (22,10) the gradient is

$$\nabla\Pi = [3(2)(22-20),\ 1.25(2)(10-8)] = 4[3,\ 1.25]$$

The gradient at (21,9) is parallel because it is

$$\nabla\Pi = [3(2)(21-20),\ 1.25(2)(9-8)] = 2[3,\ 1.25]$$

Market Segmentation with a Capacity Constraint

Obtaining the Optimality Condition with Multiple Prices

- Two vectors are parallel if they are a positive multiple of each other.
- Two vectors are perpendicular if their scalar product is zero

$$[3, 6][0; 3] = 18 \neq 0$$

$$[3, 6][-1; 0] = -3 \neq 0$$

$$[3, 6][-2; 1] = 0$$

$$[3, 6][4; -2] = 0$$

$$[3, 1.25][1.25; -3] = 0$$

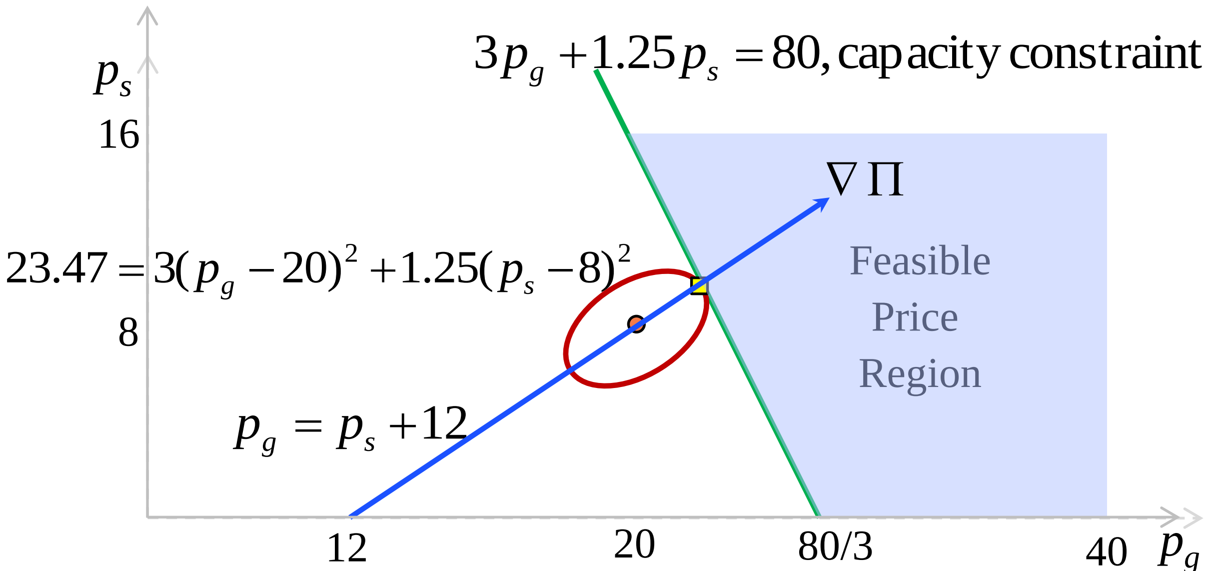
- Only the three pairs of vectors are perpendicular above.

- We want the gradient of the objective to be perpendicular to the vector representing the slope of the capacity constraint:

$$[3(2)(p_g - 20),\ 1.25(2)(p_s - 8)][1.25; -3] = 3(2)(1.25)(p_g - 20 - (p_s - 8)) = 0$$

$$\text{So } \ p_g = p_s + 12.$$

Market Segmentation with a Capacity Constraint Solution with Multiple Prices



- We intersect two lines

$$3p_g + 1.25p_s = 80 \quad \text{and} \quad p_g = p_s + 12$$

to find optimal prices p_g=22.35 and p_s=10.35. Then $\Pi(p_g=22.35, p_s=10.35) = 4.25(2.35)^2 = 23.47$.

- 52,941 tickets are sold to general public and 7,059 tickets are sold to students.
- The corresponding revenue is $1,256,471; 4.7% more than the revenue under single price