Basic Price Optimization

Outline

- Price-Response Function
- Competition
- Basic Price Optimization

Based on Phillips (2005) Chapter 3

Commodities take prices Customize to set prices

- A commodity is a standard product.
- It has a few/no features that can be used to distinguish it from others.
  - Commodities: Electricity, Drinking Water, Wheat, Bauxite (used in Aluminum production)
  - Electricity is the best commodity example as others can be classified
    - » Drinking water by its calcium/vitamin/callory content (hardness)
    - » Wheat by seed type
    - » Bauxite by purity
- Supplier of a commodity can dictate no price different than the market price as a different price can not be justified via additional features.
- Commodity suppliers are price takers.
  - Commodity prices are determined by complex market dynamics that can not be controlled even if they can be modeled.
  - Commodity prices cannot be optimized but can be estimated.
- A commodity ceases to be a commodity if customized.
  - Bottled drinking water is not a commodity. Customization by origin, hardness, vitamin levels, callory, taste, blurring differences between drinking water and soft drinks.

Commodity Example: Corn Prices per Bushel

- Corn price depends on demand and supply.
- Which of the following factors below affect the corn supply?

For high corn yield: Cool weather with ample rainfall is ideal.

US Government: "… there are concerns the government will decline a request to increase the amount of ethanol blended in gasoline to 15% from 10%."

John Kleist, broker/analyst for Allendale, a brokerage firm in McHenry, Illionis

Other Governments: China will release its reserve of 2 million ton corn.

China National Grain and Oils Information Center

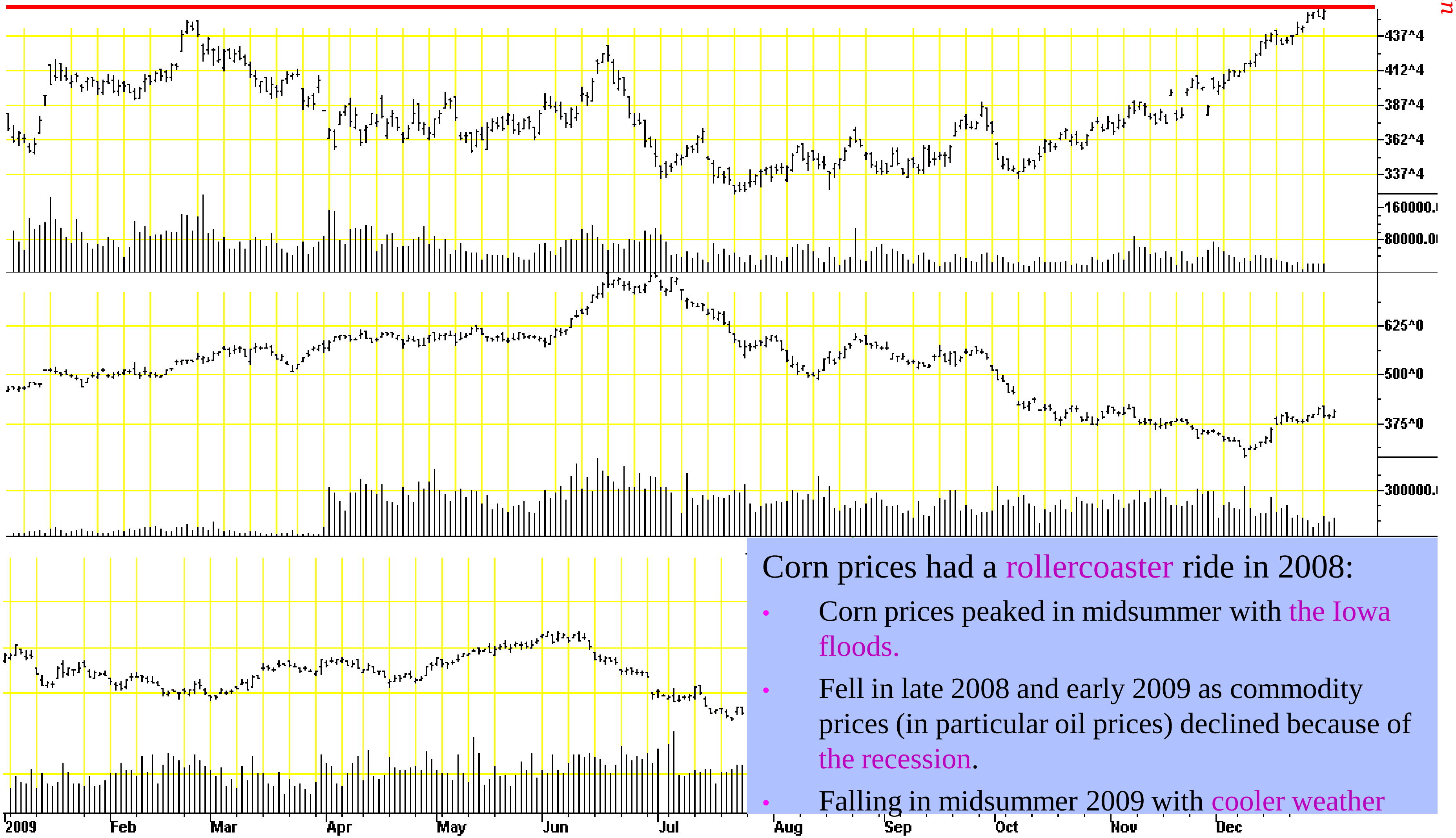
Yield: "… current models project a yield of 161.9 bushels an acre, up from the 2004 record of 160.4 bushels an acre and well above the government's most recent projection of 153.4 bushels."

Darrel Good, University of Illinois Agricultural Economist

Weather: "… sunspot cycles, the onset of El Nino and arctic air-flow patterns have combined to create the prolonged cool conditions in the U.S. corn belt [from western Ohio to eastern Nebraska and northeastern Kansas]. This cooler "bias" could remain for at least another four to six weeks. But too much cool weather could keep crop development slow, which could prompt a loss in yield if the crop hasn't finished growing by the time the first frost arrives."

Drew Lerner, Owner of World Weather Inc.,

Corn Prices:
Cents per bushel over 2007 – 2008, half of 2009

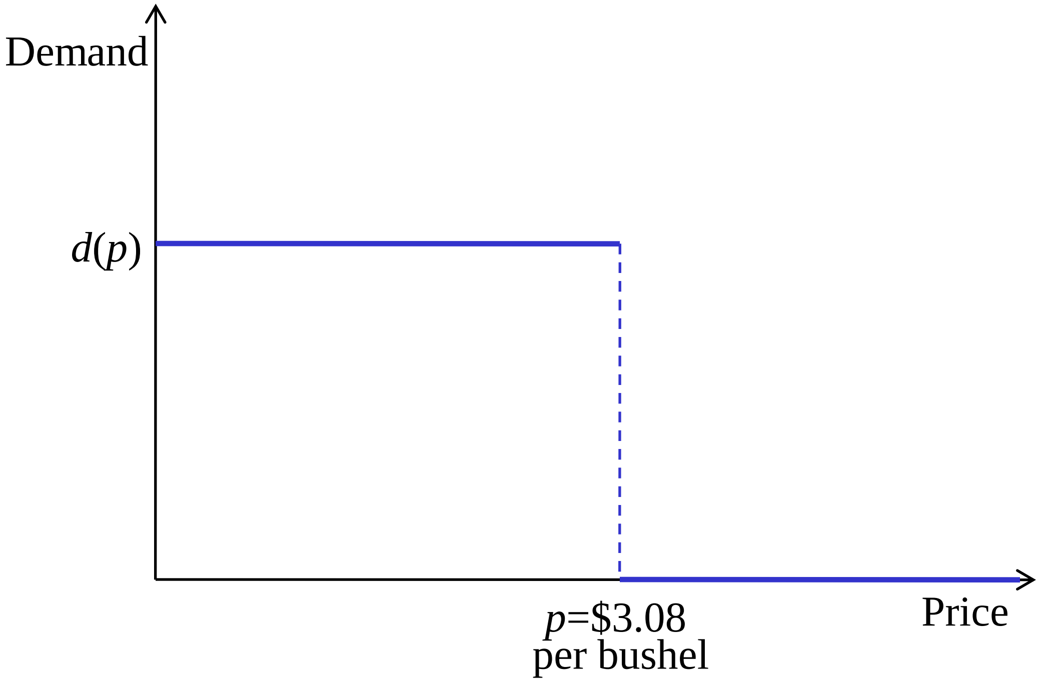


Corn prices had a rollercoaster ride in 2008:

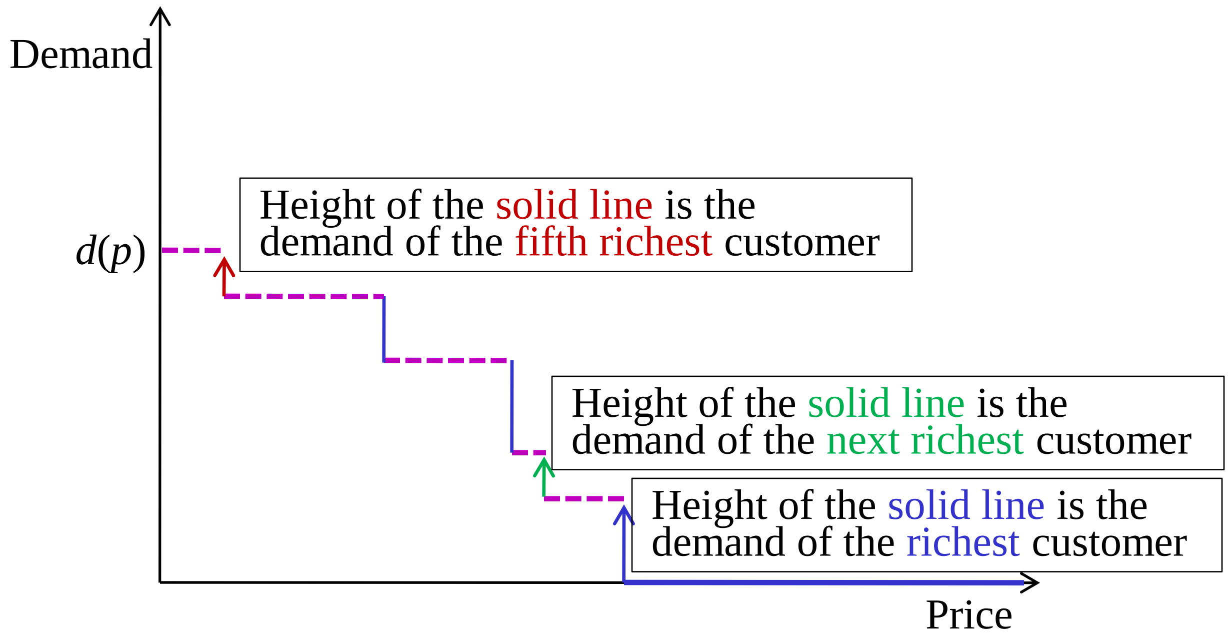
- Corn prices peaked in midsummer with the Iowa floods.
- Fell in late 2008 and early 2009 as commodity prices (in particular oil prices) declined because of the recession.
- Falling in midsummer 2009 with cooler weather

Price Response Function for Corn

- The September 2009 delivery corn contract settled at $3.08 a bushel on the Chicago Board of Trade on July 23, 2009.
- What is the corn demand for a Nebraska farmer if he prices his corn at
  - $3.09 per bushel
  - $3.07 per bushel
- All or none.



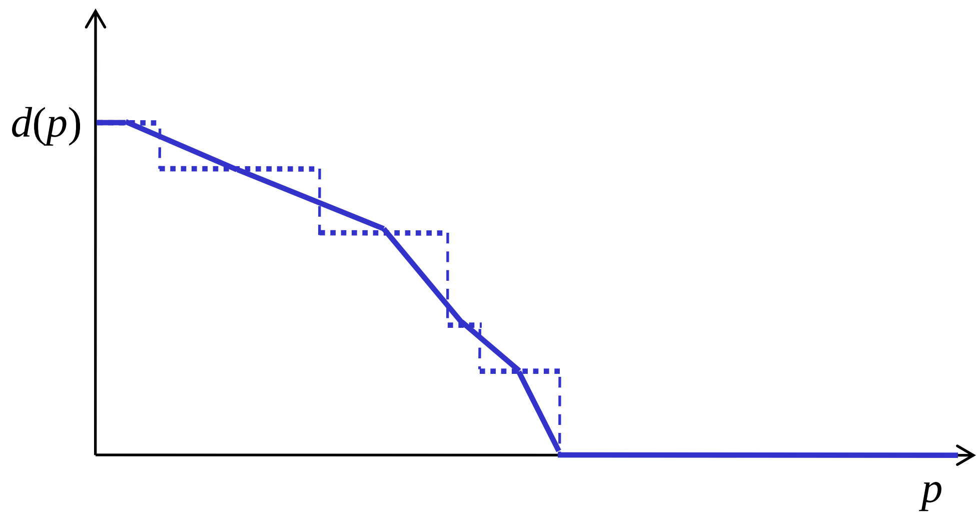
Customer Segments Smooth Price Response Function



- Consider starting with a very high price; For example $10 per bushel of corn
  - Reduce the price by say $2, the richest customer could be willing to pay $8
  - Reduce the price by $2, the next richest could pay $6
  - Reduce the price by $1, the next customer could pay $5, and so on.
- This gives a smoother price response function.
- Note that lowering price triggers another set of customers to buy.

Smooth Price Response Function

Connect the midpoints of the demands:

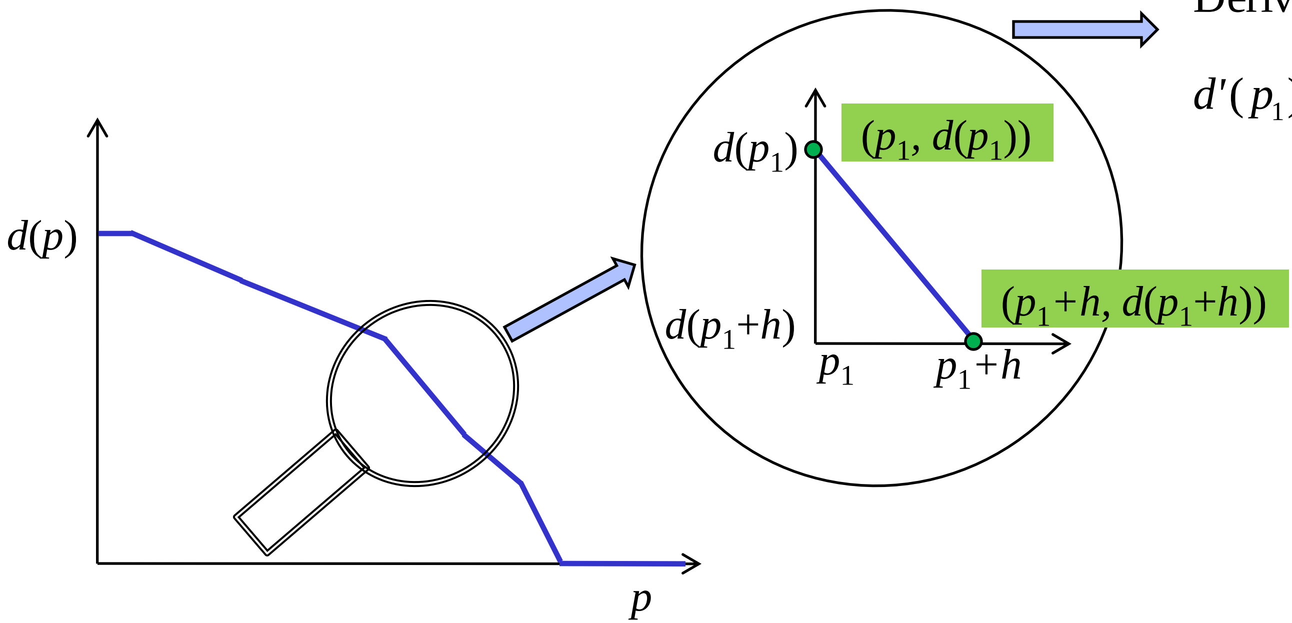


Price response curve is

- Nonnegative
- Downward slopping (in general)
- Continuous (we made it so)
- Differentiable (we wish it to be so)

Derivative of the Price Response

- While optimizing the price, the demand's response to price changes is important.
- Measure via
  - pictorially, the slope of the price response function,
  - algebraically, the derivative of the price response function, or
  - algebraically, the price sensitivity



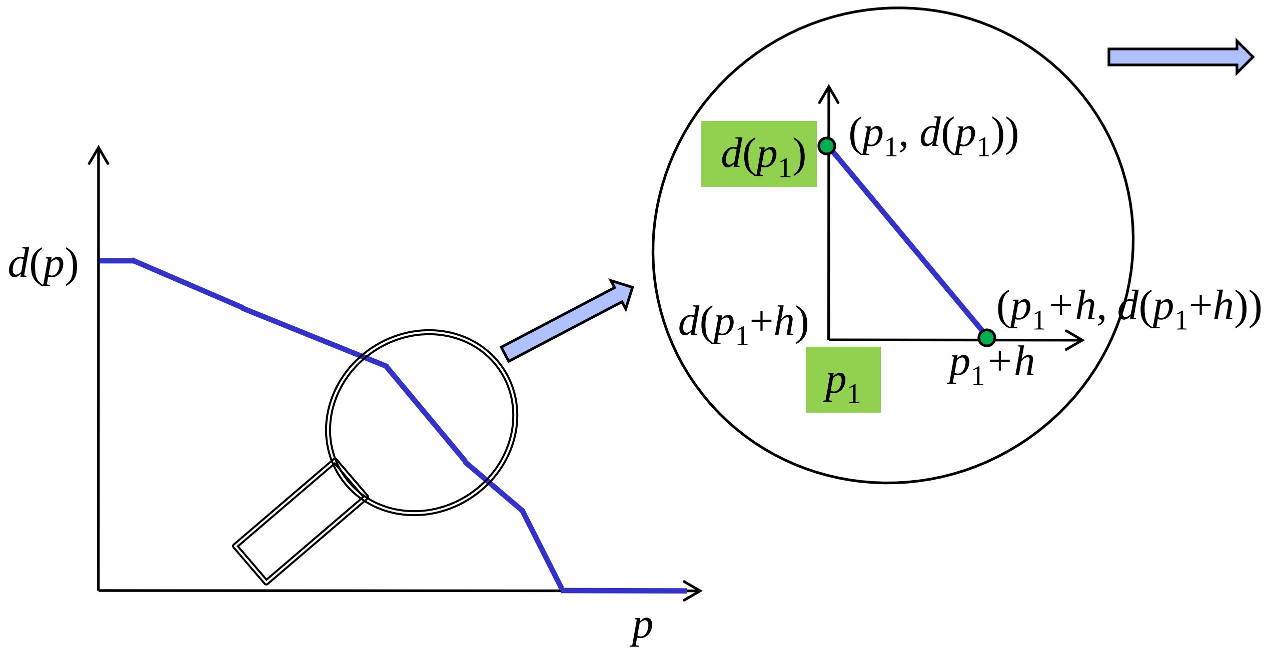
Derivative at p_1:

$$d'(p_1) = \frac{d(p_1+h) - d(p_1)}{p_1 + h - p_1}$$

as h becomes smaller

Unitless Measure of Response: Elasticity

- When computing the derivative, we divide a quantity (amount) by a monetary unit, so we end up with units such as gallon/$, tons/yuan, meters /pesos .
- Elasticity is unitless as it is a ratio of two percentages:
  - percentage change in demand
  - percentage change in price.



Elasticity at p_1:

$$\varepsilon(p_1) = -\frac{d'(p_1)}{d(p_1)/p_1}$$

Demand Sensitivity Measures: Slope, Derivative, Elasticity

- Demand sensitivity measures all tell us the sensitivity of demand to price changes. They are not independent of each other.
- When the **price increases**, the **buyer** can
  - **Buy a substitute** product; **Postpone** the purchase; or **Stop buying** the product and its substitutes, all of which decreases the demand.
- **Substitution and postponement** are more visible in the **long run**.
- The extent of substitution versus postponement affect the long-run versus short-run elasticity (sensitivity)
  - More postponement relative to substitution increases short run sensitivity
    - » E.g.: Buying an automobile: Decide to buy later when price increases.
    - » Postponement is possible in the short run and the long run.
  - Postponement may not be an option: Germany stops nuclear generator construction and plans to mainly substitute wind/solar energy to compensate for the shortfall until 2022. Wind farms are in North Sea, demand is in the industrial south, where the nuclear plants are. In the short run, the country works on increasing energy efficiency, uses thermal plants and plans to buy electric from Austria.

According to BundesNetzAgentur in "Germans Face Blackout Risk" by B. Preuschoff and J. Hromadko, WSJ Aug 31, 2011.

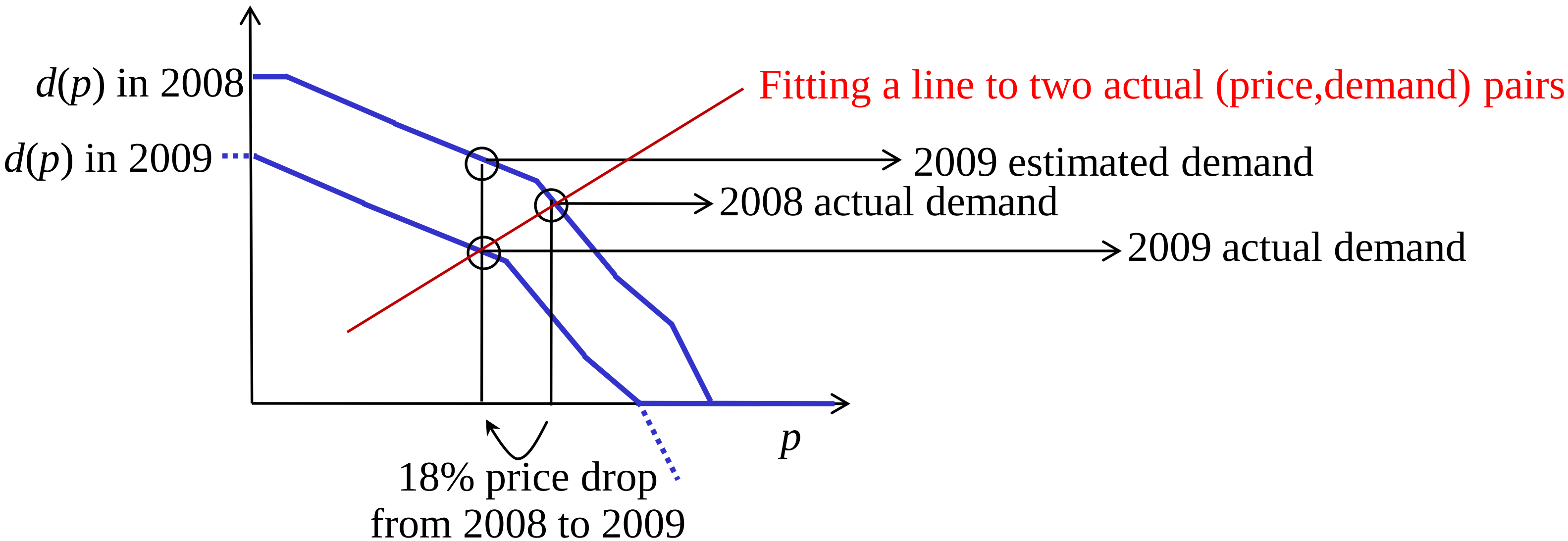
  - More substitution relative to postponement increases long run sensitivity
    - » E.g.: Vacation plans: Decide to take a cruise vacation if the ski vacation price increases.
    - » Substitution is more possible in the long run than short run.

Substitution is often a bigger factor so long term sensitivity is more than short term sensitivity.

Demand Decreasing with Price?

- "May [2009] passenger revenue was down 26% on 9.5% fewer [US Airline] passengers paying nearly 18% less per ticket …".

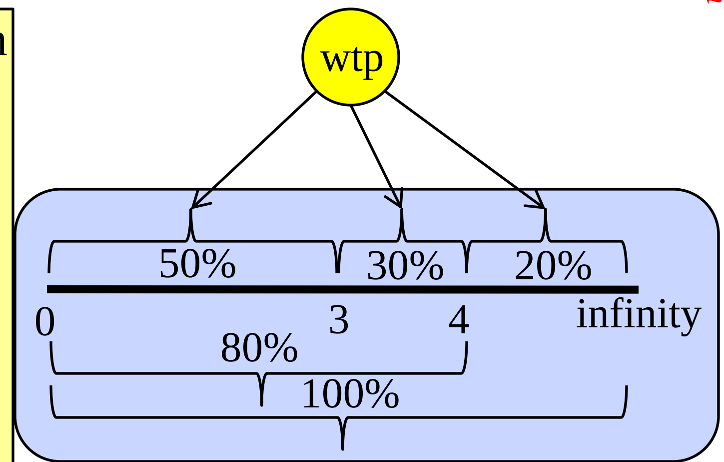
  S. Carey and M. Ested. US Airlines Fly in to Credit Squeeze. WSJ July 13, 2009
- How do we reconcile this with the downward slopping property of the price response function?
- The price response function can shift over time due to economy, income, etc.

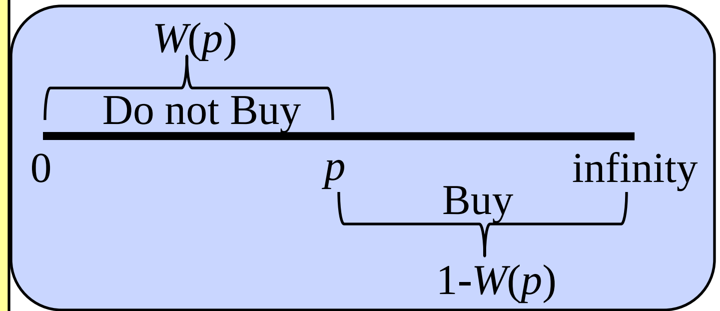


Shifts in price response function is troubling. How can we estimate the function if it can shift?
If we fit a line above, we have upward sloping function – too unlikely to happen!

Cumulative Willingness to Pay *W* (wtp)

- Write the demand *d*(*p*) as market size *D* times proportion of the consumers who are willing to pay price *p* or more.
  - This splits market size from the shifts in the purchasing power and simplifies estimation.
- *W*(*p*): the proportion who are willing to pay **at most** *p*.
  - 50% of the customers are wtp at most $3 (from 0 to 3).
  - 80% of the customers are wtp at most $4 (from 0 to 4).
    In other words, 30% are wtp between $3 and $4 (from 3 to 4).
  - 99% of the customers are wtp at most $8.
  - 100% of the customers are wtp at most infinite dollars, so 1=*W*(infinity).
- Then 1–*W*(*p*) is the proportion who are willing to pay **at least** *p* and perhaps more.
  - 50% of the customers are wtp at least $3.
  - 20% of the customers are wtp at least $4.
    - » In other words, 30% are wtp between $3 and $4.
  - 1% of the customers are wtp at least $8.
- Example: Charge $4, what percentage of customers buy and how many buy if D=1 million tons?
  - 20% and 200,000 tons.
- In general, $d(p) = D\,(1-W(p)) = D(W(\text{infinity})-W(p))$.





A customer buys if [wtp≥*p*]
Probability of a random customer purchasing at price *p* is
$P(\text{wtp} \geq p) = 1 - W(p)$.

Exact Willingness to Pay *w*

Consider the derivative of *W*: $w(p) = W'(p) = \dfrac{W(p+h) - W(p)}{h}$ for small h

- The derivative can be interpreted as the percentage willing to pay exactly *p*.
- Then $d(p) = D(W(\text{infinity}) - W(p))$ can be rewritten as

$$d(p) = D(W(\infty) - W(p)) = D\int_p^\infty w(x)dx$$

Example: Suppose $w(x)$ uniform and is given by

$$w(x) = \begin{cases} 0 & \text{if } x < 40 \\ 1/20 & \text{if } 40 \le x \le 60 \\ 0 & \text{if } x > 60 \end{cases}$$

What percentage is willing to pay more than 40? 100%

What percentage is willing to pay more than 50? 50%

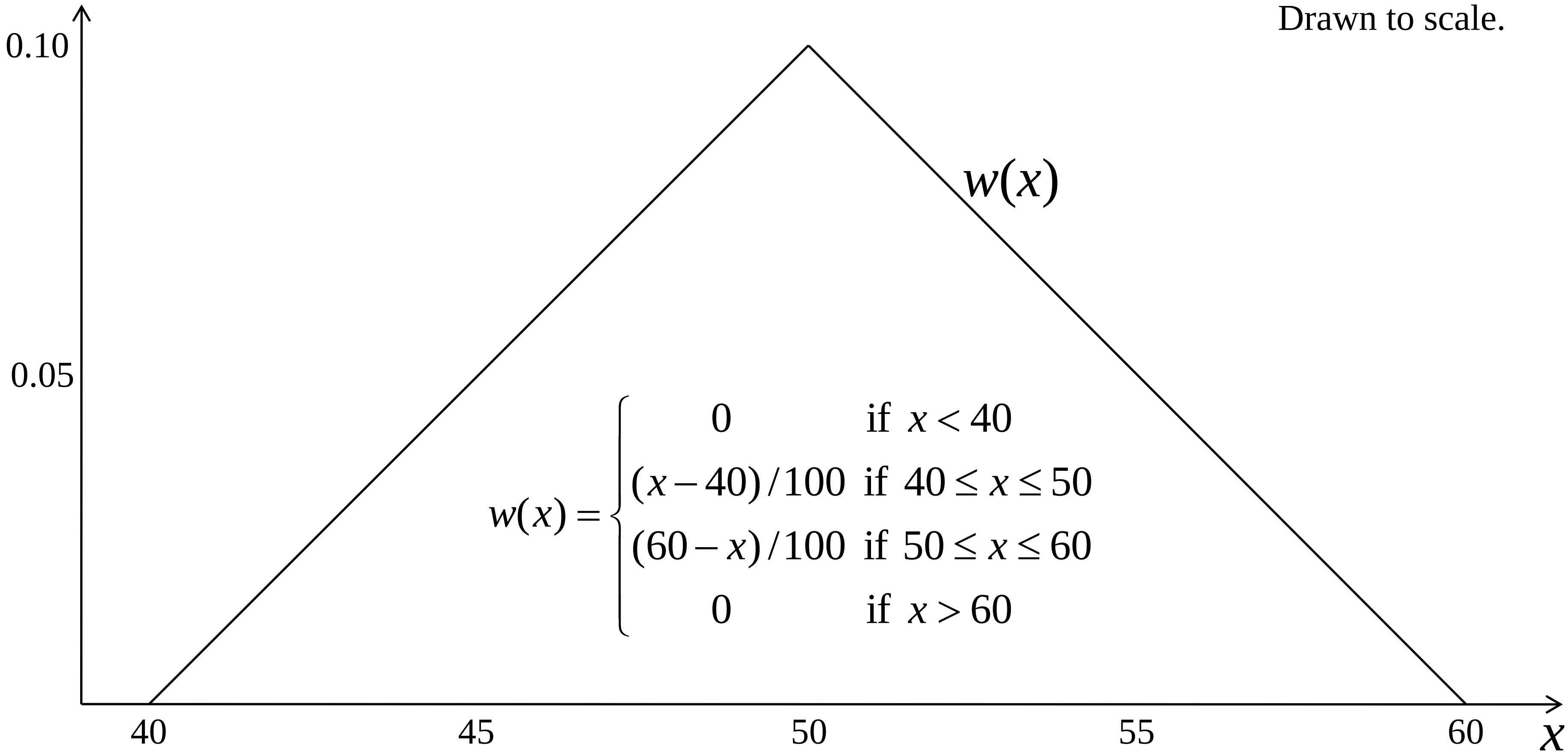
What percentage is willing to pay more than 55? 25%

What percentage is willing to pay more than 60? 0%

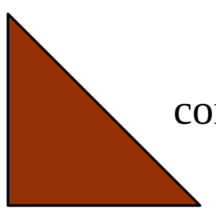
What percentage buys at the price of 47? (60-47)/20=65%

Example: Triangle Willingness to Pay

Drawn to scale.



$$w(x) = \begin{cases} 0 & \text{if } x < 40 \\ (x-40)/100 & \text{if } 40 \le x \le 50 \\ (60-x)/100 & \text{if } 50 \le x \le 60 \\ 0 & \text{if } x > 60 \end{cases}$$

A traingle of this size  corresponds to 3.125%. There are 32 of this size traingles in 100%.

Example: Willingness to Pay Geometrically

What percentage is willing to pay more than 40?

- Everybody is willing to pay more than 40, so the percentage is 100%.

What percentage is willing to pay more than 50?

- Willingness to pay is symmetric at 50; half pay more and half pays less. The percentage is 50%.

What percentage is willing to pay more than 55?

- They are located in a triangle whose corners are (55, 0), (55, 1/20) and (60,0). The height of the triangle is 1/20, the base is 5, so its area is 5/40=12.5%.

What percentage is willing to pay more than 45?

- This can be computed in two steps: Those paying more than 50 and those paying between 45 and 50. Those paying more than 50 is 50%.

Those paying between 45 and 50 are located in a trapezoid whose corners are (45, 0), (45, 1/20), (50,0) and (50,1/10). The length of the parallel sides are 1/20 and 1/10, the height is 5. The area is 15/40=37.5%. Adding this to 50%, we obtain the answer 87.5%.

- Another approach is to find the fraction willing to pay less than 45 and deduct it from 1. The fraction willing to pay less than 45 is the area of the triangle whose corners are (40,0), (45,0) and (45,0.05). The area of this triangle is 12.5%. Then 87.5%=1-12.5% is the fraction willing to pay more than 45.

Example: Willingness to Pay Algebraically

What percentage is willing to pay more than 40?

$$\int_{40}^{\infty} w(x)dx = \int_{40}^{50} \frac{x-40}{100} dx + \int_{50}^{60} \frac{60-x}{100} dx = \left.\frac{x^2-80x}{200}\right|_{x=40}^{x=50} + \left.\frac{120x-x^2}{200}\right|_{x=50}^{x=60}$$

$$= \frac{50^2-80\times 50-40^2+80\times 40}{200} + \frac{120\times 60-60^2-120\times 50+50^2}{200} = 1$$

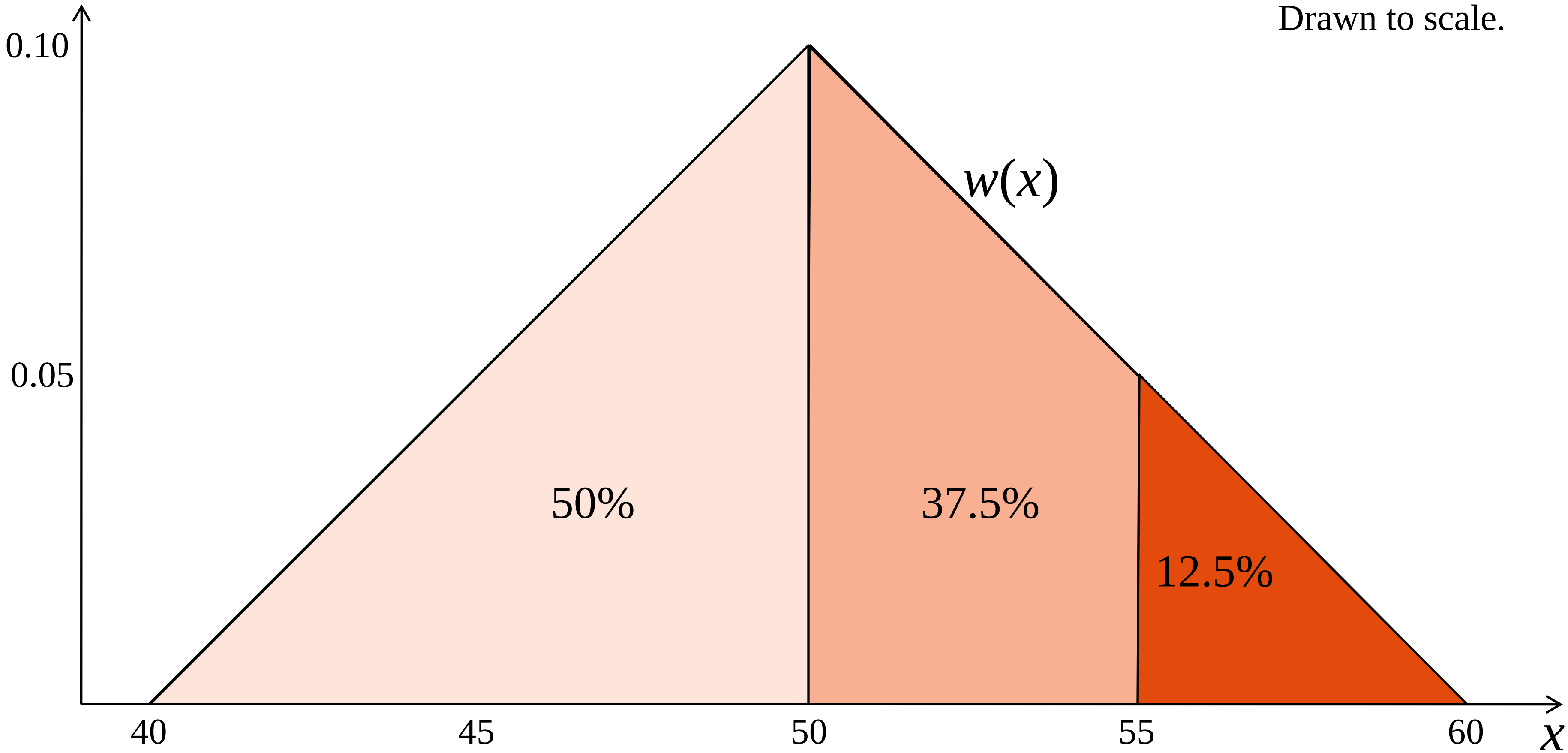
What percentage is willing to pay more than 50?

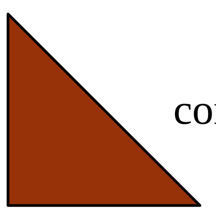
$$\int_{50}^{\infty} w(x)dx = \int_{50}^{60} \frac{60-x}{100} dx = \left.\frac{120x-x^2}{200}\right|_{x=50}^{x=60} = \frac{120\times 60-60^2-120\times 50+50^2}{200} = \frac{1}{2}$$

What percentage is willing to pay more than 55?

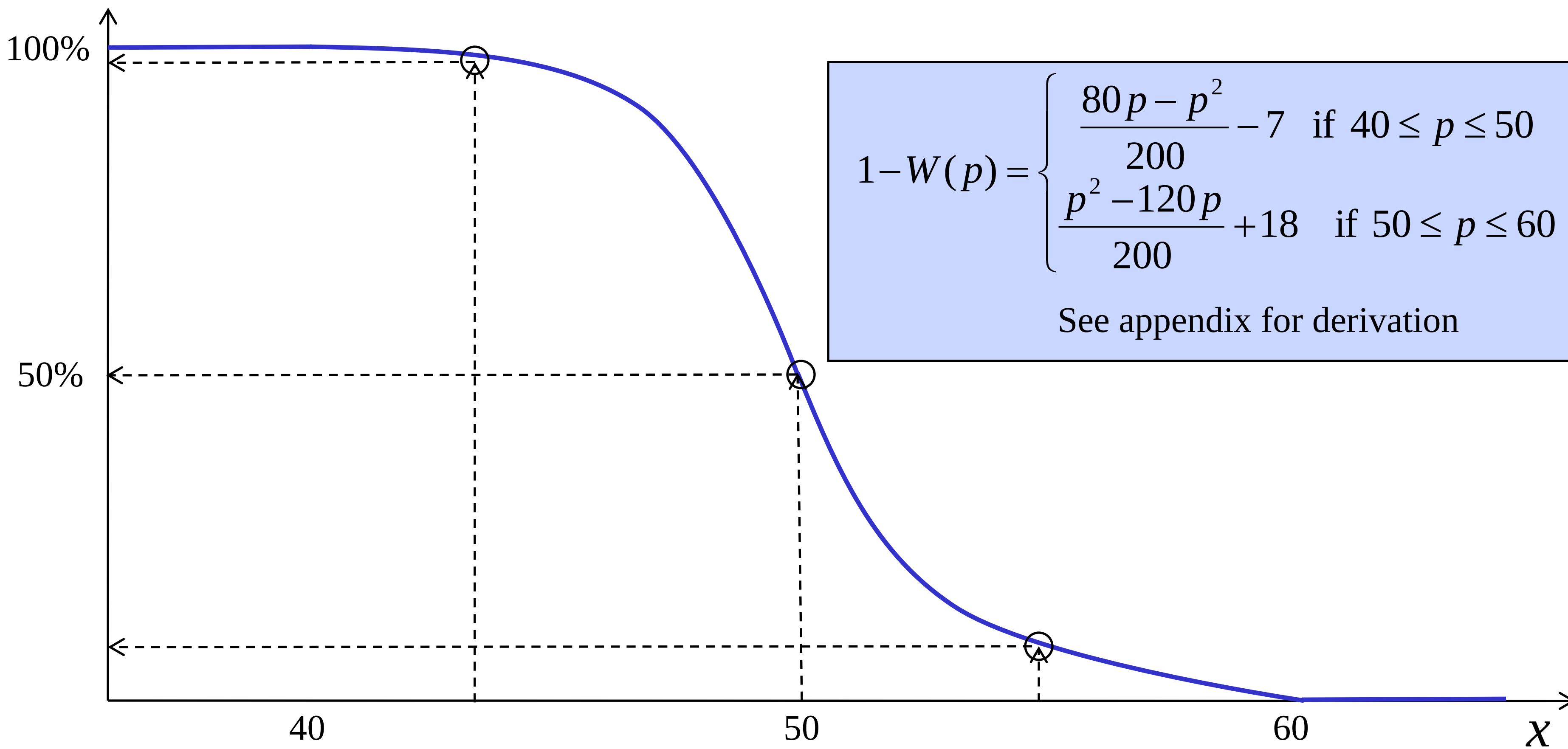
$$\int_{55}^{\infty} w(x)dx = \int_{55}^{60} \frac{60-x}{100} dx = \left.\frac{120x-x^2}{200}\right|_{x=55}^{x=60} = \frac{120\times 60-60^2-120\times 55+55^2}{200} = \frac{25}{200} = 12.5\%$$

Example: Triangle Willingness to Pay



A traingle of this size  corresponds to 3.125%. There are 32 of this size traingles in 100%.

Example: Demand from Triangle Willingness to Pay



$$1-W(p)=\begin{cases}\dfrac{80p-p^2}{200}-7 & \text{if } 40\le p\le 50\\[2ex] \dfrac{p^2-120p}{200}+18 & \text{if } 50\le p\le 60\end{cases}$$

See appendix for derivation

Charge less to capture a higher percentage of the market

Example: Demand from Constant Willingness to Pay

- Uniform willingness

  $w(p)=m/D$ for $p<D/m$

  $1-W(p)=1-mp/D$ for $p<D/m$
- Price response function

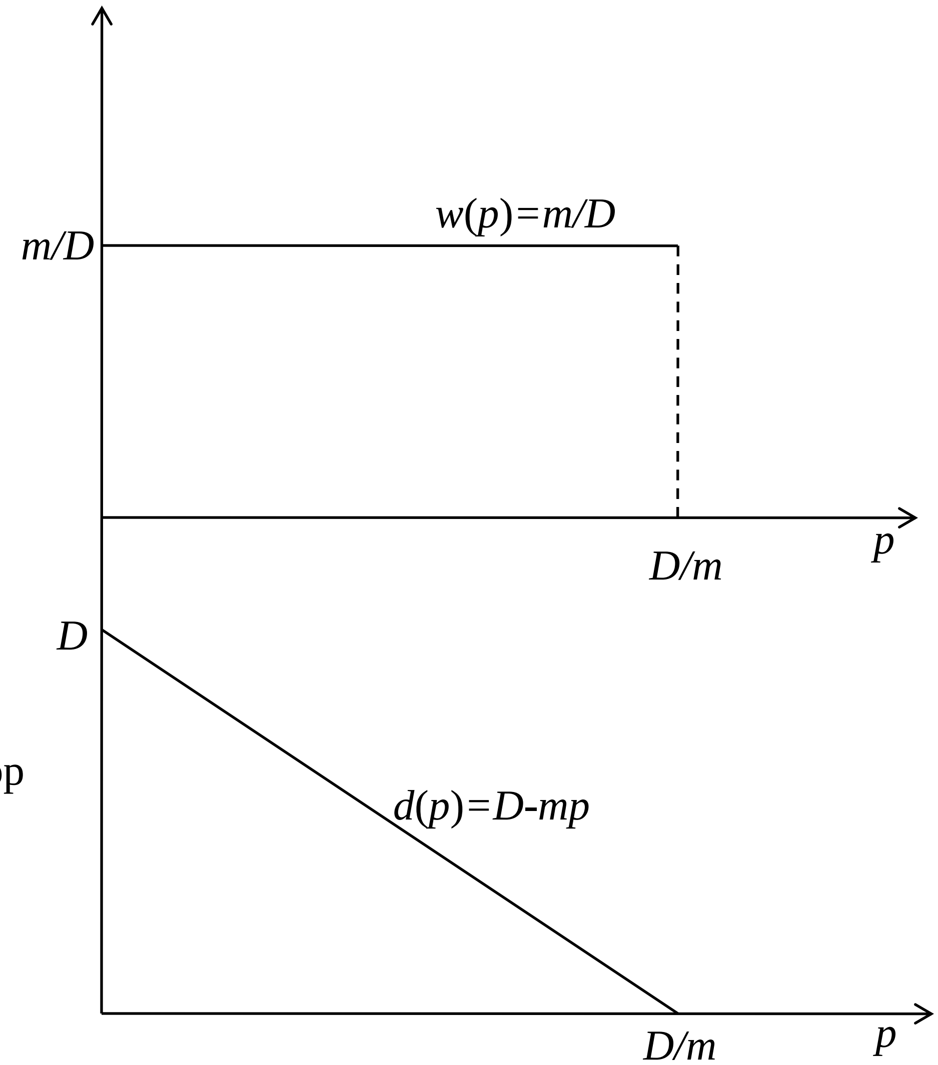
  $d(p)=D-mp$ for $p<D/m$

- Demand satiating price D/m
- Elasticity $mp/(D-mp)$

Constant $d'(p)=-m$ is not realistic.

This requires the same amount of demand drop with price increase no matter what the original price is.

This comes from constant willingness to pay.



Example: Demand from Constant Elasticity

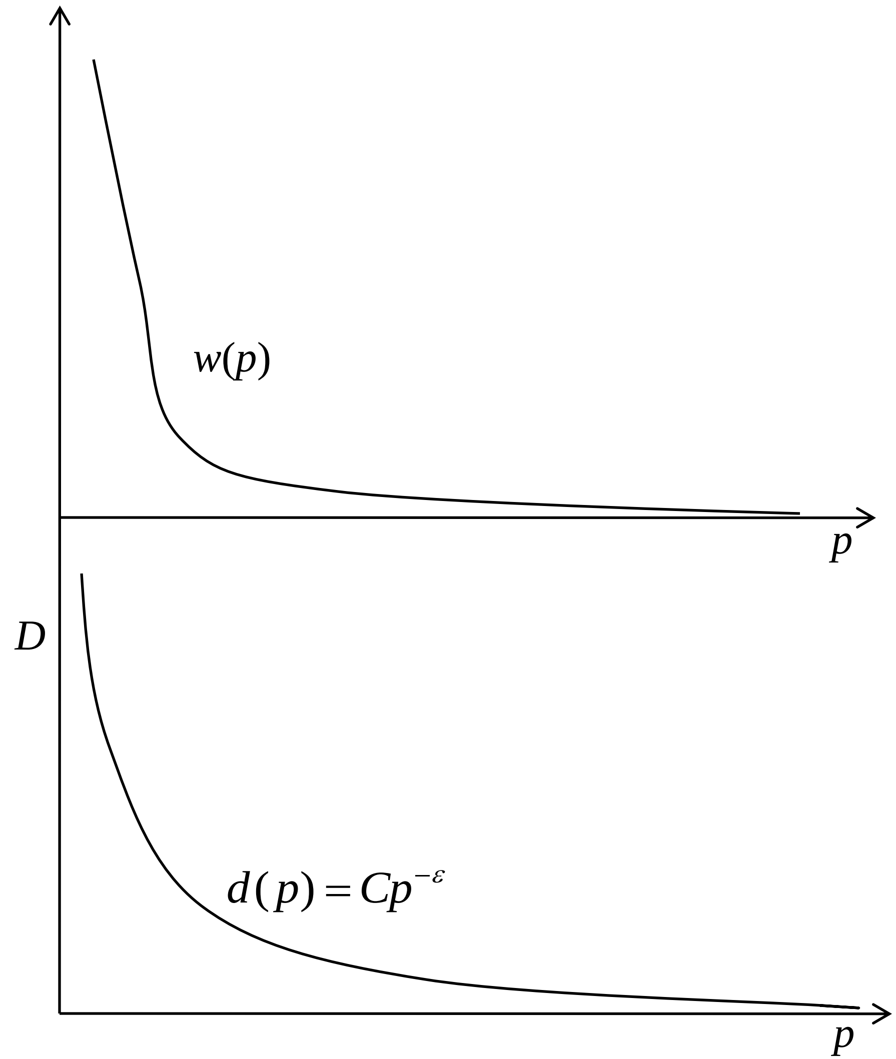
- Suppose elasticity is constant

$$\varepsilon = -\frac{d'(p)p}{d(p)}$$

- This happens only if

$$d(p) = Cp^{-\varepsilon}$$

- Easy to verify by inserting this into the elasticity formula.
- C is a market size parameter.
- Since $d(p)$ is proportional to (1-$W(p)$), $d'(p)$ is proportional to –$w(p)$. This yields, $w(x)$ proportional to $\varepsilon p^{-(\varepsilon+1)}$
- Percentage of customers willing to pay approaches infinity when price is small. This is troubling.



Example: Revenue Maximization with Constant Elasticity

- Demand with constant elasticity $d(p) = Cp^{-\varepsilon}$
- Revenue $R(p)$ with constant elasticity $R(p) = p\,Cp^{-\varepsilon} = Cp^{1-\varepsilon}$
- Derivative of the revenue $R'(p) = (1-\varepsilon)Cp^{-\varepsilon} = (1-\varepsilon)d(p)$

- Derivative is positive or negative depending on elasticity<1 and elasticity>1, respectively.
  - If elasticity<1, the revenue maximizing price is infinity
  - If elasticity>1, the revenue maximizing price is zero
  - If elasticity=1, any price maximizes the revenue
- These do not seem right, so constant elasticity at every value of price is a very strong assumption. Rather elasticity can stay constant for certain ranges of the price.

Example: Demand from Logit Price Response Function

- Let us smooth triangle willingness to pay with a bell-shaped curve.

$$w(p) = b\frac{e^{-(a+bp)}}{(1+e^{-(a+bp)})^2}$$

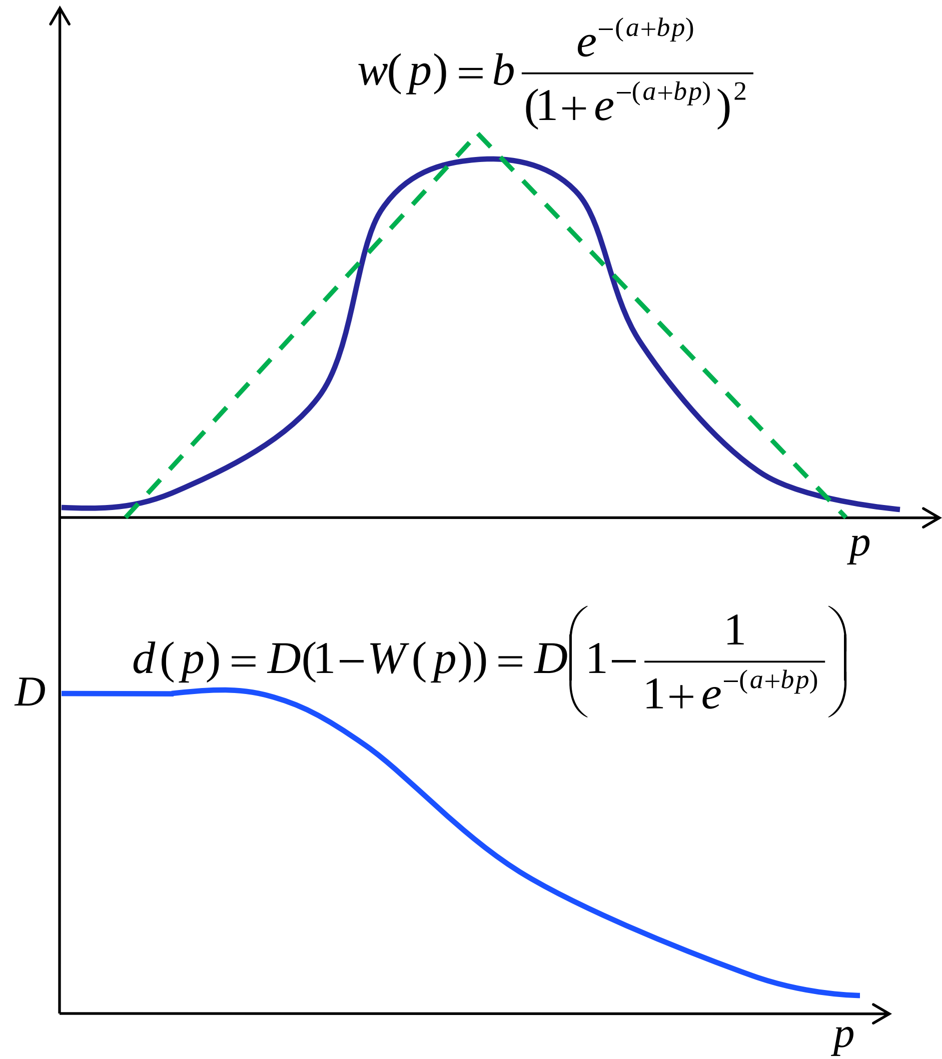
- Parameters *a*, *b* determine the shape.
- The proportion $W(p)$ willing to pay at most *p:*

$$W(p) = \frac{1}{1+e^{-(a+bp)}} - \frac{1}{1+e^{-a}}$$

- This achieves $W(0)=0$.
- Instead, to simplify, set

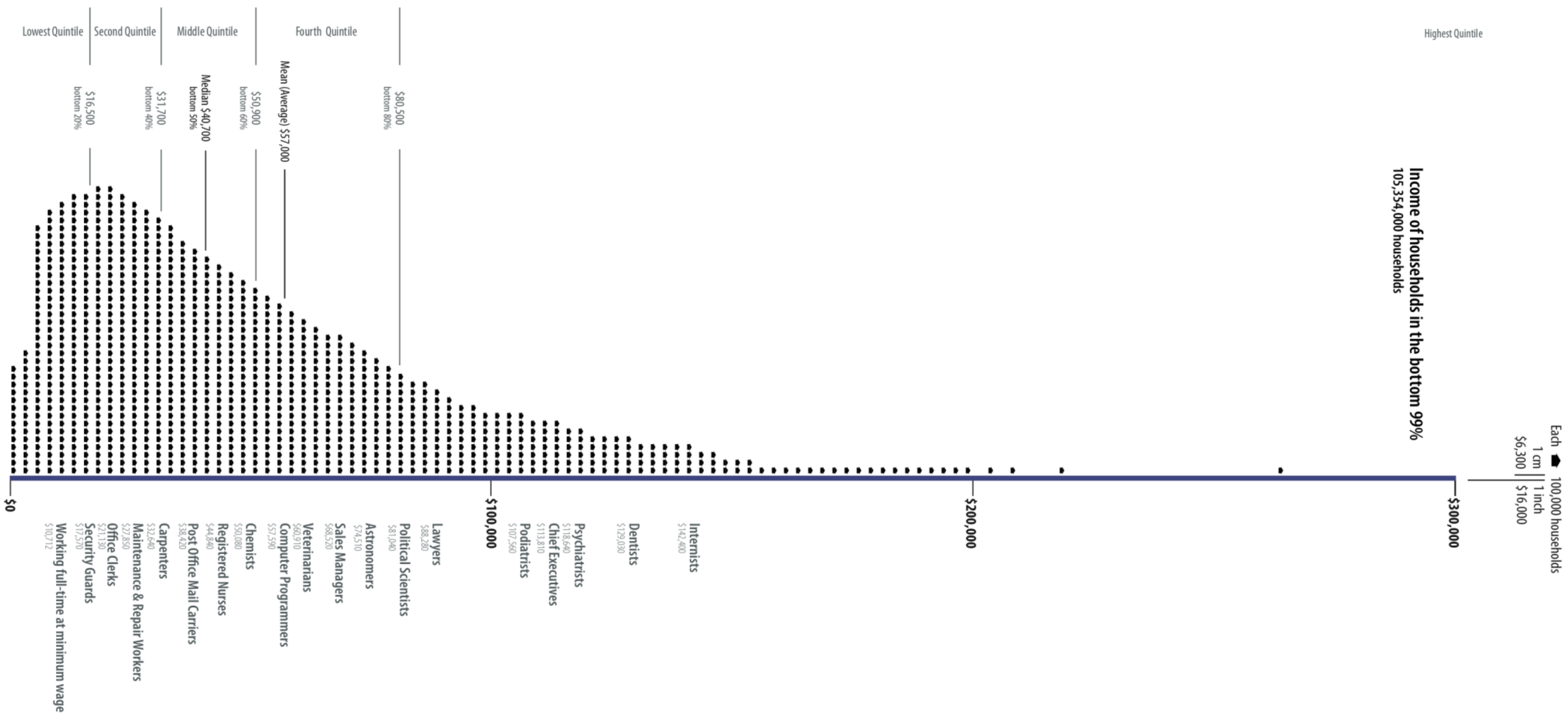
$$W(p) = \frac{1}{1+e^{-(a+bp)}}$$

- Now $W(0)>0$, so a few customers are not willing to pay even 0. No changes necessary to $w(p)$ expression.



Discussion of the Logit Price Response Function

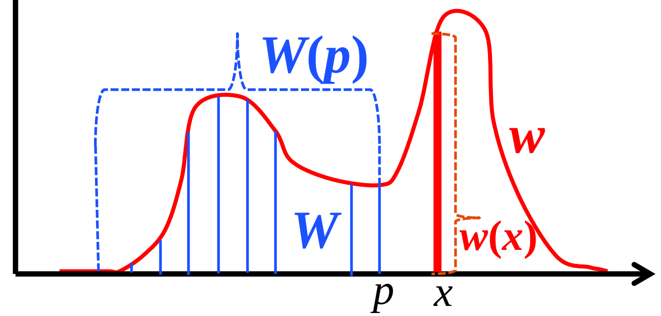
- Logit price response function is not sensitive for extremely low or high prices
  - With an extremely low price, a good portion of the market is already captured. Reducing the price further is not going to increase the demand much.
  - With an extremely high price, we are selling to a few customers who are foolishly ignorant or loyal to our brand. Increasing the price further is not going to change their ignorance or loyalty much. Hence, most will continue to buy.
  - If the current price is close to the market price, many customers can switch to buy our product with a small discount. The reverse is true as well.
- Moreover, income distribution is not uniform, nor should the willingness to pay be.



Other examples for exact willingness to pay w and its cumulative counterpart W

$$W(p) = \int_0^p w(x)\,dx$$

w is the **height** of an individual bar;
W is the **area** under the curve.



- Above relationship is the exact relationship between a probability density function (pdf) w and a cumulative density function W (cdf).
- Exact $w(x)$ can be interpreted as the wtp of a randomly chosen individual.
- Cumulative $W(p)$ can be interpreted as the probability that a randomly chosen individual is willing to pay at most p.
- Any probability density function can serve as $w(x)$ or similarly any cumulative density function can serve as $W(p)$.
- Examples for $w(x)$ and $W(p)$ from probability:
  - Uniform distribution
  - Exponential distribution
  - Normal distribution
  - Gamma distribution
  - Weibull distribution

Competition

- Competition is not ignored in pricing optimization. It can be captured at three levels of sophistication.
  - 1. Incorporating competition in the price response function
    - » Logit price response function drops fastest around the competition (market) price.
    - » Historical data used in estimating price response function already includes the effect of competition.
  - 2. Consumer-choice modeling
    - » Competition can be explicitly modeled
    - » Collect data on own price as well as competitor prices
    - » Example: 30 year fixed rate mortgage rates for $200,000 loan in Plano, TX

Lender	APR	Disc/Orig Points	Rate	Fees	Monthly payment
Quicken Loans	5.462	0.750/0.00	5.500	$1,600	$1,136
Bank of America	5.518	0.875/0.00	5.375	$1,400	$1,120
Austin First Mortgage	5.178	0.000/0.00	5.125	$1,195	$1,095
ViewPoint bank	5.406	0.000/1.00	5.250	$1,455	$1,104

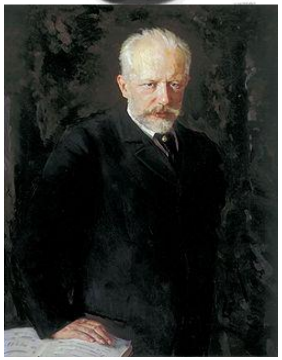
➢ Pricing of mortgage can be a project topic: Data collection easy. Consider big bank (price leader) and small bank. Study discount points vs. price of the mortgage.

  - 3. Anticipating competitive response.

2. How do Customers think?

Given a set of products of the same type, customer can come up with how much they are willing to pay.

Consider Dallas Symphony Concerts:



- Consider Tchaikovsky-lovers and their willingness to pay w:
  - » The Moody Blues on Sep 2, 2009: w=18, p=20. Surplus=-2.
  - » Mahler Symphony 4 on Sep 17, 2009: w=28, p=20. Surplus= 8.
  - » Tchaikovsky Symphony 4 September 24, 2009: w=16, p=10. Surplus= 6.
- Tchaikovsky-lovers buy Mahler Symphony 4 tickets to maximize their surplus.

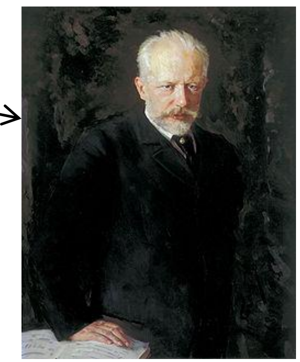
- Consider Rhythm-Blues-lovers and their willingness to pay w:
  - » The Moody Blues: w=25, p=20. Surplus= 5.
  - » Mahler Symphony 4: w=16, p=20. Surplus= -4.
  - » Tchaikovsky Symphony 4: w=14, p=10. Surplus= 4.
- Rhythm-Blues-lovers buy The Moody Blues tickets to maximize their surplus.

- Consider Romantics and their willingness to pay w:
  - » The Moody Blues: w=18, p=20. Surplus= -2.
  - » Mahler Symphony 4: w=22, p=20. Surplus= 2.
  - » Tchaikovsky Symphony 4: w=14, p=10. Surplus= -4.
- Romantics buy Mahler Symphony 4 tickets to maximize their surplus.

2. Market Share Function: $d_i(p) = D\, f_i(p)$

$f_B(p = [20, 20, 10])$

$f_M(p = [20, 20, 10])$



$f_T(p = [20, 20, 10])$

1. Market share is between 0 and 1.
2. Everybody buys exactly one product.
3. Increasing price of a product decreases its market share.
4. Increasing price of a product increases the market share of other products.

1. $0 \le f_i(p) \le 1$
2. $\sum_i f_i(p) = 1$
3. $\frac{\partial f_i(p)}{\partial p_i} \le 0$
4. $\frac{\partial f_i(p)}{\partial p_j} \ge 0$

2. Multinomial Logit

Most common consumer choice model is multinomial logit (MNL)

$$f_i(p=[p_1, p_2, ..., p_n]) = \frac{e^{-b_i p_i}}{\sum_{j=1}^{n} e^{-b_j p_j}}$$

Parameters are $b_1, \ldots, b_n$. Parameter b_i is large when product i is price sensitive.

- In the Dallas Symphony example with b_B=2.1, b_M=2, b_T=4, market shares are

$$f_B(p=[p_B=20, p_M=20, p_T=10]) = \frac{e^{-b_B p_B}}{e^{-b_B p_B} + e^{-b_M p_M} + e^{-b_T p_T}} = \frac{e^{-2.1*20}}{e^{-2.1*20} + e^{-2*20} + e^{-4*10}} = 6.30\%$$

$$f_M(p) = \frac{e^{-b_M p_M}}{e^{-b_B p_B} + e^{-b_M p_M} + e^{-b_T p_T}} = \frac{e^{-2*20}}{e^{-2.1*20} + e^{-2*20} + e^{-4*10}} = 46.85\%$$

$$f_T(p) = \frac{e^{-4*10}}{e^{-2.1*20} + e^{-2*20} + e^{-4*10}} = 46.85\%$$

2. Weaknesses of Multinomial Logit

- Market shares are very sensitive to parameters.
  - Try Dallas Symphony example with b_B=2.2 to see that Moody Blues market share goes below 1% from 6.3%. This is more than 80% drop in the market share as a result of less than 5% change in the parameter value.
- When the price changes, not only the market share function but market size can change.
  - Tchaikovsky Symphony 4 ticket price drops to $8, people who would watch a movie at $8 can choose to go to a concert. This increases the market size for concert-goers.
- Unavailable competitor price information.
  - If the competitor knows that you are pricing in response to his price, he makes every effort to hide his price.
    - » Hiding price is not easy in b2c context; much easier in b2b context.

3. Anticipating Competitive Response

- Competitors can match my price
  - Exact matching: In the mortgage industry, the late-comers (followers) match existing banks' (leaders') price, but may be with 2-3 days delay.
  - Matching with a premium: Convenience stores collect Wal-Mart (price leader) prices and add 10% premium to obtain their prices.
- Dynamic pricing says a price that is optimal today is not so tomorrow. If I frequently update my prices, the competition cannot keep up (or always late in matching prices) as there always is some delay in matching prices.
  - The discipline of Game theory applies if competitors act in expectation of each others' responses. It may not apply when price changes are frequent.
  - Moreover, Game theory seeks equilibrium prices. When prices are updated dynamically, existence, computation and implementation of equilibrium prices are much harder.
  - Game theoretic dynamic pricing is not done in practice.

Basic Price Optimization

- We use *p* for price and *c* for cost.
- Unit margin=price-cost=p-c.
- Total contribution (to margin)=$m(p)=(p-c)d(p)$.
- Basic price optimization problem:

$$\max_p \{m(p)=(p-c)d(p)\}$$

- Generally, $m(p)$ is concave (hill shaped). Taking the derivative, gives us the optimality equation for the total contribution maximizing price

$$(p-c)d'(p)+d(p)=0$$

- Writing the optimality equation as marginal revenue equal to marginal cost

$$pd'(p)+d(p)=cd'(p)$$

Simple Example: Basic Price Optimization

- Example: Suppose that c=10 and

$$d(p) = D(1-W(p)) = 200-10p \quad \text{if } 0 \le p \le 20$$

Find total margin maximizing price.

For the derivative of *d'*(*p*), we have

$$d'(p) = -10. \text{ Then } (p-c)d'(p) + d(p) = 0 \quad \text{gives}$$
$$(p-10)(-10) + 200 - 10p = 0$$
$$-20p + 300 = 0.$$

The root of this equation is 15. The derivative of the total margin is positive from 0 to 15 and negative from 15 to 20. The optimal price is 15.

Intermediate Example: Basic Price Optimization

- Example: Suppose that c=1.5 thousand dollars for a 3-credit coursed and the demand is

$$d(p) = D(1-W(p)) = 40{,}000\,p - 20{,}000\,p^2 \quad \text{if } 1 \le p \le 2$$

Find total margin maximizing price.

For the derivative of *d'*(*p*), we have

$d'(p) = 40{,}000 - 40{,}000\,p$. Then $(p-c)d'(p) + d(p) = 0$ gives

$(p-1.5)40{,}000(1-p) + 40{,}000\,p - 20{,}000\,p^2 = 0$

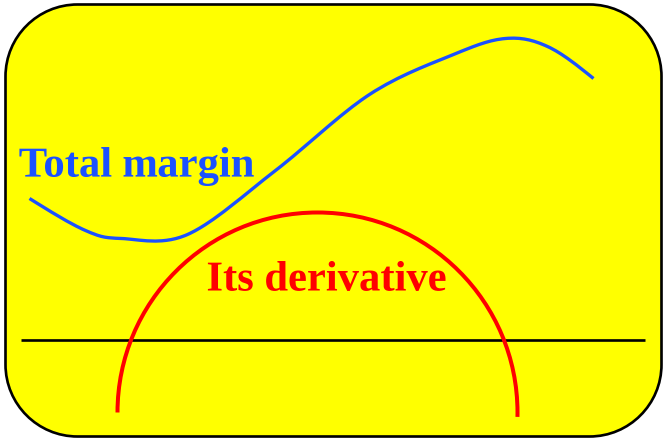
Equivalently, $(2p\text{-}3)(1\text{-}p) + 2p - p^2 = 0.$

Or, $2p - 3 - 2p^2 + 3p + 2p - p^2 = 0.$

Simplifying, $-3p^2 + 7p - 3 = 0.$

Roots of this quadratic equation are

$$\frac{-7 \pm \sqrt{7^2 - 4(-3)(-3)}}{2(\text{-}3)} = \frac{7}{6} \pm 0.6 = 1.767 \text{ or less than } 1.$$



The margin is increasing from the smaller root to the larger root 1.767. The optimal price for a 3-credit course is \$1,767.

Example: Basic Price Optimization

- Example: Suppose that c=10 and *D*=200

$$d(p) = D(1-W(p)) = 200\begin{Bmatrix} \dfrac{80p-p^2}{200}-7 & \text{if } 40 \le p \le 50 \\ \dfrac{p^2-120p}{200}+18 & \text{if } 50 \le p \le 60 \end{Bmatrix} = \begin{Bmatrix} 80p-p^2-1400 & \text{if } 40 \le p \le 50 \\ p^2-120p+3600 & \text{if } 50 \le p \le 60 \end{Bmatrix}$$

Find total margin maximizing price.

For the derivative of *d'*(*p*), we have

$$d'(p) = \begin{Bmatrix} 80-2p & \text{if } 40 \le p \le 50 \\ 2p-120 & \text{if } 50 \le p \le 60 \end{Bmatrix}. \text{ Then } (p-c)d'(p)+d(p)=0 \text{ gives}$$

$$(p-10)\begin{Bmatrix} 80-2p & \text{if } 40 \le p \le 50 \\ 2p-120 & \text{if } 50 \le p \le 60 \end{Bmatrix} + \begin{Bmatrix} -1400-p^2+80p & \text{if } 40 \le p \le 50 \\ 3600-120p+p^2 & \text{if } 50 \le p \le 60 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$-3p^2+180p-2200=0$ if $40 \le p \le 50$

The roots this quadratic equation are 17.1 and 42.9. The derivative of the total margin is positive from 40 to 42.9 and negative from 42.9 to 50. Total margin is increasing until *p*=42.9 and decreasing afterwards until *p*=50.

and

$3p^2-260p+4800=0$ if $50 \le p \le 60.$

The roots of this equation are 26.7 and 60. The derivative remains negative from 50 to 60. Revenue is decreasing from *p*=50 to *p*=60.

Alternative Interpretations of the Optimality Equation

- Using price elasticity in the optimality equation

$$m'(p) = (p-c)d'(p) + d(p) = d(p)\left(\frac{d'(p)p}{d(p)} + 1\right) - cd'(p)$$

$$= d(p)(1-\varepsilon(p)) - cd'(p) \geq d(p)(1-\varepsilon(p))$$

- If the elasticity at the current price is less than 1, increase the price to get to the optimal price. Reverse is not true.
- Optimality equation can be written as

$$d(p) = -(p-c)d'(p)$$

$$\varepsilon(p) = -\frac{d'(p)p}{d(p)} = \frac{p}{p-c}$$

- At the optimal price *p*, *p*/(*p*-*c*) is equal to the elasticity at *p*. Or equivalently, optimal price is given by

$$p = \frac{\varepsilon(p)}{\varepsilon(p)-1}c$$

Zero-Cost Products Revenue Maximization

- The marginal cost of some products are negligible
  - Examples: Software, Digital music, books, movies.
  - Read Section 3.3 on marginal costs.
- With c=0, maximizing total contribution is the same as maximizing revenue. We can adapt the optimality equation for c=0:

$$pd'(p) + d(p) = 0$$

- We can use this equation to maximize revenue with

$$d(p) = D(1 - W(p)) = 200 - 10p \quad \text{if } 0 \leq p \leq 20$$

We have $-10p + 200 - 10p = 0$

The revenue maximizing price is 10 while the total margin maximizing price was 15.

If the costs disappear, price less.

Total Margin vs. Revenue Maximization

- Maximize total margin to maximize the profit.
- Revenue maximization yields a smaller price and hence generates more demand. Use revenue maximization to capture new markets.
- You may consider a weighted combination of total margin and revenue to maximize.
  - What is the weight?
  - The resulting price is between revenue maximizing price and total margin maximizing price.

Basic Price Optimization

Summary

- **Price-Response Functions**
  - **Elasticity**
  - **Willingness to pay: Triangular, Uniform**
  - **Demands from constant elasticity and logit function.**
- **Competition: Multinomial Logit Demand Model**
- **Basic Price Optimization**

Background Material

In these slides, you see a reference to derivative. The derivative is a necessary tool for price optimization. Some quick rules of derivatives are:

Derivative of $f(x) = x^a$ is $f'(x) = ax^{a-1}$ for constant a.

Derivative of $f(x) + g(x)$ is $f'(x) + g'(x)$.

Derivative of $cf(x)$ is $cf'(x)$ for constant c.

In these slides, you see a reference to constant e=2.718 (approximately). This the base of natural logarithm. The reason why e is useful in the formulas is because

Derivative of $f(x) = e^x$ is itself $f'(x) = e^x$.

In these slides, you see a reference to integral which is the opposite of the derivative. Some quick rules of derivatives are:

$$\int x^a dx = \frac{1}{a+1} x^{a+1} \text{ for constant } a.$$

$$\int f(x) + g(x) dx = \int f(x) dx + \int g(x) dx.$$

$$\int cf(x) dx = c \int f(x) dx \text{ for constant } c.$$

Computing $W(p)$ from $w(x)$ – 1-triangle wtp

Tuition for a 3-credit course has wtp in thousand dollars as $w(x)=\begin{cases} 0 & \text{if } x<1 \\ 2(x-1) & \text{if } 1\le x\le 2 \\ 0 & \text{if } x>2 \end{cases}$

$W(p)=\int_0^p w(x)dx$ is the percentage of students willing to pay at most p.

We can use the integration rule in the background material to proceed.

$$W(p)=\begin{cases} 0 & \text{if } p<1 \\ \int_1^p 2(x-1)dx & \text{if } 1\le p\le 2 \\ 1 & \text{if } p>2 \end{cases} = \begin{cases} 0 & \text{if } p<1 \\ (x-1)^2\Big|_1^p & \text{if } 1\le p\le 2 \\ 1 & \text{if } p>2 \end{cases} = \begin{cases} 0 & \text{if } p<1 \\ (p\text{-}1)^2 & \text{if } 1\le p\le 2 \\ 1 & \text{if } p>2 \end{cases}$$

With 20,000 students at a university, the demand is

$$d(p)=D(1-W(p))=20{,}000(1-(p-1)^2)=40{,}000p-20{,}000p^2$$

Computing *W*(*p*) from *w*(*x*) – 2-triangle wtp

In the triangle wtp example, we had $w(x)=\begin{cases} 0 & \text{if } x<40 \\ (x-40)/100 & \text{if } 40\le x\le 50 \\ (60-x)/100 & \text{if } 50\le x\le 60 \\ 0 & \text{if } x>60 \end{cases}$

$W(p)=\int_0^p w(x)dx$ is the percentage of customers willing to pay at most p.

We can use the integration rule in the background material to proceed.

$$W(p)=\begin{cases} 0 & \text{if } p<40 \\ \int_{40}^{p}\frac{x-40}{100}dx & \text{if } 40\le p\le 50 \\ \frac{1}{2}+\int_{50}^{p}\frac{60-x}{100}dx & \text{if } 50\le p\le 60 \\ 1 & \text{if } p>60 \end{cases} = \begin{cases} 0 & \text{if } p<40 \\ \left.\frac{x^2-80x}{200}\right|_{40}^{p} & \text{if } 40\le p\le 50 \\ \frac{1}{2}+\left.\frac{120x-x^2}{200}\right|_{50}^{p} & \text{if } 50\le p\le 60 \\ 1 & \text{if } p>60 \end{cases}$$

Computing 1-*W*(*p*) from *w*(*x*)

$$W(p)=\begin{cases} 0 & \text{if } p<40 \\ \left.\dfrac{x^2-80x}{200}\right|_{40}^{p} & \text{if } 40\le p\le 50 \\ \dfrac{1}{2}+\left.\dfrac{120x-x^2}{200}\right|_{50}^{p} & \text{if } 50\le p\le 60 \\ 1 & \text{if } p>60 \end{cases} = \begin{cases} 0 & \text{if } p<40 \\ \dfrac{p^2-80p-(40^2-80(40))}{200} & \text{if } 40\le p\le 50 \\ \dfrac{1}{2}+\dfrac{120p-p^2-(120(50)-50^2)}{200} & \text{if } 50\le p\le 60 \\ 1 & \text{if } p>60 \end{cases}$$

$$W(p)=\begin{cases} 0 & \text{if } p<40 \\ \dfrac{p^2-80p}{200}+8 & \text{if } 40\le p\le 50 \\ 0.5+\dfrac{120p-p^2}{200}\text{-}17.5 & \text{if } 50\le p\le 60 \\ 1 & \text{if } p>60 \end{cases} = \begin{cases} 0 & \text{if } p<40 \\ \dfrac{p^2-80p}{200}+8 & \text{if } 40\le p\le 50 \\ \dfrac{120p-p^2}{200}\text{-}17 & \text{if } 50\le p\le 60 \\ 1 & \text{if } p>60 \end{cases}$$

$$1-W(p)=\begin{cases} \dfrac{80p-p^2}{200}-7 & \text{if } 40\le \mathrm{p}\le 50 \\ \dfrac{p^2-120p}{200}+18 & \text{if } 50\le \mathrm{p}\le 60 \end{cases}$$