Formulas for closed-book OPRE 6366 Exams

Reminder for Statistics and Operations Management:

- Given a population $\{X_1, X_2, \dots, X_N\}$, **Mean**: $\bar{X} = \frac{\sum_{i=1}^{N} X_i}{N}$; **Variance**: $\mathrm{Var}(X) = \frac{\sum_{i=1}^{N}(X_i - \bar{X})^2}{N}$.

Standard deviation for the population: $\sigma = \sqrt{\mathrm{Var}(X)}$. **Coefficient of Variation** for the population: $CV = \sigma/\bar{X}$.

- **Exponential** distribution fits well to interarrival times to a queue and it has a CV of 1.

$$\textbf{Prob}(\textbf{exponential random variable with mean } \mu \leq t) = 1 - e^{-t/\mu}$$

- For a **normal random variable** $N(\mu, \sigma^2)$, $(N(\mu, \sigma^2) - \mu)/\sigma = N(0, 1)$ is the **standard normal random variable**. If we **sum** L **many independent normal random variables** $N(\mu, \sigma^2)$, the sum is a normal random variable $N(L\mu, L\sigma^2)$.

$$\textbf{Excel's Normal Probability functions:} \begin{cases} normdist(x, mean, stdev, 0): & \text{normal probability density at } x, \\ normdist(x, mean, stdev, 1): & \text{normal cumulative density at } x, \\ nominv(prob, mean, stdev): & \text{inverse of the cumulative density at } prob. \end{cases}$$

Areas under the standard normal curve from $-\infty$ to z and $\mathbf{L(z)} = normdist(z, 0, 1, 0) - z * (1 - normdist(z, 0, 1, 1))$:

z	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4
Area	0.54	0.58	0.62	0.66	0.69	0.73	0.76	0.79	0.82	0.84	0.86	0.88	0.9	0.919
L(z)	0.35	0.31	0.27	0.23	0.20	0.17	0.14	0.12	0.10	0.08	0.07	0.06	0.05	0.037

z	1.5	1.6	1.65	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6
Area	0.933	0.945	0.950	0.955	0.964	0.971	0.977	0.982	0.986	0.989	0.992	0.994	0.995
L(z)	0.029	0.023	0.020	0.018	0.014	0.011	0.008	0.006	0.005	0.004	0.003	0.002	0.001

Little's Law: Average Inventory = Average Flow rate * Average Flow time; or $I = R \times T$.

Economic Production/Order Quantity model to find lot size Q:

R: Demand rate per time. P: Production rate per time. S: Fixed (Setup) cost.
h: Holding cost rate per time. C: Cost of an item.
Average Inventory=$Q/2$. **Length of an Inventory Cycle**=Q/R.

$$\textbf{Total EPQ cost per time} = C(Q; P) = \underbrace{\frac{1}{2}\frac{Q}{P}(P - R)hC}_{\textit{Inventory holding cost per time}} + \underbrace{\frac{SR}{Q}}_{\textit{Set up cost per time}} + RC$$

$$\textbf{EPQ}(P) = \sqrt{\frac{2SR}{(1 - R/P)hC}}. \qquad \textbf{EOQ}(P = \infty) = \sqrt{\frac{2SR}{hC}},$$

where we set production rate P equal to infinity to obtain EOQ. Under $P = \infty$, reorder interval length $T = \sqrt{2S/(RhC)}$ and reorder frequency $n = \sqrt{RhC/(2S)}$.

$$\textbf{Total EOQ cost per time} = C(Q; P = \infty) = \frac{1}{2}QhC + \frac{SR}{Q} + RC$$

With $Q = EOQ$, the total cost $C(Q = EOQ)$ per time becomes $C(Q = EOQ; P = \infty) = \sqrt{2SRh}$.

OPRE 6366 Formulas:

Location/Transportation Module

Transportation Problem - Contracted demand: Given m demand points, $j = 1 \dots m$ each with contracted demand D_j that must be satisfied. Given n supply points, $i = 1 \dots n$ each with capacity K_i. Each unit of shipment from supply point i to demand point j costs c_ij. Decision variable is quantity $x_{ij} \geq 0$ shipped from supply point i to demand point j.

Objective: Min $\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}$.

Demand constraint: $\sum_{i=1}^{n} x_{ij} = D_j$. Equality can be replaced by $\geq$.

Supply constraint: $\sum_{j=1}^{m} x_{ij} \leq K_i$.

Transportation Problem - Profit maximization: Let m_{ij} be the profit made from selling a unit in market j after producing it in plant i and paying for transportation/production costs. There is no contracted demand, but the size of market j is D_j.

Objective: Max $\sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij} x_{ij}$.

Demand constraint: $\sum_{i=1}^{n} x_{ij} \leq D_j$.

Supply constraint as in the Transportation Problem-Contracted demand.

Plant Location Problem with Multiple Sourcing: Cost of operating plant at location i is f_i. Additional binary decision variable: $y_i = 1$ when plant i operates.

Objective: Min $\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} + \sum_{i=1}^{n} f_i y_i$.

Demand constraint as in the Transportation Problem-Contracted demand.

Supply constraint: $\sum_{j=1}^{m} x_{ij} \leq K_i y_i$.

Plant Location Problem with Single Sourcing: New binary demand-supply assignment variable: $z_{ij} = 1$ when demand point j is satisfied from supply point i. In that case, the supply point i ships $D_j z_{ij}$ to demand point j. This handles the demand constraint.

Objective: Min $\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} D_j z_{ij} + \sum_{i=1}^{n} f_i y_i$.

Supply constraint: $\sum_{j=1}^{m} D_j z_{ij} \leq K_i y_i$.

Single sourcing constraint: $\sum_{i=1}^{n} z_{ij} = 1$.

Gravity Location Problem: Given n delivery locations with coordinates (a_i, b_i), $1 \leq i \leq n$ and tonage F_i of shipment from a new location. Find the coordinates of the new location (x, y).

Objective: Min $\sum_{i=1}^{n} F_i \sqrt{(a_i - x)^2 + (b_i - y)^2}$. Let $d_i = \sqrt{(a_i - x)^2 + (b_i - y)^2}$.

Optimality equations:

$$x = \frac{\sum_{i=1}^{n} a_i F_i / d_i}{\sum_{i=1}^{n} F_i / d_i} \quad \text{and} \quad y = \frac{\sum_{i=1}^{n} b_i F_i / d_i}{\sum_{i=1}^{n} F_i / d_i}.$$

Aggregate Planning Module

Inventory Balance Constraint: P_t, D_t are respectively the production and demand in period t. I_t, S_t are respectively inventory and backorders at the end of period t. The balance constraint sets the inputs into period t equl to the outputs: $I_{t-1} + P_t + S_t = D_t + I_t + S_{t-1}$.

Capacity expansion: δ is the rate of demand increase; x is the capacity expansion size; $f(x)$ is the cost of this expansion; r is interest rate.

$$\textbf{Total discounted expansion cost over an infinite horizon} = \sum_{k=0}^{\infty} \exp(-rkx/\delta) f(x) = \frac{f(x)}{1 - \exp(-rx/\delta)}.$$

Flexible capacity with scenarios: i is plant index, j is product index and k is scenario index; d_j^k is demand for product j in scenario k; c_{ij} is tooling cost to configure plant i to produce product j; m_j is contribution to margin of producing a unit of j; r_i is capacity at plant i; p^k is the probability of scenario k. Decision variables: binary y_{ij} if plant i is configured for product j; x_{ij}^k is the amount of product j produced at plant i in scenario k.

$$\begin{aligned}
\max - \sum_{i,j} c_{ij} y_{ij} + \sum_{i,j} p^k m_j x_{ij}^k & \\
\sum_j x_{ij}^k \le r_i & \qquad \text{for each plant } i \quad \text{and for each scenario } k \\
x_{ij}^k \le r_i y_{ij} & \qquad \text{for each plant and product } i, j \text{ and for each scenario } k \\
\sum_i x_{ij}^k \le d_j^k & \qquad \text{for each product } j \quad \text{and for each scenario } k
\end{aligned}$$

Inventory Module

Aggregation of products in a Truck: Consider N products indexed by i. For each product i, R_i is demand rate, s_i is product specific ordering cost, C_i is purchase cost, h_i is holding cost rate per time. Common transportation cost S compensates for the truck and driver regardless of the cargo.

Complete Aggregation of Products in a Truck. Let n be the frequency of truck deliveries.

$$\textbf{Total cost of compelete aggregation per time} \;=\; \frac{1}{2n}\sum_{i=1}^{N} R_i h_i C_i + n\left(\underbrace{S+\sum_{i=1}^{N} s_i}_{=S^*}\right) + \sum_{i=1}^{N} R_i C_i$$

$$\textbf{Optimal complete aggregation frequency } n^* \;=\; \sqrt{\frac{\sum_{i=1}^{N} R_i h_i C_i}{2S^*}}$$

Algorithm for Tailored Aggregation of Products on a Truck.

Step 1: Identify the most frequently ordered product and its frequency n

$$\overline{n}_i = \sqrt{\frac{R_i h_i C_i}{2(S+s_i)}} \text{ for each } i. \text{ Then } n = \max_{1\le i\le N}\{\overline{n}_i\}.$$

Step 2: Identify frequency of other products as a relative multiple m_i

$$\overline{\overline{n}}_i = \sqrt{\frac{R_i h_i C_i}{2s_i}} \text{ for each } i \text{ other than the most frequent. Then } m_i = \left\lceil \frac{n}{\overline{\overline{n}}_i} \right\rceil.$$

Step 3: Recalculate ordering frequency of most frequently ordered product

$$\textbf{Total cost of tailored aggregation per time} \;=\; \frac{1}{2n}\sum_{i=1}^{N} R_i m_i h_i C_i + n\left(S+\sum_{i=1}^{N}\frac{s_i}{m_i}\right) + \sum_{i=1}^{N} R_i C_i$$

$$\textbf{Optimal tailored aggregation frequency } n^* \;=\; \sqrt{\frac{\sum_{i=1}^{N} R_i m_i h_i C_i}{2(S+\sum_{i=1}^{N} s_i/m_i)}}$$

Step 4: Identify ordering frequencies: $n_i = n^*/m_i$ for each product other than the most frequent.

All-Unit Quantity Discount: The cost of each unit in an order of Q is the same. That cost depends on the magnitude of Q as follows.

$$\text{Price per unit} = \left\{\begin{array}{ll} C_1 & \text{if } q_0 \le Q < q_1 \\ C_2 & \text{if } q_1 \le Q < q_2 \\ \dots & \dots \\ C_N & \text{if } q_{N-1} \le Q < q_N \end{array}\right\}$$

where $q_0 = 0$ and $q_N = \infty$. Costs are ordered by $C_{n-1} \ge C_n$ for $2 \le n \le N$. If the order size Q is in region n, i.e., $q_{n-1} \le Q < q_n$, then define the total costs in region n as

$$TC_n(Q) := \frac{1}{2}(QhC_n) + \frac{SR}{Q} + RC_n. \qquad EOQ_n = \sqrt{\frac{2SR}{hC_n}}.$$

Step 1: Find EOQ_n for each region n

Step 2: If $q_{n-1} \le EOQ_n < q_n$, Candidate in this range is $\kappa_n = EOQ_n$.

If $EOQ_n < q_{n-1}$, Candidate in this range is $\kappa_n = q_{n-1}$.

If $EOQ_n \ge q_n$, Candidate in this range is $\kappa_n = q_n$.

Step 3: Optimal order size is κ_n that minimizes $TC_n(\kappa_n)$ over regions $n, 1 \le n \le N$.