OPRE 6302. OM : Newsvendor

1 Obtaining the Empirical Demand Distribution

1. Find forecasted vs. actual demand of all products for a previous period. Calculate each product's A/F ratio: A/F ratio = Actual demand / Forecasted demand.

2. Order all the products by their A/F ratio in an ascending order; then rank them from 1 to N where N is the total number of products.

3. Calculate each product's percentile: Percentile = Rank / N.

4. Multiply each product's A/F ratio by the forecasted demand for the forthcoming season. By plotting the results of these multiplications (x-axis) against the percentiles (y-axis), we obtain the empirical cumulative distribution of the demand. Let $F(Q)$ represent this empirical cumulative distribution. In other words,

$$\begin{aligned} F(Q) &= \mathrm{P}(\text{Demand} \le Q) \\ &= \frac{1}{N}\left[\begin{array}{c}\text{The rank of the product with the} \\ \text{largest (A/F ratio * Forecasted demand)} \\ \text{that is smaller than or equal to } Q\end{array}\right]. \end{aligned}$$

2 Fitting a Normal Distribution to the Empirical Distribution

To specify the Normal Distribution that fits well to the empirical cumulative demand distribuition, we need to come up with the mean μ and the standard deviation σ of the Normal distribution. A logical way is to compute the mean and the standard of the empirical distribution and set them equal to the mean and standard deviation of the Normal distribution. In other words,

$$\begin{aligned} \mu &:= \text{Average of A/F ratios} * \text{Forecasted demand}, \\ \sigma &:= \text{StDev of A/F ratios} * \text{Forecasted demand}. \end{aligned} \tag{1}$$

Since we are working with Normal distribution, let Φ be the cumulative distribution of the Standard Normal Distribution. For the Standard Normal Variate X and a constant a, we have $\mathrm{P}(X \le a) = \Phi(a) = normdist(a, 0, 1, 1)$.

Our objective while fitting a normal distribution to the empirical distribution F is to pick μ and σ appropriately so that

$$F(Q) \approx \Phi\left(\frac{Q-\mu}{\sigma}\right).$$

This is because both $F(Q)$ and $\Phi((Q-\mu)/\sigma)$ must be equal to $\mathrm{P}(\text{Demand} \le Q)$. Since we cannot hope for equality, we can only ask for approximate equality as above. The sense in which approximation is defined is a topic of Statistics. But note that we are talking about approximating one function F with another Φ. this can be done with an objective of the following form

$$\min_{\mu,\sigma \ge 0} \int_0^\infty \left[F(Q) - \Phi\left(\frac{Q-\mu}{\sigma}\right)\right] dQ.$$

3 Finding In-Stock Probability

In-stock probabilty is the probability that the inventory Q suffices for the demand during the season. In other words, we want the probability for the event [Demand $\leq Q$]. This probability is denoted by P(Demand $\leq Q$). We can think of two ways to compute this probability:

i) Use the empirical distribution F, then

$$\text{In-stock probability} = \mathrm{P}(\text{Demand} \leq Q) = F(Q).$$

ii) Use the normal approximation,

ii.a) Use Excel, then

$$\text{In-stock probability} = \mathrm{P}(\text{Demand} \leq Q) = normdist(Q, \mu, \sigma, 1).$$

ii.b) Use standard normal distribution table, then let $z := (Q - \mu)/\sigma)$ and

$$\text{In-stock probability} = \mathrm{P}(\text{Demand} \leq Q) = \Phi(z).$$

In this case, read $\Phi(z)$ from the standard normal tables which is a topic of Statistics.

4 Finding the Optimal Order Quantity

1. Calculate the underage cost per unit, c_u. This is equal to the benefit of selling one unit, or the opportunity cost of not ordering enough units.

$$c_u = \text{selling price - procurement/manufacturing cost.}$$

2. Calculate the overage cost per unit, c_o.

$$c_o = \text{procurement/manufacturing cost - salvage value,}$$

where salvage value = discount price - any holding or transportation costs per unit.

3. Calculate the critical ratio.

$$\text{Critical ratio} = \frac{c_u}{c_o + c_u}.$$

This ratio must also be equal to $F(Q^*)$ to minimize the total cost of underage and overage.

4. We can think of two ways to compute Q^*:

i) Use the empirical distribution F, then

$$Q^* = F^{-1}\left(\frac{c_u}{c_o + c_u}\right)$$

where F^{-1} is the inverse function of F.

ii) Use the normal approximation,

ii.a) Use Excel, then

$$Q^* = norminv\left(\frac{c_u}{c_o + c_u}, \mu, \sigma\right).$$

ii.b) Use standard normal distribution table, then let

$$z = \Phi^{-1}\left(\frac{c_u}{c_o + c_u}\right)$$

where Φ^{-1} is the inverse function of Φ. Whenever you are looking for the critical ratio in the standard normal table and the critical ratio falls between two z entries, choose the entry that leads to the larger order quantity. Then set

$$Q^* = \mu + z\sigma.$$

The concept of the inverse function is a topic of Calculus and must also have appeared in Statistics.

5 Finding the Expected Lost Sales

Let $ESC(Q)$ be the expected shortages in a season if the inventory is Q so

$$ESC(Q) = \mathrm{E}[\max\{D - Q, 0\}].$$

This computation can always be done exactly if D has a discrete demand distribution. This will always be the case if we are using the empirical demand distribution. If you prefer to work with the empirical distribution, first make a list of all demand values that are possible to happen. Let us call them $\{d_1, d_2, \dots, d_N\}$ where $d_1 \leq d_2 \leq \cdots \leq d_N$. These demand values can come from our A/F ratio analysis, in which case each demand value is equally likely to happen. Find the smallest demand value that is greater than Q. If there is no such value, set $ESC(Q) = 0$. If there is such demand value, let its index be i_0. Then

$$ESC(Q) = \frac{1}{N}\sum_{i \geq i_0}(d_i - Q).$$

If demand has a known continuous probability density distribution f, the expected value computation can be done, at least in theory:

$$ESC(Q) = \int_Q^{\infty}(D - Q)f(D)dD.$$

However, taking the integral may not always be possible.

An interesting special case happens when he demand is normally distributed with mean μ and satndard deviation σ. Then set $z = (Q - \mu)/\sigma$ and

$$ESC(Q) = \sigma \underbrace{\left[normdist(z,0,1,0) - z(1 - normdist(z,0,1,1))\right]}_{=:L(z)}.$$

We can think of two ways to compute $L(z)$, called the loss function.

i) Use Excel, then

$$L(z) = normdist(z,0,1,0) - z(1 - normdist(z,0,1,1).$$

ii) Use standard normal distribution loss table, then read $L(z)$ from the table.

6 Solved Exercises

6.1 Statement of Exercises

1. [Walton Bookstore] In August, Walton Bookstore must decide how many of next year's nature calendars should be ordered. Each calendar costs the bookstore \$2 and is sold for \$4.50. After January 1, any unsold calendars are returned to the publisher for a refund of \$0.75 per calendar. Walton believes that the number of calendars sold by January 1 follows the probability distribution shown in following table.

Number of Calendars Sold	100	150	200	250	300
Probability	0.3	0.2	0.3	0.15	0.05

   Walton wants to maximize the expected net profit from calendar sales. How many calendars should the bookstore order in August?

2. [Childcare] UT Dallas allows an employee to put an amount into an account at the beginning of each year, to be used for child-care expenses. This amount is not subject to federal income tax. Assume that all other income is taxed by the federal government at a 40% rate. If this amount can only be used for childcare expenses. If the amount is more than the childcare expenses, the difference is lost. If the childcare expenses are more than the amount, the employee must pay for the excess out of his/her own pocket. This excess payment can be claimed in an income tax return to receive tax credit at the rate of 25% of the excess payment. Prof. Çakanyıldırım believes that his childcare expenses for his son for the coming year will be \$3000, \$4000, \$5000, \$6000, or \$7000 with equal probabilities. At the beginning of the year, how much money should he place in the child-care account?

3. Textbook Question 11.9.

4. [Elevators] At the western side of the SOM, there are two elevators. Whenever one calls for an elevator from a certain floor, it seems like the elevators are at a different floor. An OM student team is commissioned to decide on the optimal floor for both elevators. That is, after carrying people to their destination floor, the elevators will return to their optimal floor to wait for forthcoming people unless thay are already called by some people. The OM student team found out the distribution of the elevator calls coming from the four floors of SOM:

$$\text{P(Call from the 1st floor)} = 0.4,\ \text{P(Call from the 2nd floor)} = 0.1,$$
$$\text{P(Call from the 3rd floor)} = 0.2,\ \text{P(Call from the 4th floor)} = 0.3.$$

   Suppose that the elevators, while moving, has a constant speed of 1 floor every 4 seconds. If the elavators are both placed at the 2nd floor, the expected waiting time for an elevator would be

$$\begin{aligned}\text{E[Waiting time if elevators at the 2nd floor]} &= 4[0.40|2-1| + 0.1|2-2| + 0.2|2-3| + 0.3|2-4|] \\ &= 4[0.4 + 0.2 + 0.6] = 4.8 \text{ seconds.}\end{aligned}$$

   The OM team aims to minimize the waiting time for the forthcoming elevator calls. Cast this problem as a newsvendor problem and find the optimal location of the elevators.

5. [2 Elevators] In Exercise 4, we addressed the optimal location by assuming that both elevators will wait at the same floor. Now we relax this assumption by saying that elavator A can wait at a certain floor while elevator B waits at another floor. It is easy to convince ourselves that we can obtain a better solution by having elevators wait at different floors. What are the optimal floor locations for the elevators?

6.2 Solutions

ANSWER for Exercise 1:

Let q be the number of calendars ordered in August and D be the number of calendars demanded by January 1. If $D \leq q$, the costs shown in following table are incurred where the revenue is negative cost.

For $D \leq q$	cost
Buy q calendars at \$2/calendar	$2q$
Sell D calendars at \$4.50/calendar	-4.50D
Return $q - D$ calendars at \$0.75/calendar	-0.75$(q - D)$
Total cost	$1.25q - 3.75D$

Thus, the overage cost is $c_o = 1.25$ which is the multiplier of q in the total cost above.

If $D \geq q + 1$, the costs are:

For $D \geq q + 1$	cost
Buy q calendars at \$2/calendar	$2q$
Sell D calendars at \$4.50/calendar	-4.50q
Total cost	-2.50q

From the multiplier of q, the underage cost is $c_u = 2.5$. Then

$$\frac{c_u}{c_o + c_u} = \frac{2.50}{3.75} = \frac{2}{3}.$$

Walton should order q^* calendars, where q^* is the smallest number for which $P(D \leq q^*) \geq 2/3$. Note that

$$\mathrm{P}(D \leq 100) = .30, \quad \mathrm{P}(D \leq 150) = 0.50, \quad \mathrm{P}(D \leq 200) = 0.80.$$

Hence, $q^* = 200$ calendars. □

ANSWER for Exercise 2:

Let q be the amount put aside for the childcare and d be the childcare expense. If $d \leq q$, the costs shown in following table are incurred where the revenue is negative cost.

For $d \leq q$	cost
Tax avoided	-0.4q
Lost money	q-d
Total after-tax cost	0.6q-d

Thus, the overage cost is $c_o = 0.6$ which is the multiplier of q in the total cost above.

If $d \geq q$, the costs are:

For $d \geq q$	cost
Tax avoided	-0.4q
Tax credit	-0.25(d-q)
Total after-tax cost	-0.25d-0.15q

From the multiplier of q, the underage cost is $c_u = 0.15$. Then

$$\frac{c_u}{c_o + c_u} = \frac{0.15}{0.75} = \frac{1}{5}.$$

Since P(Childcare expense = 3000) = 0.2, Prof. Çakanyıldırım should put aside 3000 for childcare. □

ANSWER for Exercise 3:

a) If 1200 units are ordered, the z-statistic is $(1200 - 980)/354 = 0.62$. Expected lost sales is $ESC = \sigma * L(Z) = 354 * 0.1633 = 57.81$. The fill rate is 1 - ESC/μ=1-57.81/980=94.1%.

b) The in-stock probability is $\Phi(z) = \Phi(0.62) = 0.7324$.

c) We begin by evaluating the underage and overage costs. The underage cost is the cost of ordering too few units (i.e., under ordering). The selling price is \$40. The variable cost of each pan is $\$16+\$3+\$0.8 = \$19.8$. We do not include the warehouse surcharge in the cost because it is unaffected by the amount we order. Leftover pans are salvaged for \$15 each. Hence, the overage cost is $c_o = 19.8 - 15 = 4.8$. The underage cost is $c_u = 40 - 19.8 = 20.2$. The critical ratio is $\frac{c_u}{c_o+c_u} = \frac{20.2}{4.8+20.2} = 0.808$. From the Standard Normal Cumulative Distribution Function Table we see $\Phi(0.87) = 0.8080$ and $\Phi(0.89) = 0.8106$, so choose $z = 0.89$, and convert to $Q = \mu + z * \sigma = 980 + 0.89 * 354 = 1292$.

d) The overage cost is still $c_o = 19.8 - 15 = 4.8$. The underage cost has changed, it is now $c_u = 40 - 19.8 + 10 = 30.2$. The critical ratio is $\frac{c_u}{c_o+c_u} = 0.8629$. From the Standard Normal Cumulative Distribution Function Table, we see $\Phi(1.09) = 0.8621$ and $\Phi(1.1) = 0.8643$, so choose $z = 1.1$. Finally, $Q = \mu + z * \sigma = 980 + 1.1 * 354 = 1369$. □

ANSWER for Exercise 4:

To cast the elevator problem as a newsvendor problem, we first let Q be the optimal floor and let D be the floor the next call comes from. The empirical distribution of D is given in the problem statement. If $Q > D$, the elevator is located above where the call happens. In this case, the elevator travels $Q - D$ in $4(Q - D)$ seconds. By increasing Q, we can reduce the waiting time by 4 seconds, so the underage cost is $c_u = 4$ seconds. Similarly, the overage cost is $c_o = 4$ seconds. Then the critical ratio is 0.5. In view of the empirical distribution of D, we have $\mathrm{P}(D \le Q) = 0.5$ when $Q = 2$. In other words, the elevator must go to the second floor to wait for forthcoming calls. □

ANSWER for Exercise 5:

Let Q_A and Q_B the waiting location of the elevators. It is safe to assume that when one calls for an elevator, the closest elevator goes to pick that person up. If two elavtors are equdistant from the person, one of the elavators can be sent arbitrarily. For $(Q_A = 1, Q_B = 2)$, we have the waiting time $W(Q_A = 1, Q_B = 2)$, whose expected value can be computed (in units of 4 seconds) as follows.

$$\mathrm{E}W(Q_A = 1, Q_B = 2) = 0.2 * 1 + 0.3 * 2 = 0.8.$$

In general,

$$\begin{aligned}\mathrm{E}W(Q_A, Q_B) &= \sum_{i=1}^{4} \mathrm{P}(D = i) \min\{|i - Q_A|, |i - Q_B|\} \\ &= 0.4 * \min\{|1 - Q_A|, |1 - Q_B|\} + 0.1 * \min\{|2 - Q_A|, |2 - Q_B|\} \\ &\quad + 0.2 * \min\{|3 - Q_A|, |3 - Q_B|\} + 0.3 * \min\{|4 - Q_A|, |4 - Q_B|\}.\end{aligned}$$

Specializing this formula to the specific Q_A and Q_B values, we obtain:

$$\begin{aligned}&\mathrm{E}W(1,3) = 0.1 * 1 + 0.3 * 1 = 0.4, \quad \mathrm{E}W(1,4) = 0.1 * 1 + 0.2 * 1 = 0.3, \\ &\mathrm{E}W(2,3) = 0.4 * 1 + 0.3 * 1 = 0.7, \quad \mathrm{E}W(2,4) = 0.4 * 1 + 0.2 * 1 = 0.6, \\ &\mathrm{E}W(3,4) = 0.4 * 2 + 0.1 * 1 = 0.9.\end{aligned}$$

Among all the floors to place the elevators, $(Q_A = 1, Q_B = 4)$ yields the lowest waiting time. Thus, one of the elevators should wait on the first floor, the other should wait on the fourth floor. □

7 Exercises

1. **[Number of Syllabi]** An instructor has a policy of bringing 25% more copies of syllabus to the first class than the number of pre-registered students. A course attracts discrete uniformly distributed number D of students between 30 and 49 inclusive, that is

$$\mathrm{P}(D = k) = \frac{1}{20} \quad \text{for } k \in \{30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49\}.$$

What is the fill rate in a class with 36 preregistered students? Hint: First The value is $\mathrm{E}\max\{D-Q, 0\}$.

2. **[UTD Bank]** The United (UTD) Bank of Dallas is examining its daily cash position. The UTD bank needs to keep enough cash at its main branch safe in downtown Dallas to meet daily withdrawal needs which are distributed normally with mean \$5000 K and standard deviation \$500 K. If the UTD bank keeps more cash than needed in a day, it loses the opportunity of investing the cash to make a daily return of 0.05%, which roughly corresponds to the internal rate of return of 18% annually. If the UTD bank does not have enough cash, it must immediately buy cash in the interbank market (basically from another bank) with the exorbitant daily interest rate of 1%.
a) What is the cost of having an open cash position by \$100 for a day?
b) What is the cost of having \$100 extra cash in the safe for a day?
c) What is the optimal amount of cash to keep in the safe?

3. **[Call Option against Discrete Valued Price]** Suppose that you own a simple call option for a stock with a strike price of Q. Suppose that the price of the underlying stock is D at the expiration time of the option. If $D < Q$, the call option has no value. Otherwise $D \geq Q$, its value is $D - Q$. What is the expected value of a call option whose strike price is \$45, written for a stock whose price at the expiration is discrete uniformly distributed number D between 30 and 49 inclusive? That is,

$$\mathrm{P}(D = k) = \frac{1}{20} \quad \text{for } k \in \{30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49\}.$$

Hint: The value is $\mathrm{E}\max\{D - Q, 0\}$.

4. **[Put Option against Discrete Valued Price]** Suppose that you own a simple put option for a stock with a strike price of Q. Suppose that the price of the underlying stock is D at the expiration time of the option. If $D \geq Q$, the put option has no value. Otherwise $D < Q$, its value is $Q - D$. What is the expected value of a put option whose strike price is \$45, written for a stock whose price at the expiration is discrete uniformly distributed number D between 30 and 49 inclusive? That is,

$$\mathrm{P}(D = k) = \frac{1}{20} \quad \text{for } k \in \{30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49\}.$$

Hint: The value is $\mathrm{E}\max\{Q - D, 0\}$.

5. **[Call Option against Normally Distributed Price]** Suppose that you own a simple call option for a stock with a strike price of Q. Suppose that the price of the underlying stock is D at the expiration time of the option. If $D < Q$, the call option has no value. Otherwise $D \geq Q$, its value is $D - Q$. What is the expected value of a call option whose strike price is \$50, written for a stock whose price at the expiration is normally distributed with mean \$51 and standard deviation \$10?

6. [**Put Option against Normally Distributed Price**] Suppose that you own a simple put option for a stock with a strike price of Q. Suppose that the price of the underlying stock is D at the expiration time of the option. If $D \geq Q$, the put option has no value. Otherwise $D < Q$, its value is $Q - D$. What is the expected value of a put option whose strike price is \$50, written for a stock whose price at the expiration is normally distributed with mean \$49 and standard deviation \$10?

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