Generalizing the Simplex Method

Up to now, we have solved LPs of the form:

$$\begin{array}{lll} \text{Maximize} & \text{A linear function of variables} & \\ \text{Subject to:} & & \\ & \text{Some linear functions of variables} \leq & \text{Some rhs's} \\ & \text{Variables} \geq 0 & \end{array}$$

1. What if the objective is to "Minimize"?
2. What if some inequalities are **not** "$\leq$"?
   - How to handle "$=$" or "$\geq$"?
3. What if some variables can take negative values?

For example how to solve:

$$\begin{array}{lrrcll} \text{Minimize} & 4x_1 & +x_2 & & & \\ \text{Subject to:} & & & & & \\ & x_1 & +x_2 & = & 3 & (1) \\ & 4x_1 & +3x_2 & \geq & 6 & (2) \\ & x_1 & +2x_2 & \leq & 3 & (3) \\ & & x_1,\, x_2 \geq 0\,. & & & \end{array}$$

Generalizing the Simplex Method: Minimization Problems

Convert Minimization to Maximization by multiplying row 0 by -1:

$$\text{Maximize} \quad -4x_1 - x_2 \,.$$

Then solve:

$$\begin{array}{lrrcrl}
\text{Maximize} & -4x_1 & -x_2 & & & \\
\text{Subject to:} & & & & & \\
 & x_1 & +x_2 & = & 3 & (1) \\
 & 4x_1 & +3x_2 & \geq & 6 & (2) \\
 & x_1 & +2x_2 & \leq & 3 & (3) \\
\end{array}$$

$$x_1,\, x_2 \geq 0 \,.$$

Remark

1. Instead of converting minimization to maximization, you may revise the rule for entering variable and optimality criterion for minimization problems. Enter the nonbasic variable with most positive coefficient in row 0 into the basis. If no positive coefficient, current basis is optimal.

Generalizing the Simplex Method: Fishing for a Basis

We must have a basic variable in each constraint. The proporty of basic variables:

A basic variable appears only once with a coefficient of 1 in a constraint.

Is there a basic variable we can associate with constraint (3)?

$$x_1 + 2x_2 \leq 3 \qquad (3)$$

Simply introduce slack s_3 into (3) as the basic variable to get:

$$x_1 + 2x_2 + s_3 = 3 \qquad (3)$$

We have found a basic variable for constraint (3).

Generalizing the Simplex Method: Fishing for a Basis

Is there a basic variable we can associate with equation (1)?

$$x_1 + x_2 = 3 \qquad (1)$$

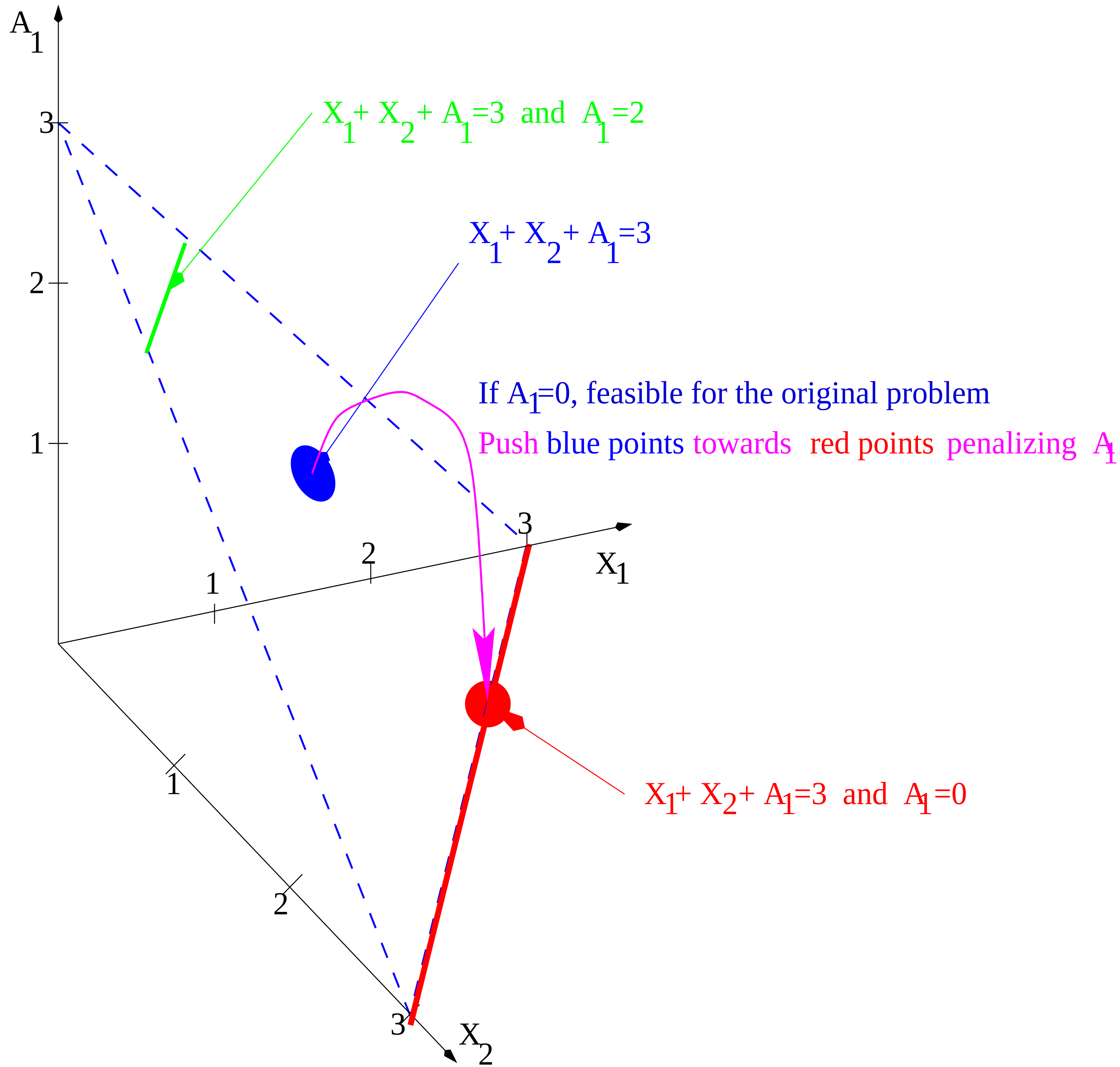
(1) is already an equality. We **cannot** add a slack s_1 as a basic variable.

We need another "simplifying complexication":

Artificially introduce a new nonnegative variable A_1 to temporarily serve as a basic variable for constraint (1). Call A_1 **Artificial variable**?

$$x_1 + x_2 + A_1 = 3 \qquad (1)$$

Add $A_1 \geq 0$ nonnegativity constraint.



A_1
3
2
1
$X_1 + X_2 + A_1 = 3$ and $A_1 = 2$
$X_1 + X_2 + A_1 = 3$
If $A_1 = 0$, feasible for the original problem
Push blue points towards red points penalizing A_1
1
2
3
X_1
1
2
3
X_2
$X_1 + X_2 + A_1 = 3$ and $A_1 = 0$

Generalizing the Simplex Method: Fishing for a Basis

- A_1 is typically positive at the start. The starting basic feasible solution is not feasible to the original problem. We are only interested in solutions that have $A_1 = 0$. We must get rid of the artificial variable.

- Penalize any positive A_1 with the objective function by adding say $1000A_1$. Since A_1 is artifical, so is its row 0 coefficient, call it M. As the Simplex searches for a solution with the smallest objective function value, it will avoid solutions with positive A_1. *Big-M method*?

Consider

1. $(x_1, x_2, A_1) = (1, 2, 0)$ with objective value $4 \cdot 1 + 1 \cdot 2 + M \cdot 0 = 6$

2. $(x_1, x_2, A_1) = (1, 1, 1)$ with objective value $4 \cdot 1 + 1 \cdot 1 + M \cdot 1 = 5 + M$

They satisfy the revised equation (1). Which is larger?

Generalizing the Simplex Method: Fishing for a Basis

Consider (2), the "$\geq$" type constraint. To create an equality, we will subtract a nonnegative *surplus variable* from the lhs of that constraint and set it equal to the rhs constant.

$$4x_1 + 3x_2 - s_2 = 6 \qquad (2)$$

Any candidate basic variables in equation (2)? Can s_2 be a basic variable?

Need to introduce A_2 to equation (2)

$$4x_1 + 3x_2 - s_2 + A_2 = 6 \qquad (2)$$

Do we have to modify the objective function by adding MA_2?

Generalizing the Simplex Method: Fishing for a Basis

Finally we obtain an LP which is likely to have a basis:

$$
\begin{array}{lrrrrrrrcrr}
\text{Minimize} & 4x_1 & +x_2 & +MA_1 & & +MA_2 & & & & & \\
\text{Subject to:} & & & & & & & & & & \\
 & x_1 & +x_2 & +A_1 & & & & = & 3 & & (1) \\
 & 4x_1 & +3x_2 & & -s_2 & +A_2 & & = & 6 & & (2) \\
 & x_1 & +2x_2 & & & & +s_3 & = & 3 & & (3)
\end{array}
$$

$$x_1,\, x_2,\, A_1,\, s_2,\, A_2,\, s_3 \geq 0\,.$$

What are the basic variables?

Generalizing the Simplex Method: Fishing for a Basis

Construct row 0 :

$$z \quad -4x_1 \quad -x_2 \quad -MA_1 \qquad -MA_2 \qquad = \quad 0\,.$$

Row 0 is not in the standard form: A_1 and A_2, target basic variables are in row 0.

Eliminate basic variables:

1. Multiply equation (1) by M and add onto equation (0) to eliminate MA_1.

2. Multiply equation (2) by M and add onto equation (0) to eliminate MA_1.

$$\begin{array}{cccccccccc}
z & -4x_1 & -x_2 & -MA_1 & & -MA_2 & = & 0 & \\
 & Mx_1 & +Mx_2 & +MA_1 & & & = & 3M & M\cdot(1) \\
 & 4Mx_1 & +3Mx_2 & & -Ms_2 & +MA_2 & = & 6M & M\cdot(2) \\
-- & --- & --- & --- & --- & --- & - & --- & \\
z & +(5M-4)x_1 & +(4M-1)x_2 & & -Ms_2 & & = & 9M &
\end{array}$$

Generalizing the Simplex Method: Fishing for a Basis

Min z

ST:

$$
\begin{array}{rrrrrrrrrl}
z & +(5M-4)x_1 & +(4M-1)x_2 & & -Ms_2 & & & = & 9M & (0) \\
 & x_1 & +x_2 & +A_1 & & & & = & 3 & (1) \\
 & 4x_1 & +3x_2 & & -s_2 & +A_2 & & = & 6 & (2) \\
 & x_1 & +2x_2 & & & & +s_3 & = & 3 & (3)
\end{array}
$$

$$x_1,\ x_2,\ A_1,\ s_2,\ A_2,\ s_3 \geq 0\,,$$

Basic Variable	z	x_1	x_2	A_1	s_2	A_2	s_3	
	1	$5M-4$	$4M-1$	0	$-M$	0	0	$9M$
A_1	0	1	1	1	0	0	0	3
A_2	0	4	3	0	-1	1	0	6
s_3	0	1	2	0	0	0	1	3

What are the current basic variables, their values and the objective value?

Should I stay or shoul I go? Pivot column? Pivot row? Pivot element?

Generalizing the Simplex Method: Big-M Method

Execute a pivot: Replace

- R_0 with $[-(5M-4)/4] \cdot R_2 + R_0$,
- R_1 with $(-1/4) \cdot R_2 + R_1$,
- R_2 with $(1/4) \cdot R_2$ and
- R_3 with $(-1/4) \cdot R_2 + R_3$.

Basic Variable	z	x_1	x_2	A_1	s_2	A_2	s_3	
	1	0	$M/4+2$	0	$M/4-1$	$-5M/4+1$	0	$3M/2+6$
A_1	0	0	$1/4$	1	$1/4$	$-1/4$	0	$3/2$
x_1	0	1	$3/4$	0	$-1/4$	$1/4$	0	$3/2$
s_3	0	0	$5/4$	0	$1/4$	$-1/4$	1	$3/2$

Should I stay or should I go?

Generalizing the Simplex Method: Big-M Method Remarks

1. You can delete any artificial variables as soon as they are out of basis.

2. Any solution with a positive artificial value is not feasible to the original problem. The bfs associated with the above tableau $(x_1, x_2, s_2, A_2, s_3) = (1, 0, 0, 2, 2)$ is not feasible.

3. If the algorithm terminates with an optimal solution that has at least one of the artificial variables being positive, then the original problem is **infeasible**.

4. What is M?

5. If our objective is to **Maximize** $4x_1 + x_2$, instead of introducing MA_1 and MA_2 into the objective function, we should introduce $-MA_1$ and $-MA_2$.

Generalizing the Simplex Method: Unrestricted and Negative Variables

Consider the linear program:

$$
\begin{array}{llrrrcrl}
\text{Minimize} & & 2x_1 & +3x_2 & -x_3 & & & \\
\text{Subject to:} & & & & & & & \\
& & x_1 & -x_2 & +2x_3 & \leq & 10 & (1) \\
& & -3x_1 & +2x_2 & -4x_3 & = & 6 & (2) \\
& & x_1 & +9x_2 & & \geq & 7 & (3)
\end{array}
$$

$x_1 \leq 0$, $x_2 \geq 0$, x_3 is unrestricted.

x_1 is nonpositive and the variable x_3 is unrestricted in sign (urs). First convert the problem into one where all variables are nonnegative.

Handling nonpositive variables is easy. $x_1' \equiv -x_1$.

Generalizing the Simplex Method: Unrestricted Variables

x_3 is urs. We can introduce two nonnegative variables: x_3^+ and x_3^-, and let $x_3 \equiv x_3^+ - x_3^-$.

If $x_3^+ = 8$ and $x_3^- = 3$, $x_3 = 5$. But there are other ways of having $x_3 = 5$.

1. $x_3^+ = 9$ and $x_3^- = 4$, $x_3 = 5$

2. $x_3^+ = 5$ and $x_3^- = 0$, $x_3 = 5$

Generalizing the Simplex Method: Unrestricted Variables

Execute conversions for x_1 and x_3 to obtain an equivalent linear program:

$$
\begin{array}{llllllll}
\text{Minimize} & -2x_1' & +3x_2 & -x_3^+ & +x_3^+ & & & \\
\text{Subject to:} & & & & & & & \\
 & -x_1' & -x_2 & +2x_3^+ & -2x_3^+ & \leq & 10 & (1) \\
 & 3x_1' & +2x_2 & -4x_3^+ & +4x_3^+ & = & 6 & (2) \\
 & -x_1' & +9x_2 & & & \geq & 7 & (3) \\
 & \multicolumn{7}{l}{x_1' \geq 0,\ x_2 \geq 0,\ x_3^+ \geq 0,\ x_3^- \geq 0\,.}
\end{array}
$$

To convert the new problem's optimal solution into one for the original problem. Consider $(x_1', x_2, x_3^+, x_3^-) = (1, 1, 1, 1)$. Since $x_1 = -x_1'$ and $x_3 = x_3^+ - x_3^-$, the corresponding solution for the original problem can be reconstructed as $(x_1, x_2, x_3) = (-1, 1, 0)$.

Can we have multiple optimal solutions such as $(x_1', x_2, x_3^+, x_3^-) = (1, 1, 1, 1)$ and $(x_1', x_2, x_3^+, x_3^-) = (1, 1, 2, 2)$ all representing $x_3 = 0$?

Generalizing the Simplex Method: Unrestricted Variables

Observe that in the new problem, terms involving x_3^+ and x_3^- always appear in the form of a pair, with opposite signs. If x_3^+ is positive, it is in the basis. It has coefficient of $+1$ in an equation. In that equation x_3^- has a coefficient of -1 (why?). Then x_3^- is nonbasic and zero.

More generally, a variable may sometimes have a given constant upper bound. For example, suppose the requirement $x_1 \leq 0$ is revised to $x_1 \leq 3$. Write $x_1 - 3 \leq 0$ and let $\bar{x}_1 \equiv x_1 - 3$, then $\bar{x}_1$ is a nonpositive variable.

Unboundedness

Consider the linear program:

$$\begin{array}{lrrcrr}
\text{Maximize} & 2x_1 & +x_2 & & & \\
\text{Subject to:} & & & & & \\
 & x_1 & -x_2 & \leq & 10 & (1) \\
 & 2x_1 & -x_2 & \leq & 40 & (2) \\
 & \multicolumn{4}{c}{x_1,\, x_2 \geq 0\,.} &
\end{array}$$

After introducing two slack variables and setting up the objective function, we obtain the following initial Simplex tableau.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	−2	−1	0	0	0
s_1	0	1	−1	1	0	10
s_2	0	2	−1	0	1	40

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	−3	2	0	20
x_1	0	1	−1	1	0	10
s_2	0	0	1	−2	1	20

Unboundedness

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	0	-4	3	80
x_1	0	1	0	-1	1	30
x_2	0	0	1	-2	1	20

i) We can move along a direction inside the feasible region as much as we want .
ii) The objective value improves along the direction.

Unbounded LP means that the feasible region is unbounded. Unbounded feasible region **does not necessarily** mean unbounded LP.

In most applications of linear programming, a problem turns out to be unbounded when a constraint has been left out.

Multiple Optimal Solutions

Consider the linear program and associated tableaus:

$$\begin{array}{llllll}
\text{Maximize} & 4x_1 & +14x_2 & & & \\
\text{Subject to:} & & & & & \\
 & 2x_1 & +7x_2 & \leq & 21 & (1) \\
 & 7x_1 & +2x_2 & \leq & 21 & (2) \\
\end{array}$$

$$x_1,\, x_2 \geq 0\,.$$

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	-4	-14	0	0	0
s_1	0	2	7	1	0	21
s_2	0	7	2	0	1	21

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	0	2	0	42
x_2	0	$2/7$	1	$1/7$	0	3
s_2	0	$45/7$	0	$-2/7$	1	15

Optimal? What is the corresponding solution $(x_1,\, x_2,\, s_1,\, s_2)$?

Multiple Optimal Solutions

The nonbasic variable x_1 has a coefficient of 0 in R_0. This implies that if we attempt to let x_1 enter the basis, then the objective-function value will not change.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	0	2	0	42
x_2	0	0	1	$7/45$	$-2/45$	$7/3$
x_1	0	1	0	$-2/45$	$7/45$	$7/3$

Optimal? What is the corresponding solution (x_1, x_2, s_1, s_2)?

Geometric interpretation. Multiple optimal solutions if objective function is paralel to one of the boundaries of the feasible region.

Degeneracy

Consider the linear program and the initial tableau:

$$
\begin{array}{llllll}
\text{Maximize} & 2x_1 & +x_2 & & & \\
\text{Subject to:} & & & & & \\
 & 4x_1 & +3x_2 & \le & 12 & (1) \\
 & 4x_1 & +x_2 & \le & 8 & (2) \\
 & 4x_1 & +2x_2 & \le & 8 & (3) \\
 & x_1,\ x_2 \ge 0\,. & & & &
\end{array}
$$

Basic Variable	z	x_1	x_2	s_1	s_3	s_3	
	1	-2	-1	0	0	0	0
s_1	0	4	3	1	0	0	12
s_3	0	4	1	0	1	0	8
s_3	0	4	2	0	0	1	8

Tie in the minimum ratio so after the pivot a basic variable will be 0 (degeneracy).

Degeneracy

Tableau I:

Basic Variable	z	x_1	x_2	s_1	s_3	s_3	
	1	0	$-1/2$	0	$1/2$	0	4
s_1	0	0	2	1	-1	0	4
x_1	0	1	$1/4$	0	$1/4$	0	2
s_3	0	0	1	0	-1	1	0

The current bfs is $(x_1, x_2, s_1, s_2, s_3) = (2, 0, 4, 0, 0)$, s_3 is degenerate.

Tableau II:

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	
	1	0	0	0	0	$1/2$	4
s_1	0	0	0	1	1	-2	4
x_1	0	1	0	0	$1/2$	$-1/4$	2
x_2	0	0	1	0	-1	1	0

The same bfs, what changed?

Geometric interpretation: Three, not two, constraint equations pass through bfs $(x_1, x_2) = (2, 0)$. They are: $x_2 = 0$, $4x_1 + x_2 = 8$, and $4x_1 + 2x_2 = 8$. 3 zeros in the augmented solution, although $n - m = 2$ zeros suffice.

Cycling Example

Is it possible for the Simplex algorithm to cycle through tableaus forever?

$$\begin{array}{llllllllll}
\text{Maximize} & & & & 3x_4/4 & -20x_5 & x_6/2 & -6x_7 & & \\
\text{Subject to:} & & & & & & & & & \\
& x_1 & & & +x_4/4 & -8x_5 & -x_6 & +9x_7 & = 0 & (1) \\
& & x_2 & & +x_4/2 & -12x_5 & -x_6/2 & +3x_7 & = 0 & (2) \\
& & & x_3 & & & x_6 & & = 1 & (3)
\end{array}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \ .$$

Since this problem has the evident starting basis $\{x_1, x_2, x_3\}$, we can directly setup the Simplex tableau and proceed as follows:

Cycling Example

Tableau I:

Basic Variable	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
	1	0	0	0	-3/4	20	-1/2	6	0
x_1	0	1	0	0	1/4	-8	-1	9	0
x_2	0	0	1	0	1/2	-12	-1/2	3	0
x_3	0	0	0	1	0	0	1	0	1

Tableau II:

Basic Variable	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
	1	3	0	0	0	-4	-7/2	33	0
x_4	0	4	0	0	1	-32	-4	36	0
x_2	0	-2	1	0	0	4	3/2	-15	0
x_3	0	0	0	1	0	0	1	0	1

Tableau III:

Basic Variable	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
	1	1	1	0	0	0	-2	18	0
x_4	0	-12	8	0	1	0	8	-84	0
x_5	0	-1/2	1/4	0	0	1	3/8	-15/4	0
x_3	0	0	0	1	0	0	1	0	1

Cycling Example

Tableau IV:

Basic Variable	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
	1	-2	3	0	1/4	0	0	-3	0
x_6	0	-3/2	1	0	1/8	0	1	-21/2	0
x_5	0	1/16	-1/8	0	-3/64	1	0	3/16	0
x_3	0	3/2	-1	1	-1/8	0	0	31/2	1

Tableau V:

Basic Variable	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
	1	-1	1	0	-1/2	16	0	0	0
x_6	0	2	-6	0	-5/2	56	1	0	0
x_7	0	1/3	-2/3	0	-1/4	16/3	0	1	0
x_3	0	-2	6	1	5/2	-56	0	0	1

Tableau VI:

Basic Variable	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
	1	0	-2	0	-7/4	44	1/2	0	0
x_1	0	1	-3	0	-5/4	28	1/2	0	0
x_7	0	0	1/3	0	1/6	-4	-1/6	1	0
x_3	0	0	0	1	0	0	1	0	1

In Tableau VI, x_2 enters into the basis and x_7 leaves and we are back to Tableau I!!

Cycling happens with Degeneracy

Simplex cycles very rarely. Ii is a challenge to construct another example that cycles.

Can Simplex ever cycle if the objective function value is increasing in every iteration? Objective value stays constant only in degenerate LPs.

Interestingly, every degenerate LP does not cycle.

Cycling prevention: the **least index rule**: Among those variable eligible to enter (or to leave) the basis choose the smallest index variable.