OPRE 6201 : 3. Special Cases

1 Initialization: The Big-M Formulation

Consider the linear program:

$$
\begin{array}{lrrcrl}
\text{Minimize} & 4x_1 & +x_2 & & & \\
\text{Subject to:} & & & & & \\
 & 3x_1 & +x_2 & = & 3 & (1) \\
 & 4x_1 & +3x_2 & \geq & 6 & (2) \\
 & x_1 & +2x_2 & \leq & 3 & (3)
\end{array}
$$

$$x_1,\, x_2 \geq 0\,.$$

Notice that there are several new features in this problem, namely: (i) the objective is to minimize; (ii) the first constraint is in equality form, but it does not have a candidate basic variable; and (iii) the second constraint is of the "$\geq$" type.

Two approaches are commonly adopted for the handling of minimization objective functions. The first is to convert the given objective function into one that is to be maximized. This is done by multiplying the original objective function by -1 and then maximizing the resulting expression. For this example, this means that we can replace the given objective function by:

$$\text{Maximize} \quad -4x_1 - x_2\,.$$

The second approach is to revise the optimality criterion in the Simplex algorithm. For maximization problems, recall that the optimality criterion is that if the coefficients of all nonbasic variables in the zeroth row of a Simplex tableau are nonnegative, then its associated basic feasible solution is optimal. If, instead, we are trying to minimize, then it is not difficult to see that we could simply revise the just-stated criterion into one that has the word "nonnegative" replaced by the word "nonpositive." Moreover, if the current solution is not optimal, then we should select the nonbasic variable that has the most-positive coefficient in the zeroth row as the entering variable. Note, however, that the remaining aspects of the Simplex algorithm, the ratio test in particular, do not require any revision. In our solution of this linear program, we will adopt the second approach. Hence, no action is necessary at this point.

Suppose a given constraint is in equality form with a nonnegative right-hand-side constant, as in equation (1) above. (If the right-hand-side constant of an equation is negative, the entire equation can be multiplied by -1 to convert the constant to a positive number.) What we need to do is to find out whether or not the constraint contains a candidate basic variable. If such a variable can be found, we simply declare it as the basic variable associated with that equation and move on to other constraints. In the event that such a variable does not exist, the idea is to artificially introduce a new nonnegative variable to serve as the basic variable associated with that equation. Such a variable will be called an *artificial variable*. Since constraint (1) above does not contain a candidate basic variable, we will revise it by introducing an artificial variable, which we denote by A_1; this results in

$$3x_1 \quad +x_2 \quad +A_1 \quad = \quad 3\,.$$

We will also add $A_1 \geq 0$ into the set of nonnegativity constraints.

It is important to realize that with the introduction of an artificial variable, the resulting new equation is *not* equivalent to the original. This is because if the artificial variable assumes a positive value, then any solution that satisfies the new equation won't satisfy the original equation. For example, if we let $A_1 = 2$ in the above equation, then, since $3x_1 + x_2 + A_1 = 3$, we must have $3x_1 + x_2 = 1$, which is in contradiction with the original equation $3x_1 + x_2 = 3$. Now, as A_1 is one of the starting basic variables, it will (typically) assume a positive value at the beginning of the Simplex iterations. Hence, the starting basic feasible solution will not be a feasible solution to the original problem. Since our aim is to derive an optimal solution that satisfies the original equality constraint, not the revised constraint, we are, in the end, only interested in solutions that have $A_1 = 0$. Therefore, an important question is: How do we get rid of this artificial variable?

One answer to this question is that we can introduce a new term MA_1, where M is a "sufficiently large" constant, into the objective function. The idea behind this approach, which is naturally called the *big-M method*, is that although the value of A_1 may be positive initially, but with this added term in the objective function, any solution that has a positive A_1 will have an associated objective-function value that is exceedingly large. Hence, as the Simplex algorithm performs its search for a solution that has the smallest objective function value, it will systematically discard or avoid solutions that have a positive A_1. In other words, the Simplex algorithm will, by design, attempt to converge to solutions that have $A_1 = 0$, and hence are feasible to the original problem.

As a numerical example, consider the solutions $(x_1, x_2, A_1) = (1/3, 2, 0)$ and $(x_1, x_2, A_1) = (1/3, 1, 1)$, which satisfy both the original and the revised equation (1). The corresponding objective-function values of these two solutions can be evaluated as: $4 \cdot (1/3) + 1 \cdot 2 + M \cdot 0 = 10/3$ and $4 \cdot (1/3) + 1 \cdot 1 + M \cdot 1 = (7/3) + M$. Notice that the outcome of the first evaluation does not involve M. It follows that the second evaluation will have a greater value, provided that M is sufficiently large (any M that is strictly greater than 1, specifically). Since any solution with $A_1 = 0$ has a smaller objective function value than any other solution with $A_1 > 0$ (when M is sufficiently large), the Simplex algorithm will attempt to weed out any solution with a positive A_1.

We now turn our attention to constraint (2), which is of the "$\geq$" type. To create an equivalent equality, we will reverse what we do in the case of a "$\leq$" constraint. That is, we will subtract a nonnegative *surplus variable*, denoted by s_1, from the left-hand side of that constraint and let the resulting expression equal to the right-hand-side constant. This yields

$$4x_1 \quad +3x_2 \quad -s_1 \quad = \quad 6\,.$$

Next, observe that none of the three variables on the left-hand side of this new equation can serve as a candidate basic variable. Therefore, similar to what we did in equation (1), we will further revise this equation by introducing another artificial variable, which we denote by A_2. This results in

$$4x_1 \quad +3x_2 \quad -s_1 \quad +A_2 \quad = \quad 6\,.$$

In addition, we will also introduce a new term MA_2 into the objective function.

Finally, since the last constraint is of the "$\leq$" type, we simply add a slack variable, denoted by s_2, to its left-hand side to convert it into an equality. In summary, we have converted the given linear program into the following form:

$$\begin{array}{llllllllll}
\text{Minimize} & 4x_1 & +x_2 & +MA_1 & & +MA_2 & & & & \\
\text{Subject to:} & & & & & & & & & \\
& 3x_1 & +x_2 & +A_1 & & & & = & 3 & (1) \\
& 4x_1 & +3x_2 & & -s_1 & +A_2 & & = & 6 & (2) \\
& x_1 & +2x_2 & & & & +s_2 & = & 3 & (3)
\end{array}$$

$$x_1,\, x_2,\, A_1,\, s_1,\, A_2,\, s_2 \geq 0\,.$$

At this point, the objective function is still not in conformance with the standard form. Following what we did in our first example, we now define a new variable z to serve as our objective function, which is to be minimized; and we will introduce a zeroth constraint, namely

$$z \quad -4x_1 \quad -x_2 \quad -MA_1 \qquad -MA_2 \qquad = \quad 0\,,$$

into the constraint set.

Furthermore, notice that the artificial variables A_1 and A_2, which are targeted to serve as basic variables in equations (2) and (3), also participate in this new constraint. Since this is not allowed in the standard form, we will have to eliminate them. This is done in two steps. First, we multiply equation (1) by M and add the outcome into equation (0); this eliminates MA_1. Next, we repeat the same operation with equation (2) to eliminate MA_2. These two steps, together, yield

$$z \quad +(7M-4)x_1 \quad +(4M-1)x_2 \qquad -Ms_1 \qquad = \quad 9M\,.$$

With these further revisions, we finally arrive at

$$\begin{array}{llllllllll}
\text{Minimize} & z & & & & & & & & \\
\text{Subject to:} & & & & & & & & & \\
& z & +(7M-4)x_1 & +(4M-1)x_2 & & -Ms_1 & & & = 9M & (0) \\
& & 3x_1 & +x_2 & +A_1 & & & & = 3 & (1) \\
& & 4x_1 & +3x_2 & & -s_1 & +A_2 & & = 6 & (2) \\
& & x_1 & +2x_2 & & & & +s_2 & = 3 & (3) \\
& & \multicolumn{6}{l}{x_1,\ x_2,\ A_1,\ s_1,\ A_2,\ s_2 \geq 0\,,} & &
\end{array}$$

which is now ready for the Simplex algorithm.

In tabular form, the above problem becomes:

Basic Variable	z	x_1	x_2	A_1	s_1	A_2	s_2	
	1	$7M-4$	$4M-1$	0	$-M$	0	0	$9M$
A_1	0	3	1	1	0	0	0	3
A_2	0	4	3	0	-1	1	0	6
s_2	0	1	2	0	0	0	1	3

Notice that the introduction of the artificial variables allows us to conveniently declare the basis associated with this tableau as A_1, A_2, and s_2 (listed on the left margin). Therefore, the initial basic feasible solution is $(x_1,\ x_2,\ A_1,\ s_1,\ A_2,\ s_2) = (0,\ 0,\ 3,\ 0,\ 6,\ 3)$, with a corresponding objective-function value of $9M$. Since M is "big," the coefficients of x_1 and x_2 in R_0, namely $7M-4$ and $4M-1$, are both positive, implying that the current solution is not optimal. Moreover, a big M also implies that $7M-4$ is strictly larger than $4M-1$. Hence, x_1 is the entering variable, and the x_1-column is the pivot column.

A comparison of the three ratios 3/3, 6/4, and 3/1 shows that R_1 is the pivot row, and hence A_1 is the leaving variable. This also identifies the entry "3," located at the intersection of the pivot column and the pivot row, as the pivot element.

We now execute a pivot. An examination of the x_1-column shows that we need to go through the following row operations: $[-(7M-4)/3] \cdot R_1 + R_0$, $(1/3) \cdot R_1$, $(-4/3) \cdot R_1 + R_2$, and $(-1/3) \cdot R_1 + R_3$. These four sets of operations will produce new versions of equations (0), (1), (2), and (3), respectively; and these equations constitute the new tableau below.

Basic Variable	z	x_1	x_2	A_1	s_1	A_2	s_2	
	1	0	$(5M+1)/3$	$-(7M-4)/3$	$-M$	0	0	$2M+4$
x_1	0	1	$1/3$	$1/3$	0	0	0	1
A_2	0	0	$5/3$	$-4/3$	-1	1	0	2
s_2	0	0	$5/3$	$-1/3$	0	0	1	2

Since the coefficient of x_2 in R_0 is positive, this tableau is not optimal, and hence more iterations are necessary. We will not complete the remaining iterations, since they are now straightforward.

1.1 Remarks

1. At the end of the above iteration, the artificial variable A_1 is no longer in the basis; that is, its value has been driven to zero by the algorithm. Since artificial variables are not part of the original problem, they can be discarded from further consideration as soon as they leave the basis. This serves to reduce the amount of computation. In fact, we can do even better by discarding the leaving artificial variable (A_1, in this example) at the start (as opposed to the end) of the pivot. One should be careful not to discard any of the original variables, however.

2. Any solution that contains a positive value for any of the artificial variable is not feasible to the original problem. For example, the basic feasible solution associated with the above tableau is $(x_1, x_2, s_1, A_2, s_2) = (1, 0, 0, 2, 2)$, where we have removed A_1. Since $A_2 = 2$ is positive, the current solution is not feasible to the original problem. This will continue to be the case until all artificial variables are driven out.

3. In general, it is possible for a given linear program not to have any feasible solution. In such a case, the Simplex algorithm will not be able to succeed in driving out all of the artificial variables. Thus, if the algorithm terminates with an optimal solution that has at least one of the artificial variables being positive, then the original problem is *infeasible.*

4. Throughout our computation, we did not assign a specific value for M. We simply treated it as a large number *operationally.* This means that whenever M is compared against another number, we will let M be the larger of the two. This seems convenient, but can pose a challenge in a computer implementation of the algorithm.

5. If our objective is to *maximize* $4x_1 + x_2$, then, instead of introducing MA_1 and MA_2 into the objective function, we should introduce $-MA_1$ and $-MA_2$.

2 Unrestricted and Other Variable Types

Consider the linear program:

$$\begin{array}{llllll}
\text{Minimize} & 2x_1 & +3x_2 & -x_3 & & \\
\text{Subject to:} & & & & & \\
& x_1 & -x_2 & +2x_3 & \leq 10 & (1) \\
& -3x_1 & +2x_2 & -4x_3 & = 6 & (2) \\
& x_1 & +9x_2 & & \geq 7 & (3)
\end{array}$$

$$x_1 \leq 0,\ x_2 \geq 0,\ x_3 \text{ is unrestricted.}$$

In this problem, the variable x_1 is nonpositive and the variable x_3 is unrestricted in sign. Before the launch of the Big-M procedure, we need to first convert the problem into one where all variables are nonnegative.

Handling nonpositive variables is quite easy. The idea is to define a new variable that equals the negative of the original variable. In this case, we can denote the new variable by x_1' (say) and let $x_1' \equiv -x_1$. Since the original variable is nonpositive, it follows that the new variable x_1' is nonnegative. To implement this change of variable, we simply replace every instance of x_1 in the given problem by $-x_1'$.

More generally, a variable may sometimes have a given constant upper bound. For example, suppose the requirement $x_1 \leq 0$ is revised to $x_1 \leq 3$. Observe that the latter can be rewritten as $x_1 - 3 \leq 0$, which suggests that if we define $\bar{x}_1 \equiv x_1 - 3$, then $\bar{x}_1$ is a nonpositive variable. Hence, we can replace every instance of x_1 in the given problem by $\bar{x}_1 + 3$ to create an equivalent problem with $\bar{x}_1 \leq 0$. Notice that doing so will create an extra constant in the objective function. The constant will not have any impact on the Simplex algorithm, but it is recommended that it be set aside temporarily until the end of the solution procedure. Of course, a variable with a constant lower bound, say $x_1 \geq 5$, can be handled similarly: Replace every instance of x_1 by $\bar{x}_1 + 5$, where $\bar{x}_1 \geq 0$.

We now turn our attention to the handling of unrestricted variables, x_3 in this case. The idea is to rewrite such a variable as the difference between two nonnegative variables. That is, we can introduce two nonnegative variables, say x_3^+ and x_3^-, and let $x_3 \equiv x_3^+ - x_3^-$.

To see how this works, consider a numerical example. Suppose the value of x_3 is to equal 5; then, the assignments $x_3^+ = 8$ and $x_3^- = 3$ will yield a difference of 5. Such assignments are clearly not unique. Other examples are: $x_3^+ = 18$ and $x_3^- = 13$; and $x_3^+ = 11$ and $x_3^- = 6$. In fact, such assignments are innumerable. Of particular interest is the assignments $x_3^+ = 5$ and $x_3^- = 0$. Notice that in this case, we have $x_3^+ = \max[x_3, 0]$ and $x_3^- = -\min[x_3, 0]$. When these two relations hold, we say that x_3^+ is the *positive part* of x_3 and x_3^- is the *negative part* of x_3. Thus, if the variable x_3 equals 5, then its positive part is 5 and its negative part is 0.

As another numerical example, suppose x_3 is equal to -5. Then, the positive part of x_3, denoted by x_3^+, equals 0; and the negative part of x_3, denoted by x_3^-, equals 5. Indeed, $x_3 = 0 - 5 = -5$; and both the positive part and the negative part are nonnegative.

We now execute the proposed conversions for x_1 and x_3. With x_1 replaced by $-x_1'$ and x_3 replaced by $x_3^+ - x_3^-$, the given problem is equivalent to the linear program below.

$$
\begin{array}{llllllll}
\text{Minimize} & -2x_1' & +3x_2 & -x_3^+ & +x_3^- & & & \\
\text{Subject to:} & & & & & & & \\
 & -x_1' & -x_2 & +2x_3^+ & -2x_3^- & \leq & 10 & (1) \\
 & 3x_1' & +2x_2 & -4x_3^+ & +4x_3^- & = & 6 & (2) \\
 & -x_1' & +9x_2 & & & \geq & 7 & (3) \\
 & \multicolumn{7}{l}{x_1' \geq 0,\ x_2 \geq 0,\ x_3^+ \geq 0,\ x_3^- \geq 0\,.}
\end{array}
$$

Thus, the new problem has 4 variables, all of which are nonnegative; and it is now ready for the application of the Big-M procedure.

After solving the new problem, it is very easy to convert its optimal solution into one for the original problem. As an example, consider an arbitrarily chosen feasible solution to the new problem, say $(x_1', x_2, x_3^+, x_3^-) = (1, 1, 1, 1)$. Since $x_1 = -x_1'$ and $x_3 = x_3^+ - x_3^-$, the corresponding feasible solution for the original problem can be reconstructed as $(x_1, x_2, x_3) = (-1, 1, 0)$. A similar conversion can, of course, be executed for the optimal solution.

Finally, we observe that in the new problem, terms involving x_3^+ and x_3^- always appear in the form of a pair, with opposite signs. This property has an interesting consequence. Recall that all of the operations in the Simplex algorithm are *row* operations. This implies that the above property will be preserved throughout the Simplex iterations. Now, suppose x_3^+ is a basic variable in a tableau. Then, since every column associated with a basic variable must have its entries equal to 0 except for one that equals 1, it follows that the x_3^--column assumes the form

x_3^-
0
0
$\vdots$
0
-1
0
$\vdots$
0

where the coefficient -1 is located at some row. Clearly, the variable associated with such a column cannot be basic. Therefore, if x_3^+ happens to be basic, then x_3^- must be nonbasic; and conversely, if x_3^- happens to be basic, then x_3^+ must be nonbasic. Of course, both of these variables can be nonbasic at the same time. In summary, the conclusion is that, in any basic feasible solution, at most one of the two values assigned to x_3^+ and x_3^- can be positive. Remarkably, this is in complete agreement with the (intended) interpretation of x_3^+ and x_3^- as the positive part and the negative part of the variable x_3.

3 Unboundedness

Consider the linear program:

$$
\begin{array}{llrcrl}
\text{Maximize} & 2x_1 & +x_2 & & & \\
\text{Subject to:} & & & & & \\
& x_1 & -x_2 & \leq & 10 & \quad (1) \\
& 2x_1 & -x_2 & \leq & 40 & \quad (2) \\
& x_1,\ x_2 \geq 0\,. & & & &
\end{array}
$$

Again, we will first apply the Simplex algorithm to this problem. The algorithm will take us to a tableau that indicates unboundedness of the problem. We will then examine the geometrical origin of unboundedness with the help of the graphical representation of this problem.

After introducing two slack variables and setting up the objective function, we obtain the following initial Simplex tableau.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	-2	-1	0	0	0
s_1	0	1	-1	1	0	10
s_2	0	2	-1	0	1	40

With x_1 as the entering variable, it is easily seen that R_1 is the pivot row. After executing a pivot, we obtain the tableau below.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	-3	2	0	20
x_1	0	1	-1	1	0	10
s_2	0	0	1	-2	1	20

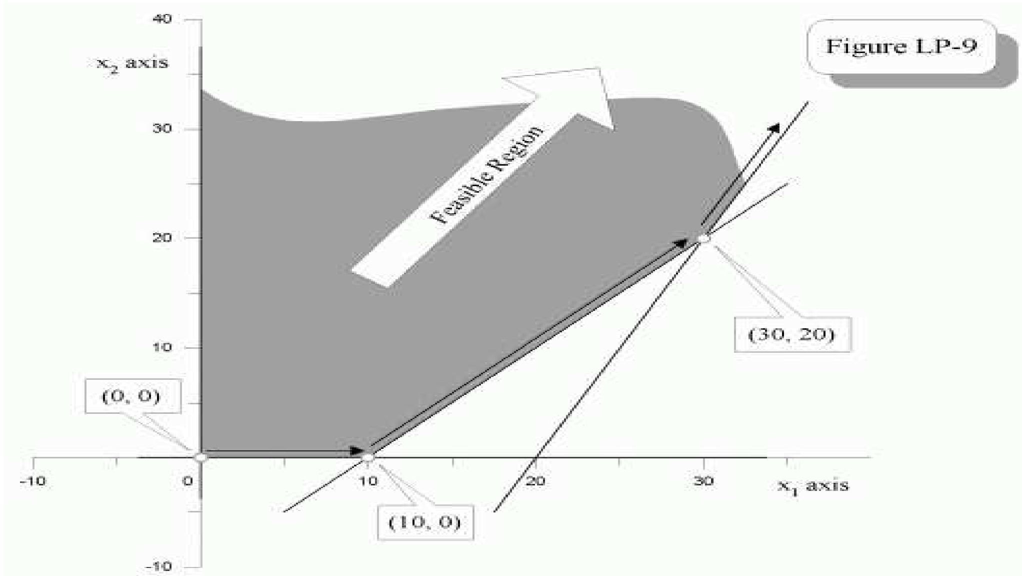

Figure 1: Feasible region of an unbounded LP.

Since x_2 has a negative coefficient in R_0, this tableau is not optimal. Another pivot takes us to the next tableau.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	0	-4	3	80
x_1	0	1	0	-1	1	30
x_2	0	0	1	-2	1	20

This tableau again is not optimal. However, at this point, we are unable to perform further iterations, because as we attempt to carry out a ratio test with s_1 as the entering variable, it turns out that there is no ratio to compute. What this means is that as we attempt to bring s_1 in as a basic variable, none of the constraints will stop us from increasing its value to infinity. Now, as the value of s_1 increases, the objective-function value will also increase correspondingly at a rate of 4. It follows that the problem does not have an optimal solution.

The feasible region of this problem is depicted in Figure 1. There, we see that the Simplex algorithm starts with the point (0, 0), follows the x_1-axis to the point (10, 0), rises diagonally to the point (30, 20), and then takes off to infinity along an infinite "ray" that emanates from (30, 20).

More formally, what we have is that for any nonnegative δ, the solution $(x_1, x_2, s_1, s_2) = (30 + \delta, 20 + 2\delta, \delta, 0)$ is feasible. Since this solution has a corresponding objective-function value of $80 + 4\delta$, we see that the problem is unbounded.

Clearly, unboundedness of a problem can occur only when the feasible region is unbounded, which, unfortunately, is something we cannot tell in advance of the solution attempt. In the above example, we detected unboundedness when we encountered a pivot column that does not contain any positive entry. More generally, we can in fact declare a problem as unbounded if *any* (nonbasic) column, not necessarily associated with the entering variable, is identified to have the above-stated property at the end of an iteration. Referring back to the initial tableau, we see that, indeed, the x_2-column had this property. Therefore, we could have concluded that the problem is unbounded at the outset. The difference is that the algorithm would then follow the x_2-axis to infinity. (Of course, another difference is the amount of effort.)

The corresponding condition for unboundedness in a minimization problem is slightly different: We should look for a nonbasic column with a positive coefficient in R_0 and with all other entries nonpositive.

In most applications of linear programming, if a problem turns out to be unbounded, it is often due to the fact that at least one relevant constraint has been left out during the formulation stage. Therefore, one should carefully reexamine the original formulation.

4 Multiple Optimal Solutions

Consider the linear program:

$$\begin{array}{llrcrl} \text{Maximize} & 4x_1 & +14x_2 & & & \\ \text{Subject to:} & & & & & \\ & 2x_1 & +7x_2 & \leq & 21 & (1) \\ & 7x_1 & +2x_2 & \leq & 21 & (2) \\ & x_1,\, x_2 \geq 0\,. & & & & \end{array}$$

As before, we will first apply the Simplex algorithm to this problem. The algorithm will take us to a tableau that indicates that alternative optimal solutions exist. We will then examine the geometrical origin behind the existence of alternative optimal solutions, with the help of the graphical representation of this problem.

After introducing two slack variables and setting up the objective function, we obtain the following initial Simplex tableau.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	-4	-14	0	0	0
s_1	0	2	7	1	0	21
s_2	0	7	2	0	1	21

With x_2 as the entering variable, it is easily seen that R_1 is the pivot row. After executing a pivot, we obtain the tableau below.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	0	2	0	42
x_2	0	2/7	1	1/7	0	3
s_2	0	45/7	0	$-2/7$	1	15

At this point, since every nonbasic variable has a nonnegative coefficient in R_0, the current solution $(x_1,\, x_2,\, s_1,\, s_2) = (0,\, 3,\, 0,\, 15)$ is optimal. However, notice that the nonbasic variable x_1 has a coefficient of 0 in R_0. This implies that if we attempt to let x_1 enter the basis, then the objective-function value will not change. Indeed, after a pivot with the x_1-column as the pivot column, we obtain the following new tableau.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	0	2	0	42
x_2	0	0	1	7/45	$-2/45$	7/3
x_1	0	1	0	$-2/45$	7/45	7/3

With the same objective-function value, the new solution $(x_1,\, x_2,\, s_1,\, s_2) = (7/3,\, 7/3,\, 0,\, 0)$ is, of course, also optimal. Note that a further attempt at a pivot in the s_2-column will take us back to the previous solution. We will therefore not pursue things further.

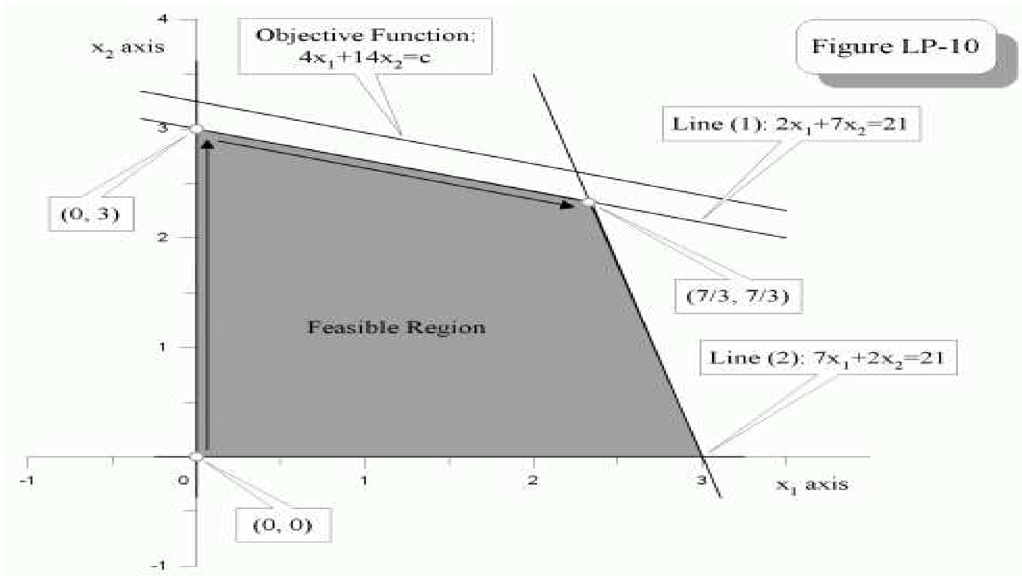

Figure 2: An LP with multiple optimal solutions.

The feasible region of this problem is depicted in Figure 2. There, we see that the Simplex algorithm starts with the point (0, 0), travels along the x_2-axis to the first optimal solution at (0, 3), and then continues on to the second optimal solution at (7/3, 7/3). Notice that the objective-function line $4x_1 + 14x_2 = c$ (for any c) is parallel to the edge that begins at (0, 3) and ends at (7/3, 7/3). Hence, every point on this edge is optimal.

In general, if we are given two optimal solutions to a linear program, then an infinite number of optimal solutions can be constructed. In this example, both (0, 3) and (7/3, 7/3) are optimal. Therefore, every point on the edge connecting these two points will also be optimal. Formally, points on this edge are traced out by solutions of the form:

$$(x_1,\, x_2) = \delta \cdot (0,\, 3) + (1 - \delta) \cdot (7/3,\, 7/3)\,,$$

where δ is any value in the interval [0, 1]. As specific examples, if we let $\delta = 1$, then we have the point (0, 3); if we let $\delta = 0$, then we have the point (7/3, 7/3); and if we let $\delta = 1/2$, then we have the point (7/6, 8/3), which is half way between (0, 3) and (7/3, 7/3).

5 Degeneracy and Cycling

Consider the linear program:

$$\begin{array}{lrrcrl}
\text{Maximize} & 2x_1 & +x_2 & & & \\
\text{Subject to:} & & & & & \\
 & 4x_1 & +3x_2 & \leq & 12 & (1) \\
 & 4x_1 & +x_2 & \leq & 8 & (2) \\
 & 4x_1 & +2x_2 & \leq & 8 & (3) \\
 & \multicolumn{5}{l}{x_1,\, x_2 \geq 0\,.}
\end{array}$$

We will first apply the Simplex algorithm to this problem. After a couple of iterations, we will hit a degenerate solution, which is why this example is chosen. We will then examine the geometrical origin of degeneracy and the related issue of "cycling" in the Simplex algorithm, with the help of the graphical representation of this problem.

After introducing three slack variables and setting up the objective function, we obtain the following initial Simplex tableau.

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	
	1	-2	-1	0	0	0	0
s_1	0	4	3	1	0	0	12
s_2	0	4	1	0	1	0	8
s_3	0	4	2	0	0	1	8

With x_1 as the entering variable, there is a tie for the minimum ratio, at R_2 and R_3. This implies that after a pivot with either R_2 or R_3 as the pivot row, the resulting tableau will have a degenerate basic variable. Let us choose R_2 (say) as the pivot row. Then, after executing a pivot, we obtain the tableau below.

Tableau I:

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	
	1	0	$-1/2$	0	$1/2$	0	4
s_1	0	0	2	1	-1	0	4
x_1	0	1	$1/4$	0	$1/4$	0	2
s_3	0	0	1	0	-1	1	0

The current basic feasible solution is $(x_1, x_2, s_1, s_2, s_3) = (2, 0, 4, 0, 0)$, where s_3 is (as expected) a degenerate basic variable. The next pivot column and pivot row will be the x_2-column and R_3, respectively. After executing another pivot, we obtain the following tableau.

Tableau II:

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	
	1	0	0	0	0	$1/2$	4
s_1	0	0	0	1	1	-2	4
x_1	0	1	0	0	$1/2$	$-1/4$	2
x_2	0	0	1	0	-1	1	0

Again, the current basic feasible solution is $(x_1, x_2, s_1, s_2, s_3) = (2, 0, 4, 0, 0)$. However, the identity of the degenerate basic variable has switched from s_3 to x_2. Note that this tableau happens to be optimal (independent of the phenomenon of degeneracy).

To understand what it means to have a degenerate solution, let us now refer to the graphical representation of this problem, which is shown in Figure 3. Notice that three, not two, constraint equations pass through the corner-point solution $(x_1, x_2) = (2, 0)$. These equations are: $x_2 = 0$, $4x_1 + x_2 = 8$, and $4x_1 + 2x_2 = 8$. Since only two lines are needed to define such an intersection, we see that degeneracy is a manifestation of redundancy in information. That is, we can choose to let any pair of these equations (out of

$$\binom{3}{2} = 3$$

combinations) to define this intersection. For example, if we choose $x_2 = 0$ and $4x_1 + x_2 = 8$ as the defining equations, then, since the solution to this pair of equations will automatically satisfy equation $4x_1 + 2x_2 = 8$, the value of the slack variable associated with the inequality $4x_1 + 2x_2 \leq 8$, namely s_3, must turn out to be 0. This accounts for the appearance of the degenerate basic variable s_3 in Tableau I. Similarly, if we choose $4x_1 + x_2 = 8$ and $4x_1 + 2x_2 = 8$ as the defining equations, then the inequality constraint $x_2 \geq 0$ will turn out to be binding. This accounts for the fact that x_2 is a degenerate basic variable in Tableau II.

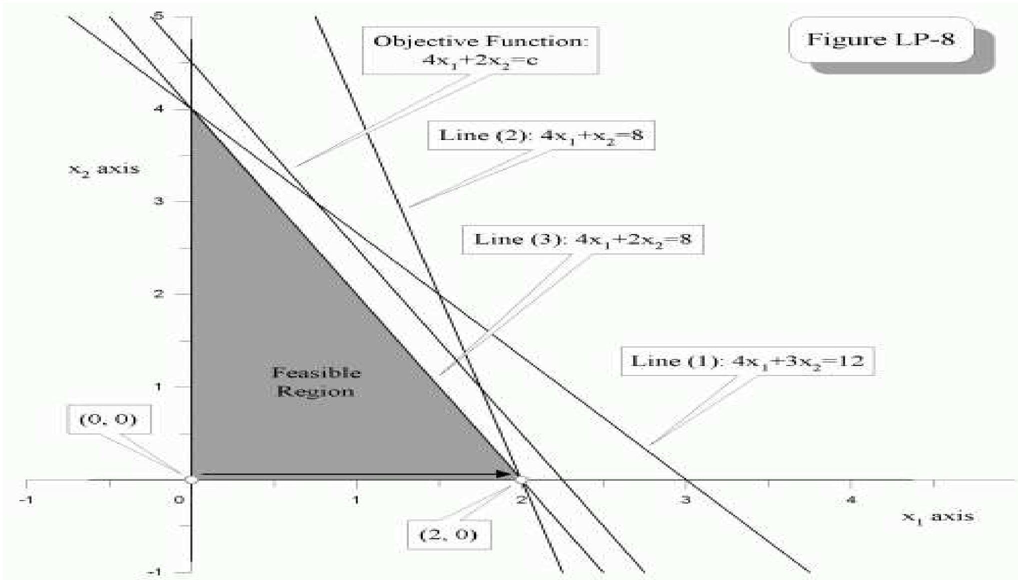

Figure 3: Feasible region of a degenerate LP.

What will happen if we choose $x_2 = 0$ and $4x_1+2x_2 = 8$ as the defining equations? A careful examination of Tableau II shows that if we choose the s_2-column and R_3 as the pivot column and the pivot row, then the following tableau results after a pivot.

Tableau III:

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	
	1	0	0	0	0	1/2	4
s_1	0	0	1	1	0	-1	4
x_1	0	1	1/2	0	0	1/4	2
s_2	0	0	-1	0	1	-1	0

This new tableau, again, corresponds to the solution $(x_1, x_2, s_1, s_2, s_3) = (2, 0, 4, 0, 0)$. Notice however that the slack variable s_2 associated with the inequality $4x_1 + x_2 \leq 8$ has indeed replaced x_2 as the degenerate basic variable.

Recall that in the last pivot, the pivot element is a negative number, -1. This will never occur during ordinary Simplex iterations. (Why?) Our purpose for carrying out such a pivot is to show that there is indeed a third tableau that is associated with the corner-point solution (2, 0).

Now, with three tableaus all corresponding to the same set of coordinates, the question is: Is it possible for the Simplex algorithm to cycle through these (or a subset of these) tableaus forever? Theoretically, the answer is yes. However, this happens rarely in practice, and can in fact be avoided. The following example is given by Beale and it illustrates Simplex cycling at nonoptimal bases:

$$\begin{array}{llllllllll}
\text{Maximize} & & & & 3x_4/4 & -20x_5 & x_6/2 & -6x_7 & & \\
\text{Subject to:} & & & & & & & & & \\
& x_1 & & & +x_4/4 & -8x_5 & -x_6 & +9x_7 & = 0 & (1) \\
& & x_2 & & +x_4/2 & -12x_5 & -x_6/2 & +3x_7 & = 0 & (2) \\
& & & x_3 & & & x_6 & & = 1 & (3)
\end{array}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \ .$$

Since this problem has the evident starting basis $\{x_1, x_2, x_3\}$, we can directly setup the Simplex tableau and proceed as follows:

Tableau I:

Basic Variable	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
	1	0	0	0	-3/4	20	-1/2	6	0
x_1	0	1	0	0	1/4	-8	-1	9	0
x_2	0	0	1	0	1/2	-12	-1/2	3	0
x_3	0	0	0	1	0	0	1	0	1

Tableau II:

Basic Variable	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
	1	3	0	0	0	-4	-7/2	33	0
x_4	0	4	0	0	1	-32	-4	36	0
x_2	0	-2	1	0	0	4	3/2	-15	0
x_3	0	0	0	1	0	0	1	0	1

Tableau III:

Basic Variable	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
	1	1	1	0	0	0	-2	18	0
x_4	0	-12	8	0	1	0	8	-84	0
x_5	0	-1/2	1/4	0	0	1	3/8	-15/4	0
x_3	0	0	0	1	0	0	1	0	1

Tableau IV:

Basic Variable	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
	1	-2	3	0	1/4	0	0	-3	0
x_6	0	-3/2	1	0	1/8	0	1	-21/2	0
x_5	0	1/16	-1/8	0	-3/64	1	0	3/16	0
x_3	0	3/2	-1	1	-1/8	0	0	31/2	1

Tableau V:

Basic Variable	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
	1	-1	1	0	-1/2	16	0	0	0
x_6	0	2	-6	0	-5/2	56	1	0	0
x_7	0	1/3	-2/3	0	-1/4	16/3	0	1	0
x_3	0	-2	6	1	5/2	-56	0	0	1

Tableau VI:

Basic Variable	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
	1	0	-2	0	-7/4	44	1/2	0	0
x_1	0	1	-3	0	-5/4	28	1/2	0	0
x_7	0	0	1/3	0	1/6	-4	-1/6	1	0
x_3	0	0	0	1	0	0	1	0	1

In Tableau VI, x_2 enters into the basis and x_7 leaves. Carrying out the row operations we obtain Tableau I once more. Repeating the pivots in the example, Simplex can cycle without ever reaching the optimal solution $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (3/4, 0, 0, 1, 0, 1, 0)$.

Simplex cycles very rarely and Beale's example is one of those rare cases. We will know make couple observations to understand why Simplex cycled. Let us first ask the following question: Can Simplex ever cycle if the objective function value is increasing in every iteration? The answer is no because, objective function value can increase from one basis to another only if bases are different in each iteration.

Once we agree that the objective value stays constant as long as Simplex cycles, we realize that this is the case only if the right-hand side of the pivot row is zero. That in turn implies that before the pivot at least one of the basic variables was zero. We call an LP **degenerate** if one of the basic variables assume the value of zero during Simplex iterations. Geometrically, degeneracy happens when more than n equations are intersecting in n dimensions, see Figure 3.

Studying each tableau above, it is evident that in every iteration at least one basic variable is zero. Thus each tableau is degenerate. For example, in Tableau I $x_1 = x_2 = 0$ and in Tableau II $x_2 = x_4 = 0$. More interestingly, each tableau corresponds to the same solution $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (0, 0, 1, 0, 0, 0, 0)$ although this solution is obtained with different bases.

Not every degenerate LP cycles. Actually there are rules to pick entering and leaving variables to avoid cycling. The most popular rule is the **least index rule**: Among those variable eligible to enter (and to leave) the basis choose the smallest index variable. This simple rule avoids cycling for degenerate LP's.

6 Solved Exercises

1. Set up the Simplex tableau for the following problem:

$$\begin{array}{lrrrcll}
\text{Minimize} & 3x_1 & & -x_3 & & & \\
\text{Subject to:} & & & & & & \\
& 2x_1 & +x_2 & +x_3 & = & 6 & (1) \\
& x_1 & & +3x_3 & \leq & 8 & (2) \\
& x_1 & & +x_3 & \geq & 3 & (3)
\end{array}$$

$$x_3 \text{ free}, \ x_1, x_2 \leq 0\,.$$

Note the meaning of **free** is the same as **unrestricted**. Is the tableau optimal? Write the associated basic feasible solution for this tableau. (Do not do any Simplex iterations.)

Solution: Let $x_1' = -x_1$, $x_2' = -x_2$ and $x_3 = x_3^+ - x_3^-$.

$$\begin{array}{lrrrrcll}
\text{Minimize} & -3x_1' & & -x_3^+ & +x_3^- & & & \\
\text{Subject to:} & & & & & & & \\
& -2x_1' & -x_2' & +x_3^+ & -x_3^- & = & 6 & (1) \\
& -x_1' & & +3x_3^+ & -3x_3^- & \leq & 8 & (2) \\
& -x_1' & & +x_3^+ & -x_3^- & \geq & 3 & (3)
\end{array}$$

$$x_1', x_2', x_3^+, x_3^- \geq 0\,.$$

Although x_2' appears only in constraint (1), it is coefficient is -1. Then we cannot use it as a basic variable. We introduce artificial variables A_1 and A_3 for constraints (1) and (3). Since the objective is minimization, we add MA_1 and MA_3 to it.

Minimize z

Subject to:

$$\begin{array}{rrrrrrrrrrl}
z & +3x_1' & & +x_3^+ & -x_3^- & -MA_1 & & & -MA_3 & = & 0 \quad (0)\\
 & -2x_1' & -x_2' & +x_3^+ & -x_3^- & +A_1 & & & & = & 6 \quad (1)\\
 & -x_1' & & +3x_3^+ & -3x_3^- & & +s_2 & & & = & 8 \quad (2)\\
 & -x_1' & & +x_3^+ & -x_3^- & & & -s_3 & +A_3 & = & 3 \quad (3)
\end{array}$$

$$x_1', x_2', x_3^+, x_3^-, A_1, s_2, s_3, A_3 \geq 0\,.$$

To eleminate A_1 and A_3, we add M times constraint (1) and M times constraint (2) to row (0) to obtain the tableau below:

Basic Variable	z	x_1'	x_2'	x_3^+	x_3^-	A_1	s_2	s_3	A_3	rhs
	1	3-3M	-M	1+2M	-1-2M	0	0	-M	0	9M
A_1	0	-2	-1	1	-1	1	0	0	0	6
s_2	0	-1	0	3	-3	0	1	0	0	8
A_3	0	-1	0	1	-1	0	0	-1	1	3

2. Use the Tableau method to solve the following LP to optimality:

Maximize $a + 2b + c$

Subject to:

$$\begin{array}{rrrcll}
a & +b & +c & = & 4 & (1)\\
a & -b & +c & \geq & 2 & (2)
\end{array}$$

$$a, b, c \geq 0\,.$$

Solution: Because of $=$ and $\geq$ constraints, this is a BigM problem. However, it does not hurt to realize that we can set $x = a + c$ and $y = b$ to obtain an equivalent LP:

Maximize $x + 2y$

Subject to:

$$\begin{array}{rrcll}
x & +y & = & 4 & (1)\\
x & -y & \geq & 2 & (2)
\end{array}$$

$$x, y \geq 0\,.$$

Now introduce artificial variable A_1 into (1), and artificial variable A_2 and slack variable s_2 into (2):

$$x + y + A_1 = 4 \text{ and } x - y - s_2 + A_2 = 2.$$

Also penalize artificial variables in the objective function:

$$\text{Maximize z where } z - x - 2y + MA_1 + MA_2 = 0.$$

We eliminate A_1 and A_2 by adding $-M$ times the first equation and $-M$ times the second equation on this equation:

$$z - (2M + 1)x - 2y + Ms_2 = 0.$$

Basic Variable	z	x	y	s_2	A_1	A_2	rhs
	1	-(2M+1)	-2	M	0	0	-6M
A_1	0	1	1	0	1	0	4
A_2	0	1	-1	-1	0	1	2

Clearly x enters into the basis and A_2 exits:

Basic Variable	z	x	y	s_2	A_1	A_2	rhs
	1	0	-(2M+3)	-(M+1)	0	2M+1	-2M+2
A_1	0	0	2	1	1	-1	2
x	0	1	-1	-1	0	1	2

We can delete the column of A_2. y enters into the basis and A_1 exits:

Basic Variable	z	x	y	s_2	A_1	rhs
	1	0	0	1/2	M+3/2	5
y	0	0	1	1/2	1/2	3
x	0	1	0	-1/2	1/2	1

This tableau is optimal. Optimal solution is $x = 3$, $y = 1$ with objective value of 5. The optimal solution to the original problem is $b = 1$ and nonegative a, c such that $a + c = 3$. In other words, this problem has multiple optimal solutions.

7 Solving LPs with LINDO

Consider the linear program:

$$\begin{array}{llllllll} \text{Minimize} & z & = & 84x_1 & +72x_2 & +60x_3 & & \\ \text{Subject to:} & & & & & & & \\ & & & 90x_1 & +20x_2 & +40x_3 & \geq & 200 \quad (1) \\ & & & 30x_1 & +80x_2 & +60x_3 & \geq & 180 \quad (2) \\ & & & 10x_1 & +20x_2 & +60x_3 & \geq & 150 \quad (3) \end{array}$$

$$x_1,\, x_2,\, x_3 \geq 0\,.$$

This problem comes from Exercise 3.6-4. Briefly, the scenario that gives rise to this formulation is as follows. A farmer wishes to determine the quantities of the available types of feed, which are corn, tankage, and alfalfa, that should be given to each pig. Since pigs will eat any mix of these feed types, the objective is to determine which mix will meet a given set of basic nutritional requirements at a minimum cost. The decision variables are:

x_1 = kilograms of corn,
x_2 = kilograms of tankage, and
x_3 = kilograms of alfalfa.

The basic nutritional ingredients are: carbohydrates, protein, and vitamins; and their respective requirements are expressed in constraints (1)–(3) above. Finally, the costs per kilogram of feed types, 84, 72, and 60, are in cents.

Now, to launch LINDO and enter the problem as:

```
MIN 84 X1 + 72 X2 + 60 X3
ST
90 X1 + 20 X2 + 40 X3 > 200
30 X1 + 80 X2 + 60 X3 > 180
10 X1 + 20 X2 + 60 X3 > 150
END
```

where X1, X2, and X3 correspond, respectively, to x_1, x_2, and x_3. Next, click on the "Solve" menu and then select "Solve". Click on "Yes" in the "Do Range (Sensitivity) Analysis?" dialog box. The following report should now be displayed in the "Reports Window":

```
LP OPTIMUM FOUND AT STEP      2

        OBJECTIVE FUNCTION VALUE

        1)      241.7143

  VARIABLE        VALUE          REDUCED COST
        X1         1.142857          0.000000
        X2         0.000000         17.714285
        X3         2.428571          0.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000         -0.771429
        3)         0.000000         -0.485714
        4)         7.142857          0.000000

 NO. ITERATIONS=       2


 RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
       X1       84.000000        51.000000        37.199997
       X2       72.000000         INFINITY        17.714285
       X3       60.000000        11.272726        22.666666

                           RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                   RHS           INCREASE         DECREASE
        2      200.000000        24.999998        80.000000
        3      180.000000       120.000000         6.000000
        4      150.000000         7.142857         INFINITY
```

We will interpret this output from top to bottom. The Simplex algorithm went through two iterations to produce the optimal solution. The optimal objective-function value is 241.7143 (cents). LINDO automatically introduces slack or surplus variables to convert inequality constraints into equalities. The numbering of the rows starts with the number 1, as opposed to 0. Thus, ROW 1 refers to the objective-function row, ROW 2 refers to (functional) constraint (1), and so on; correspondingly, the slack or surplus variables will also be numbered this way. For this problem, three surplus variables are introduced, and they are named SLK 2, SLK 3, and SLK 4 (LINDO uses SLK to denote both slack and surplus variables). In our standard notation, these correspond to s_1, s_2, and s_3; therefore, the optimal solution is (x_1, x_2, x_3, s_1, s_2, s_3) = (1.142857, 0, 2.428571, 0, 0, 7.142857). The optimal basis is $\{x_1, x_3, s_3\}$.

To explain the columns under REDUCED COST and DUAL PRICES, we need to refer to the final tableau. To generate the final tableau, click on the title bar at the top of the problem window (to return focus to that window), click on the "Reports" menu, and then click "Tableau". The final tableau now appears at the bottom of the previous "Reports Window". It is pasted below.

```
THE TABLEAU

    ROW  (BASIS)          X1         X2         X3   SLK    2   SLK    3
      1 ART            0.000     17.714      0.000      0.771      0.486
      2       X3       0.000      1.571      1.000      0.007     -0.021
      3       X1       1.000     -0.476      0.000     -0.014      0.010
      4 SLK    4       0.000     69.524      0.000      0.286     -1.190


    ROW   SLK    4
      1      0.000  -241.714
      2      0.000     2.429
      3      0.000     1.143
      4      1.000     7.143
```

This tableau is in our standard format, except that it overflows into a second line. Here, the variable ART refers to z. (Recall that z was artificially created for convenience.) The basic variables are X3 (for ROW 2), X1 (for ROW 3), and SLK 4 (for ROW 4); and they assume the corresponding values listed in the RHS column. Note, however, that the RHS constant in ROW 1 is listed as -241.714. The negative sign here is a consequence of converting a minimization problem into a maximization problem, which LINDO does automatically. (Working with maximization problems only simplifies interpretation.) Therefore, this negative sign should be reversed in the final report, which, again, the program does automatically. In ROW 1, the coefficients of X2, SLK 2, and SLK 3, the nonbasic variables, are all positive; this indicates that what we have is indeed the (unique) final tableau.

We now return to the report above. The values listed in the REDUCED COST column are taken from the coefficients of X1, X2, and X3 in ROW 1, in the final tableau. In other words, X1 and X3 have a reduced cost of 0, whereas X2 has a reduced cost of 17.714. Formally (for a maximization problem), the **reduced cost** for a nonbasic variable is defined as the amount by which the value of z will decrease if we increase the value of that nonbasic variable by 1 (while holding all other nonbasic variables at 0). The adjective "reduced" is used because such a cost is relative to a specific tableau, i.e., is from the viewpoint of the particular current basic feasible solution. The 1-unit increment in the nonbasic variable is nominal, in that we are only contemplating an increase, even when it is not feasible to do so. The reduced cost for a basic variable is defined as 0. Mechanically, this is because basic variables always have a coefficient of 0 in the objective-function row; and conceptually, this is because the basic variables are already "participating" in the current solution (and therefore we do not attempt to bring them into the basis).

For example, the final tableau tells us that it is not optimal to include any tankage in the mix. Moreover, if we insist on having tankage in the mix, then the cost per kilogram of tankage is 17.714 cents. Like the concept of shadow price, this cost is relative to our current optimal solution, i.e., it has nothing to do with the "market" cost of tankage (which is at 72 cents per kilogram).

Next, we move on to the values listed in the DUAL PRICES column. The term "dual prices" is equivalent to shadow prices. (Every linear program has an associated dual linear program, and the concept of dual prices originates from the dual linear program. We will not discuss the dual of a linear program, as it is a more-advanced topic.) Formally speaking, the shadow price associated with the RHS constant of an original constraint (or with the availability of a resource) is defined as the amount by which the optimal objective-function value will improve if we increase the value of that constant by 1. (Again, this 1-unit increment is nominal.) In the current problem, the original functional constraints are of the "$\geq$" type. For such a constraint, an increase in the RHS constant corresponds to a tightening of that constraint; hence, the increase will (typically) result in a degradation of the optimal objective-function value. Indeed, the reported dual prices for ROW 2 and ROW 3 are negative.

For example, for every unit of increase in the nutritional requirement for carbohydrates, the cost of the *optimal* mix will increase by 0.771429. Similarly, the corresponding increase in cost associated with protein is 0.485714. That the dual price for vitamins equals 0 is a consequence of the fact that the optimal mix already exceeds the vitamins requirement by a margin of 7.142857 (that is, the surplus variable SLK 4 equals 7.142857 in the optimal solution).

Next, we examine the RANGES IN WHICH THE BASIS IS UNCHANGED. Here, THE BASIS refers to the optimal basis, i.e. the set of basic variables in the optimal solution; and two sets of ranges are displayed, one for the original objective-function coefficients and one for the original RHS constants. For example, if we increase the current objective function coefficient of X1 by 51 or decrease it by 37.19, the optimal basis still remains as $\{x_1, x_3, s_3\}$. Since we are changing only the objective function, the numerical values of $\{x_1, x_3, s_3\}$ do not change either. As another example, if we increase the current RHS of Row 2 by 25 or decrease it by 80, the optimal basis does not change. However, the numerical values of the basic solution can change. All this analysis is related to the topic of Sensitivity Analysis, we will not provide more details here.

8 Exercises

1. Refer to Beale's example that makes Simplex cycle.
   a) Write down the augmented solutions for all the tableaus.
   b) Point out the entering and exiting variables in each iteration. Until which tableau entering and exiting variables are chosen with the least index rule?
   c) Perform two iterations with the least index rule starting with Tableau IV. Do you reach the optimal solution with these two iterations?

2. Refer to the alternative formulation of production planning problem. Use your answer to a) to answer parts b), c) and d).
   a) Suppose that we are solving a 3-month problem and modify the formulation accordingly.
   b) Introduce slack, excess, artificial variables as necessary to make up a starting basis.
   c) Put your constraints into a format such that each basic variable appears only once with a coefficient of 1. Point out the associated basic variable for each constraint.
   d) Explain variables defined in b) (except artificial ones) in your words within the context of production planning.

3. Suppose we have a primitive LP software that solves only maximization problems with only nonnegative variables and without any equality constraints. Modify the following LP so that it can be solved with this software:

$$\begin{array}{llllll} \text{Minimize} & 6x_1 & +14x_2 & & & \\ \text{Subject to:} & & & & & \\ & 2x_1 & +7x_2 & = & 21 & (1) \\ & 7x_1 & +2x_2 & \leq & 21 & (2) \\ & \multicolumn{5}{l}{x_1 \text{ free}, \; x_2 \leq 0\,.} \end{array}$$

4. Use Lindo or Excel to solve the investment problem of the formulation chapter, see the excel template. Report the objective value and the values of the decision variables.

a) Report how does the solution change if \$200 is available initially.
b) Report how does the solution change if \$100 is available initially but there is 2% interest per period.

5. Textbook H-L: pp. 96-97, 3.4-13 c).

6. In a minimization problem with some "=" and "≥" constraints the following tableau is obtained, find the optimal solution for this problem, or establish that it is unbounded or infeasible.

Basic Variable	z	x_1	x_2	A_1	s_2	A_2	s_3	
	1	0	$M/4+2$	0	$M/4-1$	$-5M/4+1$	0	$3M/2+6$
A_1	0	0	1/4	1	1/4	−1/4	0	3/2
x_1	0	1	3/4	0	−1/4	1/4	0	3/2
s_3	0	0	5/4	0	1/4	−1/4	1	3/2

7. Perform a single Simplex iteration starting from the initial tableau of solved exercise 1 in the Section 6.

8. Solve the DART route investment, Problem 3 of Chapter 1 Section 7 using Excel. Report your results.