The Transportation Problem: An LP Formulation

Suppose a company has m warehouses and n retail outlets. Products shiped from the warehouses to the retailers.

- The total supply of the product from warehouse i is a_i, where $i = 1,\ 2,\ \ldots,\ m$.
- The total demand for the product at outlet j is b_j, where $j = 1,\ 2,\ \ldots,\ n$.
- The cost of sending one unit of product from warehouse i to retailer j is c_{ij}, where $i = 1,\ 2,\ \ldots,\ m$ and $j = 1,\ 2,\ \ldots,\ n$. The total cost of a shipment is linear in the size of the shipment.

Determine an optimal transportation scheme between the warehouses and the outlets, subject to the specified supply and demand constraints.

The Transportation Problem: An LP Formulation

The Decision Variables

x_{ij} = the size of the shipment from warehouse i to outlet j, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

The Objective Function

Consider the shipment from warehouse i to outlet j. The transportation cost per unit is c_{ij}; and the size of the shipment is x_{ij}. The total cost of this shipment is given by $c_{ij}x_{ij}$.

$$\text{Minimize} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \,.$$

The Transportation Problem: An LP Formulation

The Constraints

Outgoing shipment from warehouse i is the sum $x_{i1} + x_{i2} + \cdots + x_{in}$. This shipment cannot exceed the total supply.

$$\sum_{j=1}^{n} x_{ij} \leq a_i \,, \quad \text{for } i = 1,\, 2,\, \ldots,\, m.$$

Incoming shipment at outlet j is the sum $x_{1j} + x_{2j} + \cdots + x_{mj}$. In summation notation, this is written as $\sum_{i=1}^{m} x_{ij}$. This shipment cannot be below the demand.

$$\sum_{i=1}^{m} x_{ij} \geq b_j \,, \quad \text{for } j = 1,\, 2,\, \ldots,\, n.$$

The Transportation Problem: An LP Formulation

LP Formulation

Minimize $$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to:

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &\leq a_i && \text{for } i = 1, 2, \ldots, m \\ \sum_{i=1}^{m} x_{ij} &\geq b_j && \text{for } j = 1, 2, \ldots, n \end{aligned}$$

$$x_{ij} \geq 0 \quad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n.$$

The Transportation Problem: A Numerical Example

We formulate a transportation problem for $m = 3$ and $n = 2$; $a_1 = 45$, $a_2 = 60$, and $a_3 = 35$; $b_1 = 50$ and $b_2 = 60$; and finally, $c_{11} = 3$, $c_{12} = 2$, $c_{21} = 1$, $c_{22} = 5$, $c_{31} = 5$, and $c_{32} = 4$.

$$
\begin{array}{lcccccccll}
\text{Minimize} & 3x_{11} & +2x_{12} & +x_{21} & +5x_{22} & +5x_{31} & +4x_{32} & & & \\
\text{Subject to:} & & & & & & & & & \\
 & x_{11} & +x_{12} & & & & & \leq & 45 & (1) \\
 & & & x_{21} & +x_{22} & & & \leq & 60 & (2) \\
 & & & & & x_{31} & +x_{32} & \leq & 35 & (3) \\
 & x_{11} & & +x_{21} & & +x_{31} & & \geq & 50 & (4) \\
 & & x_{12} & & +x_{22} & & +x_{32} & \geq & 60 & (5)
\end{array}
$$

$$x_{ij} \geq 0 \quad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2.$$

For an example of an explicit transportation scheme, let $x_{11} = 20$, $x_{12} = 20$, $x_{21} = 20$, $x_{22} = 20$, $x_{31} = 10$, and $x_{32} = 20$. Feasible? Cost?

$$3 \cdot 20 + 2 \cdot 20 + 1 \cdot 20 + 5 \cdot 20 + 5 \cdot 10 + 4 \cdot 20 = 350.$$

The Transportation Problem: An Equivalent Formulation

1. For a given problem to have any feasible solution, the total supply must not be less than the total demand.

2. If the total supply happens to be equal to the total demand, then **any** feasible solution must satisfy all of the inequality constraints as equalities. (For example, this would be the case if a_1, a_2, and a_3 had been 40, 40, and 30, respectively.)

3. If the total supply happens to be equal to the total demand, after replacing inequalities with equalities, there is no longer any need for introducing slack or surplus variables.

How to equalize supply and demand? Dummy sink! For this specific example, we can introduce a third outlet to serve as the dummy sink; and let $b_3 = 30$ and $c_{13} = c_{23} = c_{33} = 0$:

The Transportation Problem: An Equivalent Formulation

$$\begin{array}{lccccccccccc}
\text{Minimize} & 3x_{11} & +2x_{12} & & +x_{21} & +5x_{22} & & +5x_{31} & +4x_{32} & & & \\
\text{Subject to:} & & & & & & & & & & & \\
 & x_{11} & +x_{12} & +x_{13} & & & & & & & = & 45 \\
 & & & & x_{21} & +x_{22} & +x_{23} & & & & = & 60 \\
 & & & & & & & x_{31} & +x_{32} & +x_{33} & = & 35 \\
 & x_{11} & & & +x_{21} & & & +x_{31} & & & = & 50 \\
 & & x_{12} & & & +x_{22} & & & +x_{32} & & = & 60 \\
 & & & x_{13} & & & +x_{23} & & & +x_{33} & = & 30
\end{array}$$

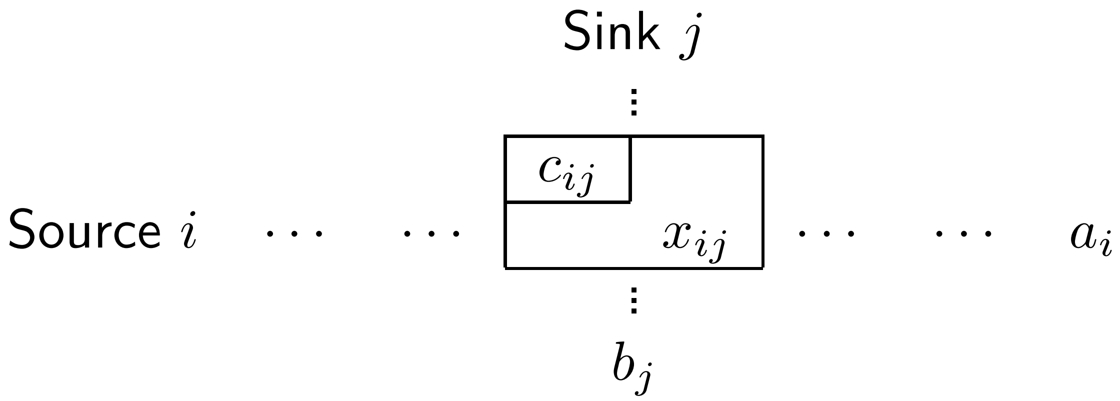
$x_{ij} \geq 0$ for $i = 1, 2, 3$ and $j = 1, 2, 3.$

The **standard form** of the transportation problem is under the assumption that

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

The Transportation Problem: The Transportation Tableau

A table with m rows (one for each source) and n columns (one for each sink).



Construct the tableau for our numerical example:

		Sinks			
		1	2	3	
Sources	1	3	2	0	45
	2	1	5	0	60
	3	5	4	0	35
		50	60	30	

The Transportation Problem: The Transportation Tableau

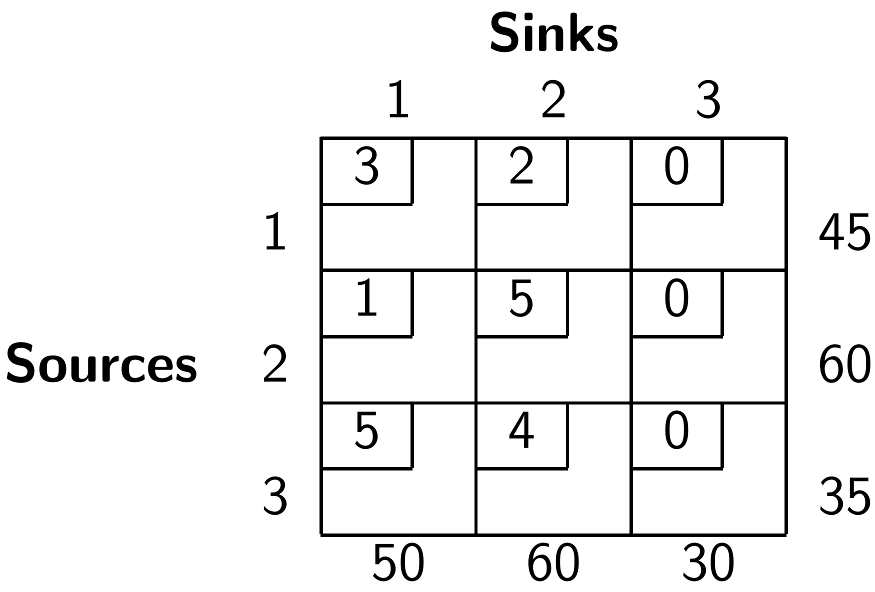
As a specific example, $x_{11} = 20$, $x_{12} = 20$, $x_{13} = 5$, $x_{21} = 20$, $x_{22} = 20$, $x_{23} = 20$, $x_{31} = 10$, $x_{32} = 20$, and $x_{33} = 5$ would be entered as:

			Sinks		
		1	2	3	
	1	3 20	2 20	0 5	45
Sources	2	1 20	5 20	0 20	60
	3	5 10	4 20	0 5	35
		50	60	30	

Remarks

1. The optimal solution of a transportation problem is integer-valued.
2. When an oversupply exists, we can accommodate the inventory-holding costs as part of the objective function by reinterpreting them as transportation costs to the dummy sink.

The Transportation Problem: Constructing an Initial BFS



Sinks

Sources	1	2	3	
1	3	2	0	45
2	1	5	0	60
3	5	4	0	35
	50	60	30	

6 equality constraints and 9 variables in the algebraic formulation.

Reduce the number of constraint equations by 1. Notice that the three supply constraints sum up to

$$x_{11} \quad +x_{12} \quad +x_{13} \quad +x_{21} \quad +x_{22} \quad +x_{23} \quad +x_{31} \quad +x_{32} \quad +x_{33} \quad = \quad 140 \ ;$$

deduct any two demand constraints from this to get the third demand constraint. Any solution to any five of the equalities also satisfies the sixth (redundant) equality.

The Transportation Problem: Constructing an Initial BFS

With one equation removed, a bfs will have 5 basic variables and 4 nonbasic variables. If we are to solve this problem by the standard Simplex method, then our next task is to introduce 5 artificial variables. Is there another way?

Our first observation is that it is quite easy to construct a feasible solution to the problem. Arbitrarily choose an empty cell in a tableau and assign a value to the x_{ij} such that

- The sum of the x_{ij}'s in every row equals the specified supply at the right margin of that row;

- The sum of the x_{ij}'s in every column equals the specified demand at the bottom margin of that column.

The Transportation Problem: Constructing an Initial BFS

		Sinks			
		1	2	3	
	1	3 20	2 20	0 5	45
Sources	2	1 20	5 20	0 20	60
	3	5 10	4 20	0 5	35
		50	60	30	

Is this a basic solution?

Constructing an Initial BFS: Northwest Corner Rule

Start from the most northwest cell to assign values and assign as much as possible. Reduce the supply and the demand accordingly. When a source or a sink depleted cross that out.

Sinks

		1	2	3	
	1^*	3 45	2	0	45 → 0
Sources	2	1	5	0	60
	3	5	4	0	35
		50 → 5	60	30	

Exactly one row or one column is crossed out after each value assignment.

Constructing an Initial BFS: Northwest Corner Rule

Sinks

		1*	2	3	
Sources	1*	3 45	2	0	45 → 0
	2	1 5	5	0	60 → 55
	3	5	4	0	35
		50 → 5 → 0	60	30	

Sinks

		1*	2	3	
Sources	1*	3 45	2	0	45 → 0
	2*	1 5	5 55	0	60 → 55 → 0
	3	5	4	0	35
		50 → 5 → 0	60 → 5	30	

Constructing an Initial BFS: Northwest Corner Rule

Sinks

		1^*	2^*	3	
	1^*	3 45	2	0	45 → 0
Sources	2^*	1 5	5 55	0	60 → 55 → 0
	3	5	4 5	0	35 → 30
		50 → 5 → 0	60 → 5 → 0	30	

Sinks

		1^*	2^*	3	
	1^*	3 45	2	0	45 → 0
Sources	2^*	1 5	5 55	0	60 → 55 → 0
	3^*	5	4 5	0 30	35 → 30 → 0
		50 → 5 → 0	60 → 5 → 0	30 → 0	

Constructing an Initial BFS: Northwest Corner Rule

The explicit assignments $x_{11} = 45$, $x_{21} = 5$, $x_{22} = 55$, $x_{32} = 5$, and $x_{33} = 30$. The objective-function value of this solution:

$$3 \cdot 45 + 1 \cdot 5 + 5 \cdot 55 + 4 \cdot 5 + 0 \cdot 30 = 435.$$

Why were the final remaining supply and the final remaining demand exhausted simultaneously?

Did we ever look at the costs, should we?

Constructing an Initial BFS: The Least-Cost Method

Always select the cell with the smallest c_{ij} value among all remaining cells as the cell to assign some flow.

Sinks

		1	2	3^*	
	1	3	2	0 30	45 → 15
Sources	2	1	5	0	60
	3	5	4	0	35
		50	60	30 → 0	

Sinks

		1^*	2	3^*	
	1	3	2	0 30	45 → 15
Sources	2	1 50	5	0	60 → 10
	3	5	4	0	35
		50 → 0	60	30 → 0	

Constructing an Initial BFS: The Least-Cost Method

			Sinks		
		1*	2	3*	
	1*	3	2 15	0 30	45 → 15 → 0
Sources	2*	1 50	5 10	0	60 → 10 → 0
	3*	5	4 35	0	35 → 0
		50 → 0	60 → 45 10 → 0	30 → 0	

The least-cost method assignments are: $x_{12} = 15$, $x_{13} = 30$, $x_{21} = 50$, $x_{22} = 10$, and $x_{32} = 35$ with an objective-function value of 270.

Finding the Optimal Solution: the Stepping-Stone Method

Go back to our numerical example:

		Sinks 1	2	3	
	1	3	2 15	0 30	45
Sources	2	1 50	5 10	0	60
	3	5	4 35	0	35
		50	60	30	

Is the current solution optimal? Optimality test: check to see if there are cycles along which increasing the flow decrease the total transportation cost. If no such cycles, the current tableau is optimal.

Finding such cycles is entirely equivalent to finding variables to enter the basis.

Finding the Optimal Solution: the Stepping-Stone Method

Take nonbasic variable x_{11} and increase its value to bring it into the basis:

		Sinks			
		1	2	3	
Sources	1	3 $+\delta$	2 $15-\delta$	0 30	45
	2	1 $50-\delta$	5 $10+\delta$	0	60
	3	5	4 35	0	35
		50	60	30	

The only cells "visited" are: $(1, 1)$, $(1, 2)$, $(2, 2)$, and $(2, 1)$.

$$\begin{tikzcd}
(1, 1)^* \arrow[r] & (1, 2) \arrow[d] \\
(2, 1) \arrow[u] & (2, 2) \arrow[l]
\end{tikzcd}$$

The direction of visits is irrelevant. Always make a 90-degree turn after stepping on a cell. Exactly one stepping-stone path can originate from any nonbasic cell.

Finding the Optimal Solution: the Stepping-Stone Method

By increasing the value of δ, generate a family of solutions of the form:

$$(x_{11},\, x_{12},\, x_{13},\, x_{21},\, x_{22},\, x_{23},\, x_{31},\, x_{32},\, x_{33}) = (\delta,\, 15-\delta,\, 30,\, 50-\delta,\, 10+\delta,\, 0,\, 0,\, 35,\, 0)\,.$$

Geometrically, we are following an edge of the feasible region, starting from the corner-point solution

$$(x_{11},\, x_{12},\, x_{13},\, x_{21},\, x_{22},\, x_{23},\, x_{31},\, x_{32},\, x_{33}) = (0,\, 15,\, 30,\, 50,\, 10,\, 0,\, 0,\, 35,\, 0)\,.$$

What is the greatest value of δ which keeps the solution feasible?

$$(x_{11},\, x_{12},\, x_{13},\, x_{21},\, x_{22},\, x_{23},\, x_{31},\, x_{32},\, x_{33}) = (15,\, 0,\, 30,\, 35,\, 25,\, 0,\, 0,\, 35,\, 0)\,;$$

This is another corner-point solution, adjacent to the previous one.

The Stepping-Stone Method: Reduced Costs

With $x_{11} = \delta$, costs change by: $3 \cdot \delta$ for cell $(1,\, 1)$, $2 \cdot (-\delta)$ for cell $(1,\, 2)$, $5 \cdot \delta$ for cell $(2,\, 2)$, and $1 \cdot (-\delta)$ for cell $(2,\, 1)$. With $\delta = 1$, the total rate of the cost change:

$$\begin{aligned} 3 \cdot \delta + 2 \cdot (-\delta) + 5 \cdot \delta + 1 \cdot (-\delta) &= 3 \cdot 1 + 2 \cdot (-1) + 5 \cdot 1 + 1 \cdot (-1) \\ &= 3 - 2 + 5 - 1 \\ &= 5\,. \end{aligned}$$

Should we bring x_{11} into the basis?

Denote the reduced cost associated with x_{ij} by $\bar{c}_{ij}$. As an example:

$$\begin{aligned} \bar{c}_{11} &= c_{11} - c_{12} + c_{22} - c_{21} \\ &= 3 - 2 + 5 - 1 \\ &= 5\,, \end{aligned}$$

The Stepping-Stone Method: Reduced Costs

Other nonbasic cells: $(2,\ 3)$, $(3,\ 1)$, and $(3,\ 3)$. Check their reduced costs.

The stepping-stone path associated with cell $(2,\ 3)$ is:

$$\begin{tikzcd}
(1,\ 2) \arrow[r] & (1,\ 3) \arrow[d] \\
(2,\ 2) \arrow[u] & (2,\ 3)^* \arrow[l]
\end{tikzcd}$$

After picking up the c_{ij}'s (and alternating their signs) in the order $(2,\ 3)$, $(2,\ 2)$, $(1,\ 2)$, and $(1,\ 3)$, the reduced cost associated with x_{23} can be computed as:

$$\begin{aligned}
\bar{c}_{23} &= c_{23} - c_{22} + c_{12} - c_{13} \\
&= 0 - 5 + 2 - 0 = -3\,.
\end{aligned}$$

Negative reduced cost, a candidate to enter into the basis, solution is not optimal.

The Stepping-Stone Method: Reduced Costs

For cell $(3,\ 1)$:

$$\begin{tikzcd}
(2,\ 1) \arrow[r, "\longrightarrow" description, phantom] & (2,\ 2) \arrow[d] \\
(3,\ 1)^* \arrow[u] & (3,\ 2) \arrow[l]
\end{tikzcd}$$

The reduced cost associated with x_{31} can be computed as:

$$\bar{c}_{31} = c_{31} - c_{21} + c_{22} - c_{32} = 5 - 1 + 5 - 4 = 5.$$

Should we bring x_{31} into the basis?

How about x_{33}? For cell $(3,\ 3)$:

$$\begin{tikzcd}
(1,\ 2) \arrow[r] & (1,\ 3) \arrow[d] \\
\bullet \arrow[u] & \bullet \arrow[d] \\
(3,\ 2) \arrow[u] & (3,\ 3)^* \arrow[l]
\end{tikzcd}$$

$$\begin{aligned}
\bar{c}_{33} &= 0 - 4 + 2 - 0 \\
&= -2\,.
\end{aligned}$$

The Stepping-Stone Method: Ratio Test

Since $\bar{c}_{23} < \bar{c}_{33}$, bring x_{23} into the basis. In the Simplex language, x_{23}-column becomes the pivot column.

		Sinks			
		1	2	3	
	1	3	2 $15+\delta$	0 $30-\delta$	45
Sources	2	1 50	5 $10-\delta$	0 $+\delta$	60
	3	5	4 35	0	35
		50	60	30	

What is the maximum value of δ to keep the solution feasible? In the Simplex language, we are doing a ratio test. The minimum δ comes from the cell (2,2). In the Simplex language, the row where x_{22} is basic becomes the pivot row. Substitute x_{22} with x_{23} in the basis. In the Simplex language, perform row operations.

The Stepping-Stone Method: Second Iteration

Sinks

		1	2	3	
Sources	1	3	2 25	0 20	45
	2	1 50	5	0 10	60
	3	5	4 35	0	35
		50	60	30	

Optimal? For cell (1,1):

$$\begin{tikzcd}
(1,\ 1)^* \arrow[r] & \bullet \arrow[r] & (1,\ 3) \arrow[d] \\
(2,\ 1) \arrow[u] & \bullet \arrow[l] & (2,\ 3) \arrow[l]
\end{tikzcd}$$

$$\bar{c}_{11} = 3 - 0 + 0 - 1 = 2$$

The Stepping-Stone Method: Second Iteration

For cell (2,2):

$$\begin{tikzcd}
(1,\ 2) \arrow[r] & (1,\ 3) \arrow[d] \\
(2,\ 2)^* \arrow[u] & (2,\ 3) \arrow[l]
\end{tikzcd}$$

$$\bar{c}_{22} = 5 - 2 + 0 - 0 = 3$$

For cell (3,1):

$$\begin{tikzcd}
 & (1,\ 2) \arrow[d] & (1,\ 3) \arrow[l] \\
(2,\ 1) \arrow[r] & \bullet \arrow[r] \arrow[d] & (2,\ 3) \arrow[u] \\
(3,\ 1)^* \arrow[u] & (3,\ 2) \arrow[l] &
\end{tikzcd}$$

$$\bar{c}_{31} = 5 - 1 + 0 - 0 + 2 - 4 = 2$$

For cell (3,3):

$$\begin{tikzcd}
(1,\ 2) \arrow[r] & (1,\ 3) \arrow[d] \\
\bullet \arrow[u] & \bullet \arrow[d] \\
(3,\ 2) \arrow[u] & (3,\ 3)^* \arrow[l]
\end{tikzcd}$$

$$\bar{c}_{33} = 0 - 4 + 2 - 0 = -2$$

Since $\bar{c}_{33} < 0$, the current solution is not optimal. Enter x_{33} into the basis.

The Stepping-Stone Method: Second Iteration

We consider solutions of the form below.

			Sinks		
		1	2	3	
	1	3	2 $25 + \delta$	0 $20 - \delta$	45
Sources	2	1 50	5	0 10	60
	3	5	4 $35 - \delta$	0 $+\delta$	35
		50	60	30	

What is the largest δ? Perform a pivot:

The Stepping-Stone Method: Second Iteration

Sinks

		1	2	3	
	1	3	2 45	0	45
Sources	2	1 50	5	0 10	60
	3	5	4 15	0 20	35
		50	60	30	

Optimal? The values of reduced costs are: $\bar{c}_{11} = 4$, $\bar{c}_{13} = 2$, $\bar{c}_{22} = 1$ and $\bar{c}_{31} = 4$. The current solution is optimal.

$$(x_{11},\ x_{12},\ x_{13},\ x_{21},\ x_{22},\ x_{23},\ x_{31},\ x_{32},\ x_{33}) = (0,\ 45,\ 0,\ 50,\ 0,\ 10,\ 0,\ 15,\ 20)\,.$$

This bfs has an objective-function value of 200.

The Assignment Problem

Assign m warehouses to m retailers: The material flow is only allowed between warehouses and retailers that are assigned to each other. The cost of sending materials from warehouse i to retailer j is c_{ij}, irrespective of how much materials are sent. We are interested in determining the lowest cost assignment of retailers to warehouses.

Every assignment problem is a special transportation problem where both supplies and demands are 1 unit.

$$\text{Minimize} \qquad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to:

$$\begin{aligned}
\sum_{j=1}^{m} x_{ij} &= 1 \qquad \text{for } i = 1,\, 2,\, \ldots,\, m \;\; (1)\\
\sum_{i=1}^{m} x_{ij} &= 1 \qquad \text{for } j = 1,\, 2,\, \ldots,\, m \;\; (2)\\
x_{ij} \geq 0 & \qquad \text{for } i = 1,\, 2,\, \ldots,\, m \text{ and } j = 1,\, 2,\, \ldots,\, m.
\end{aligned}$$

The Assignment Problem

For a 3 warehouse and 3 retailer example, a feasible solution

$$(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) = (0, 1, 0, 1, 0, 0, 0, 0, 1)$$

implies that warehouse 1, warehouse 2 and warehouse 3 are assigned to retailer 2, retailer 1 and retailer 3 respectively. The cost of such an assignment would be $c_{12} + c_{21} + c_{33}$. In general $x_{ij} = 1$ if and only if warehouse I is assigned to retailer j.

How about a solution like

$$(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) = (0, 1/2, 1/2, 1/2, 1/2, 0, 1/2, 0, 1/2)$$

Can you read of the assignments from this solution?

Do not worry: All basic feasible solutions of the assignment problem will take integer values because of the special structure of the constraints.

The Assignment Problem: A Numerical Example

An example for $m = 3$ with costs:

$$\mathbf{c} = [c_{11}\ c_{12}\ c_{13};\ c_{21}\ c_{22}\ c_{23};\ c_{31}\ c_{32}\ c_{33}] = [10\ 12\ 8;\ 6\ 8\ 10;\ 6\ 4\ 8].$$

$$\begin{array}{llllllllllll}
\text{Min} & 10x_{11} & +12x_{12} & +8x_{13} & +6x_{21} & +8x_{22} & +10x_{23} & +6x_{31} & +4x_{32} & 8x_{33} & & \\
\text{St:} & & & & & & & & & & & \\
 & x_{11} & +x_{12} & +x_{13} & & & & & & & = & 1 \\
 & & & & x_{21} & +x_{22} & +x_{23} & & & & = & 1 \\
 & & & & & & & x_{31} & +x_{32} & +x_{33} & = & 1 \\
 & x_{11} & & & +x_{21} & & & +x_{31} & & & = & 1 \\
 & & x_{12} & & & +x_{22} & & & +x_{32} & & = & 1 \\
 & & & x_{13} & & & +x_{23} & & & +x_{33} & = & 1 \\
 & & & & & & x_{ij} \geq 0 & \multicolumn{5}{l}{\text{for } i = 1, 2, 3 \text{ and } j = 1, 2, 3}
\end{array}$$

The Shortest Path Problem

Given a network with a set of nodes and arcs where an arc going from node i to node j has a length denoted by c_{ij} (≥ 0).

A path is a sequence of arcs. The length of a path is the sum of the lengths of the arcs in the path. Find the path with the shortest length between two nodes.

Principle of optimality: Portions of a shortest path are shortest paths themselves.

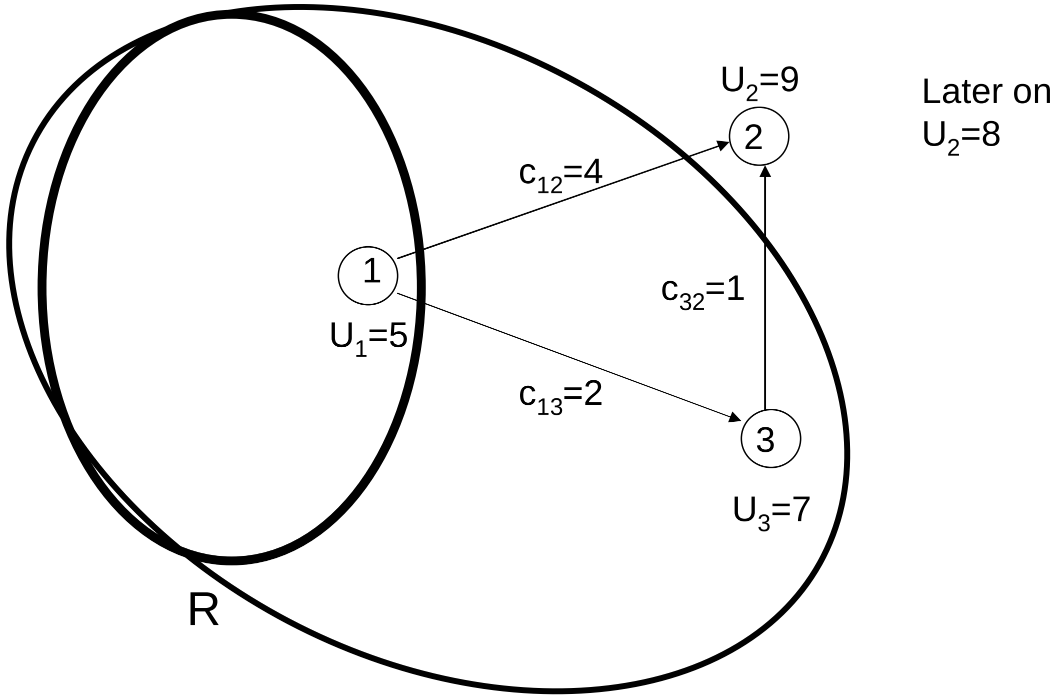
u_j: the length of the **shortest path** from source node to node j. u_j are correct only for some nodes, i.e. those only in node set R. We know shortest paths to nodes in R Then we try to reach to a node j outside R $(j \notin R)$, going through a node k inside R $(k \in R)$. We do a computation:

$$u_j = \min_{k \in R}\{u_k + c_{kj}\} \quad j \notin R.$$

At this step for $j \notin R$, u_j is the shortest path to node j using **only** the nodes in R. At later steps when we put more nodes in R, some u_j may decrease so all u_j $(j \notin R)$ values are phony except for the minimum of them, say $u_{\bar{j}}$

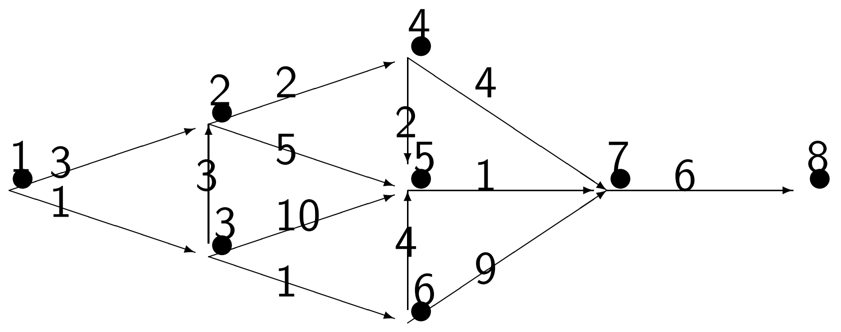
$$u_{\bar{j}} = \min_{j \notin R} u_j \text{ update } R \text{ as } R \leftarrow R \cup \{\bar{j}\}$$

At each iteration one more node is added to this list R



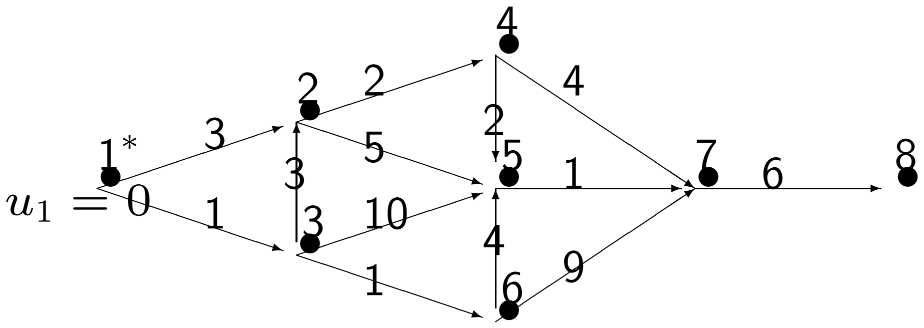
The Shortest Path Problem: A Numerical Example

Consider highway network from Ithaca to dallas. In this network 1=Ithaca, 2=Columbus, 3=Pittsburgh, 4=Evansville, 5=Nashville, 6=Lexington, 7=Memphis and 8=Dallas. Numbers on the arcs are arc lengths.



Initially $R = \{1\}$, $u_1 = 0$ and $u_2 = u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = \infty$. Put * on every node in R.

The Shortest Path Problem: A Numerical Example

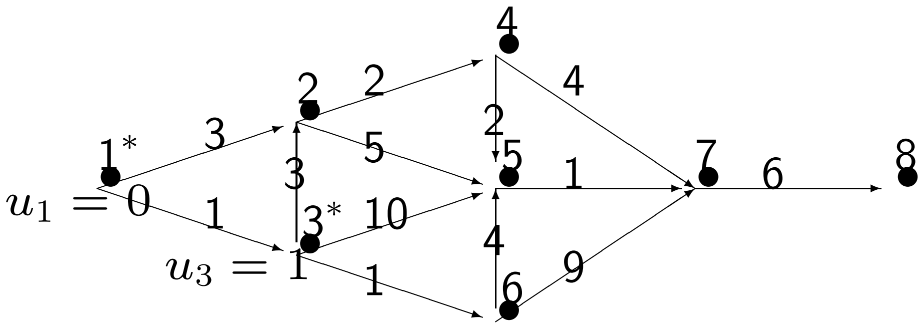


Now compute distances:

$$u_2 = \min\{u_1 + c_{12}\} = \min\{0 + 3\} = 3$$
$$u_3 = \min\{u_1 + c_{13}\} = \min\{0 + 1\} = 1$$
$$u_4 = u_5 = u_6 = u_7 = u_8 = \infty$$

Add node 3 to R.

The Shortest Path Problem: A Numerical Example



At the beginning of the second iteration, we compute the distances once more:

$$u_2 = \min\{u_1 + c_{12}, u_3 + c_{32}\} = \min\{0+3, 1+3\} = 3$$

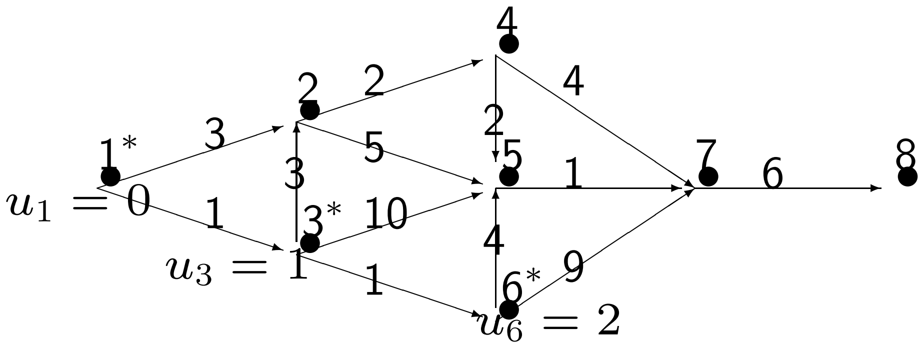
$$u_5 = \min\{u_3 + c_{35}\} = \min\{1+10\} = 11$$

$$u_6 = \min\{u_3 + c_{35}\} = \min\{1+1\} = 2$$

$$u_4 = u_7 = u_8 = \infty$$

We add node 6 to R.

The Shortest Path Problem: A Numerical Example



In the third iteration, we compute the distances again:

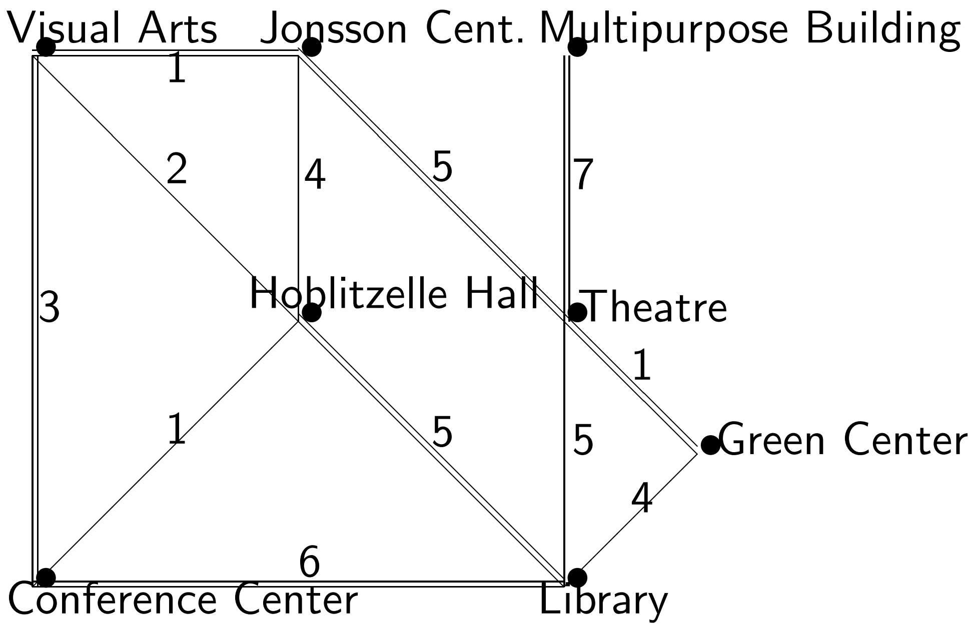
$$u_2 = \min\{u_1 + c_{12}, u_3 + c_{32}\} = \min\{0 + 3, 1 + 3\} = 3$$
$$u_5 = \min\{u_3 + c_{35}, u_6 + c_{65}\} = \min\{1 + 10, 2 + 4\} = 4$$
$$u_7 = \min\{u_6 + c_{67}\} = \min\{2 + 9\} = 11$$
$$u_4 = u_8 = \infty$$

Add node 2 to R.

Balloon analogy.

The Minimum Spanning Tree Problem

Below is a map of UTD campus. Double lines are communication links.

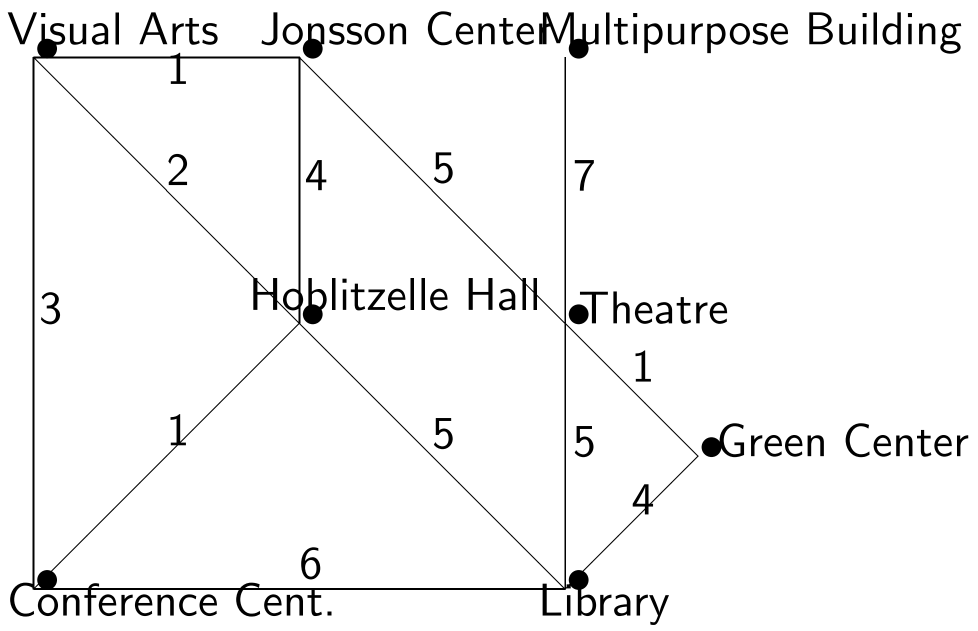


The current installment costs 1+3+6+5+5+1+7=\$28. Pose this problem as a minimum spanning tree problem. **Tree? Spanning Tree? Cost of a tree?**

Keep a list of the arcs T and add arcs to this list. Initially, there are no links installed so $T = \emptyset$. Arcs in T are the connections in the optimal plan.

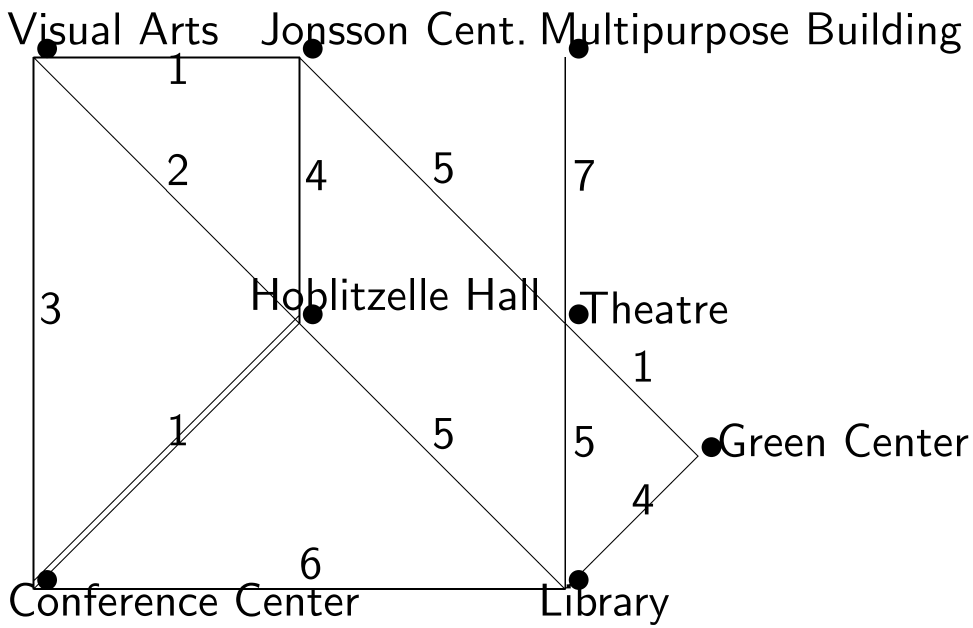
Add arc (i, j) to T if c_{ij} is the smallest among the arcs not considered yet and if i and j nodes are not already connected to each other with the arcs in the current T.

The Minimum Spanning Tree Problem



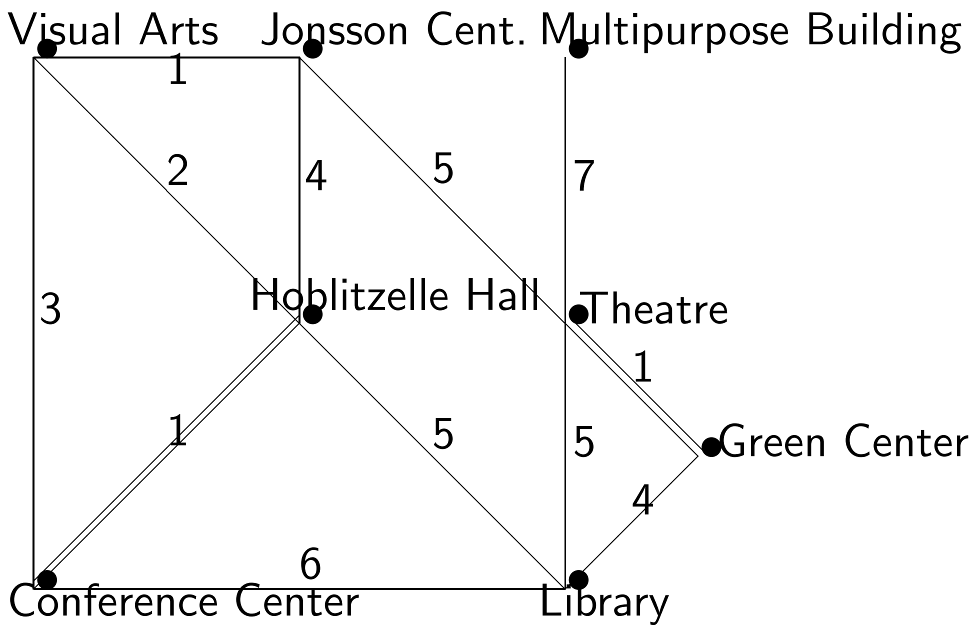
We search for the minimum cost arc. Actually there is a tie; H-C (Hoblitzelle Hall-Conference Center), T-G (Theatre-Green Center) and V-J (Visual Arts-Jonsson center) all have cost of 1. We break ties arbitrarily and pick H-C arc to add to T.

The Minimum Spanning Tree Problem



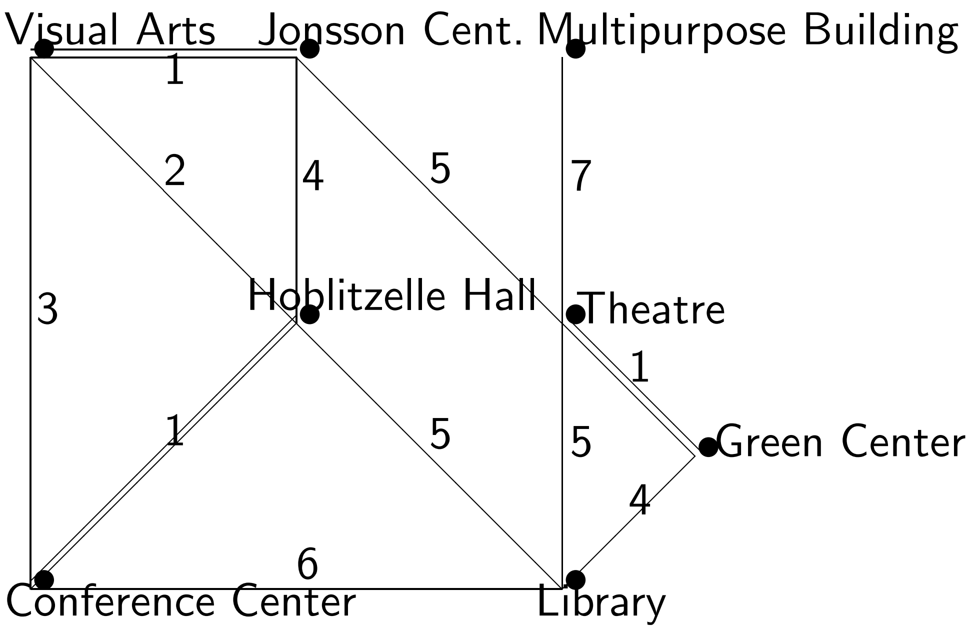
In the second iteration, again search for the minimum cost arc. Tie between T-G and V-J arcs. Arbitrarily pick T-G arc. T-G is not connecting already connected C and H so it can be added to T. Then $T = \{H - C, T - G\}$.

The Minimum Spanning Tree Problem



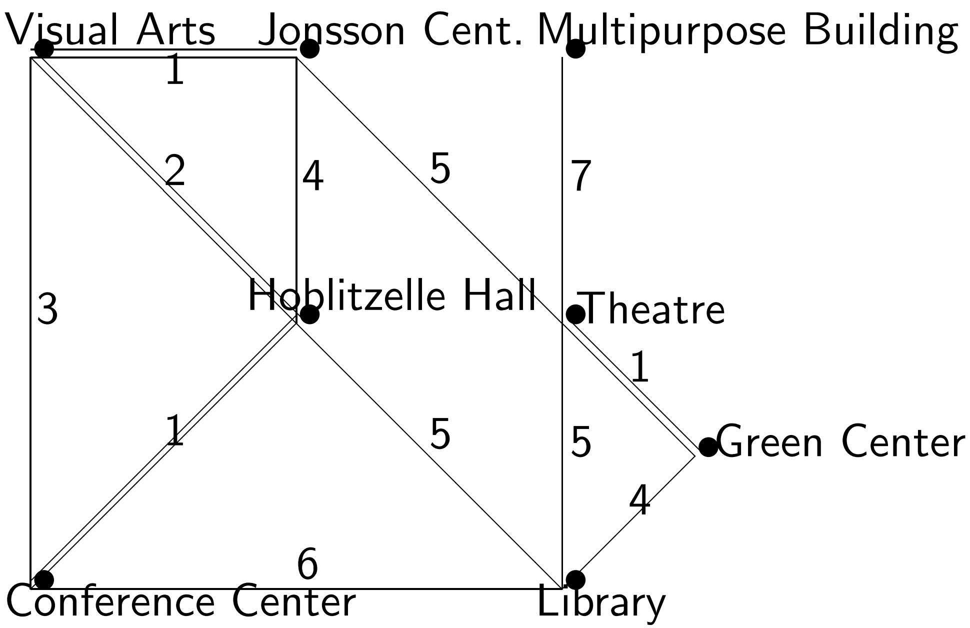
In the third iteration search the minimum cost arc, it is V-J. Moreover, V-J is not connecting any of the already connected nodes, so it can be added to T. At the end of the third iteration, $T = \{H - C,\ T - G,\ V - J\}$.

The Minimum Spanning Tree Problem



In iteration four, arc V-G has the minimum cost and does not connect already connected arcs. Then $T = \{H-C,\ T-G,\ V-J,\ V-G\}$.

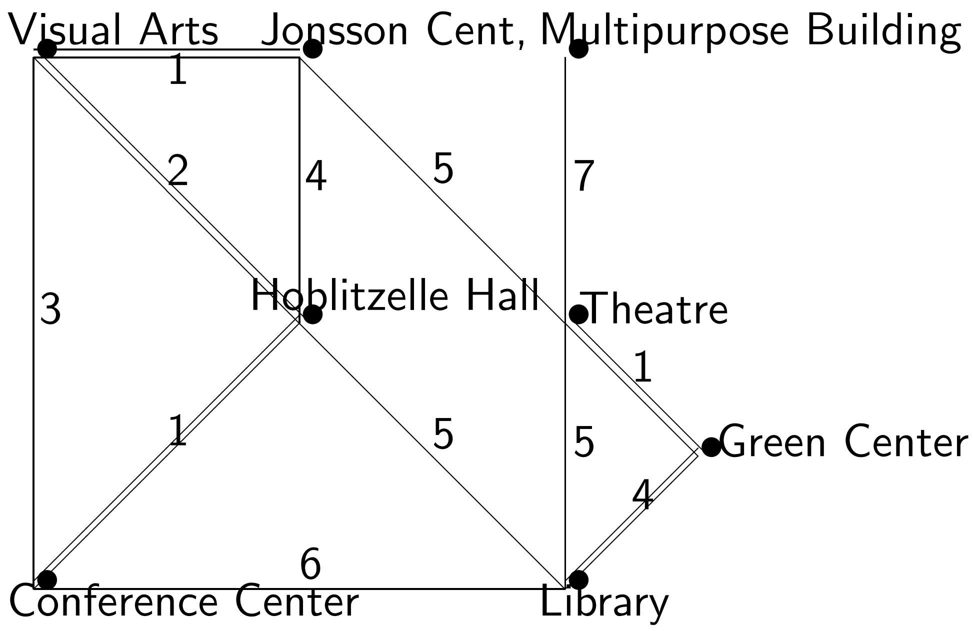
The Minimum Spanning Tree Problem



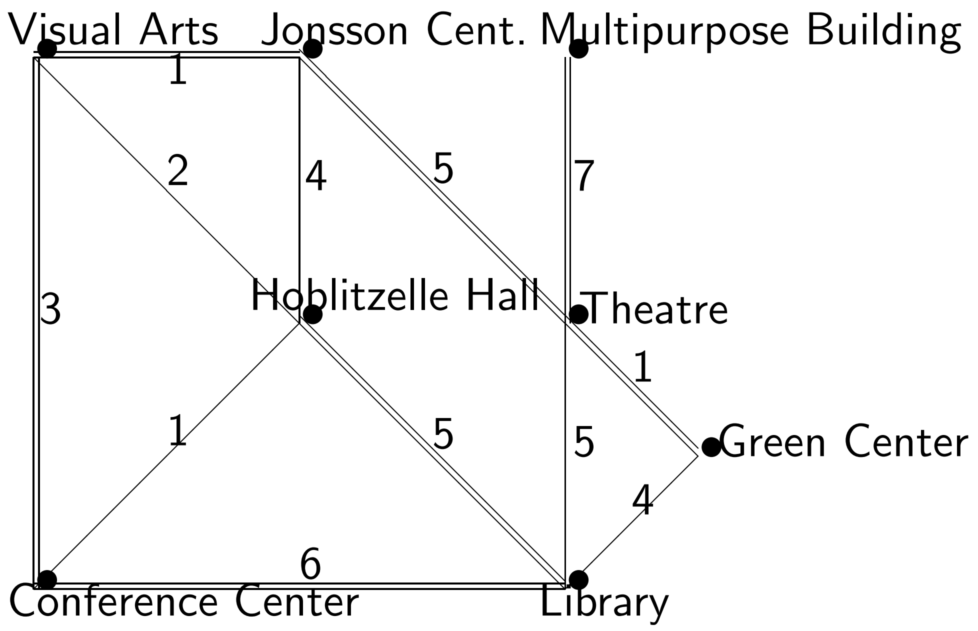
Fifth iteration, the minimum cost arc is V-C. It **cannot** be added to T. It is connecting already connected V and C. Thus, discard V-C.

Look for the next minimum cost arc. Tie between J-H and L-G, arbitrarily choose to consider J-H first however, it is connecting already connected J and H. Thus, discard J-H as well. L-G can be added to T so $T = \{H-C,\ T-G,\ V-J,\ V-H,\ L-G\}$.

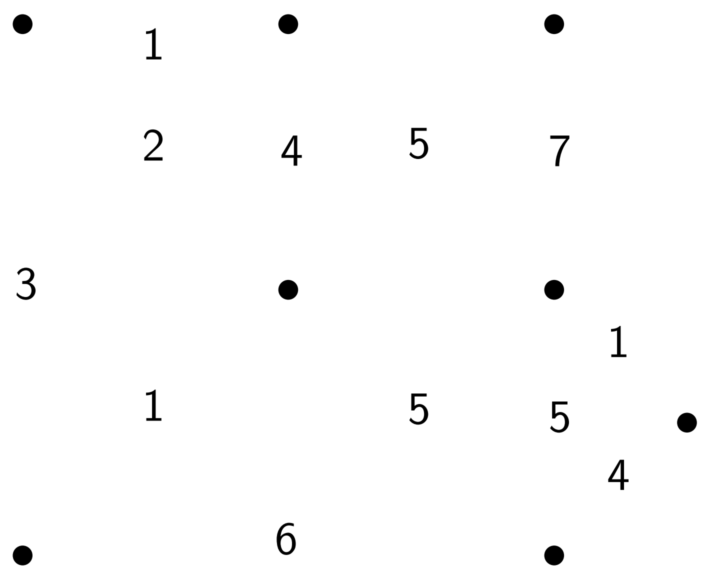
The Minimum Spanning Tree Problem



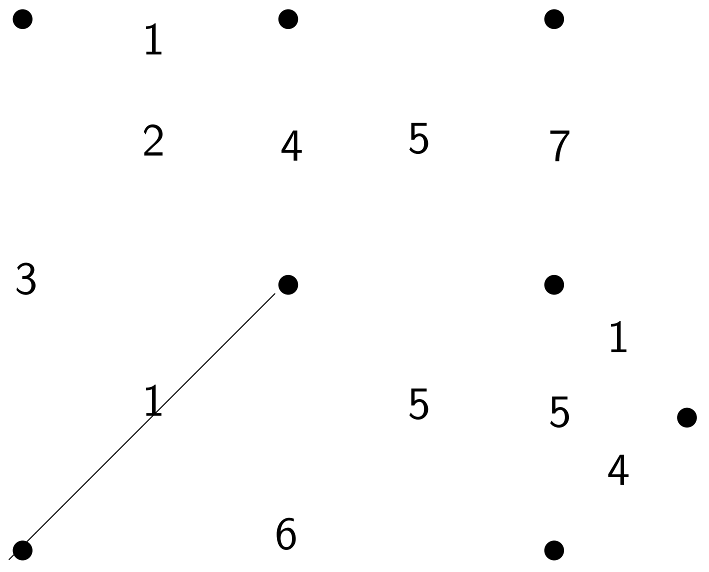
The Minimum Spanning Tree Problem: Optimal Solution



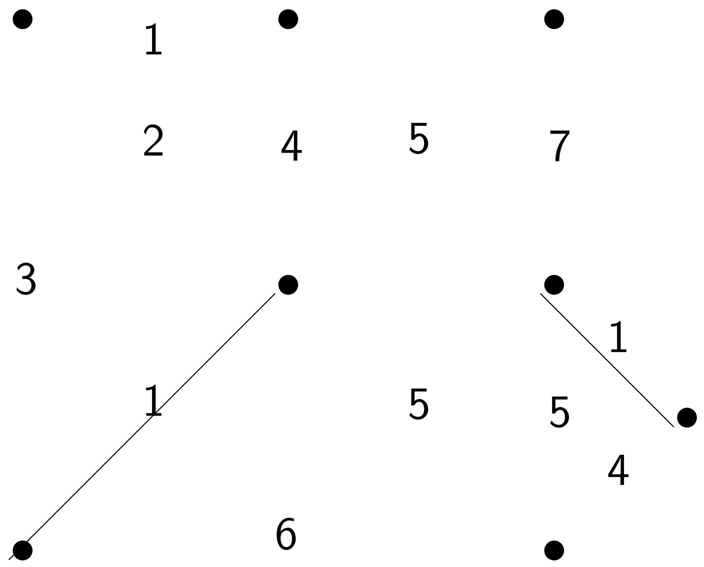
Cracking but Not Breaking Ice Analogy



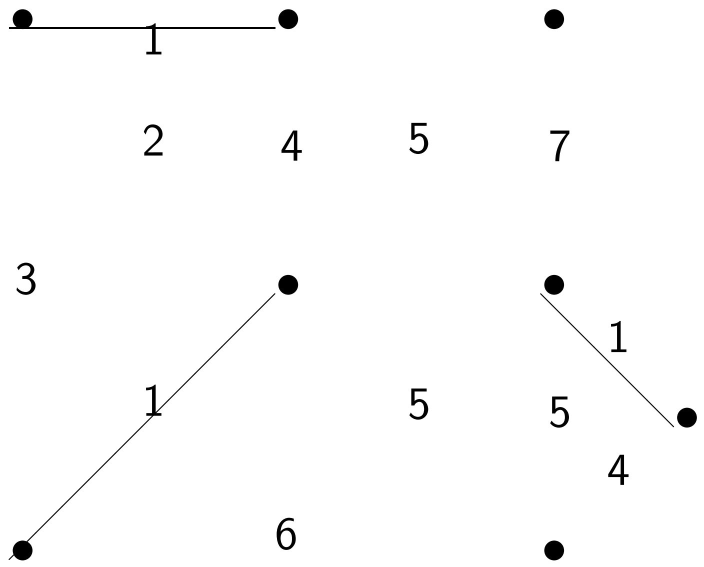
Cracking but Not Breaking Ice Analogy



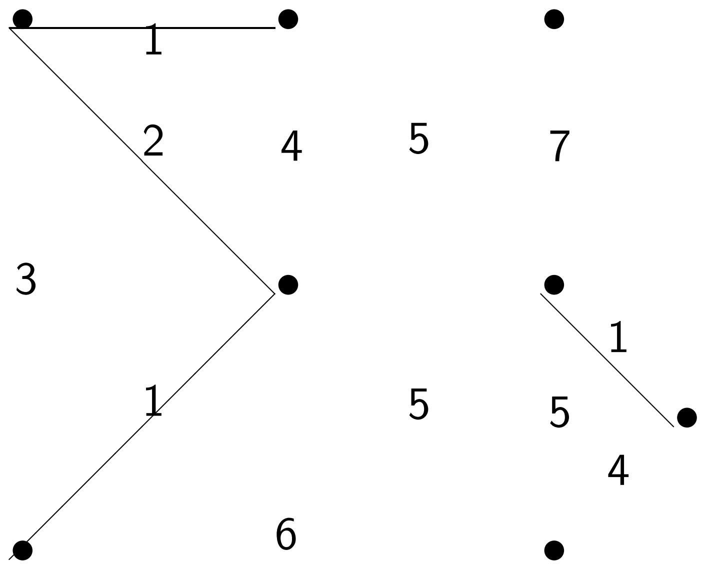
Cracking but Not Breaking Ice Analogy



Cracking but Not Breaking Ice Analogy



Cracking but Not Breaking Ice Analogy



Cracking but Not Breaking Ice Analogy