COHORT MBA

MATH review (part2)

by Lucian Mitroiu

The LOG and EXP functions

The exponential function $\exp : R \to (0,+\infty)$ defined as

$$\exp(x) = e^x$$

Properties:

$e^x e^y = e^{x+y}$

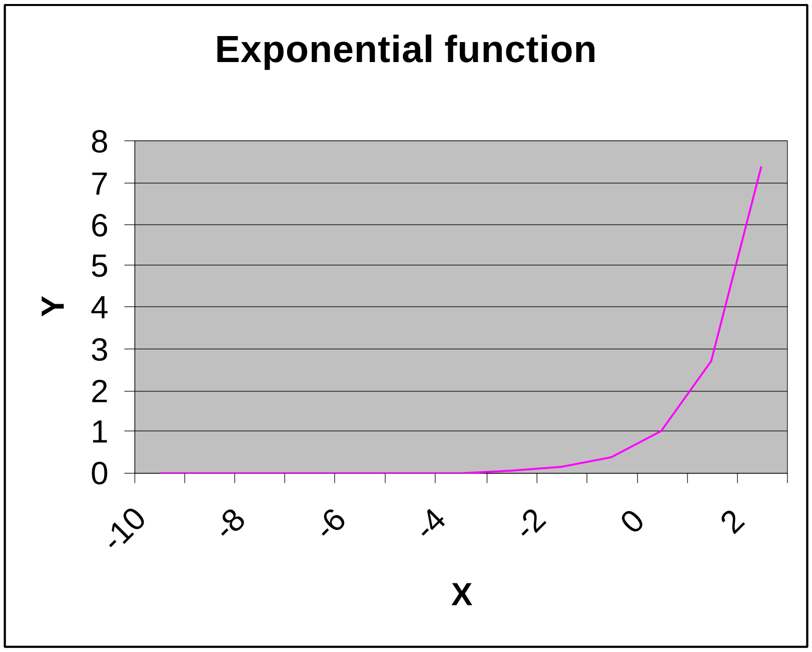
$\frac{e^x}{e^y} = e^{x-y}$

$e^x > 0$

$e^0 = 1$

$\lim_{x\to-\infty} e^x = 0$

$\lim_{x\to\infty} e^x = \infty$



The natural logarithm function ln (in US- log): $(0,\infty) \to R$

$function : \ln(x)$

COHORT MBA

Properties:

$\ln(x * y) = \ln(x) + \ln(y)$

$\ln(\frac{x}{y}) = \ln(x) - \ln(y)$

$\ln(x^{y}) = y\ln(x)$

$\ln(1) = 0$

$\lim_{x \to 0} \ln(x) = -\infty$

$\lim_{x \to \infty} \ln(x) = \infty$

LINEAR ALGEBRA

Vectors: Geometrically speaking a vector is an oriented segment, which means that two points and an orientation defines uniquely a vector. The length of the vector is the length of the segment. In a 2-dimensional system of coordinates XOY, given two points A(s,t) (the origin) and B(u,v) (the end point) and orientation from A to B we can build vector $\overrightarrow{AB}$ and its length is

$$\| \overrightarrow{AB} \| = \sqrt{(u-s)^2 + (v-t)^2}$$

If X and Y are 2 points in a n-dimensional system of coordinates then, for X(x1,x2,..,xn) and Y(y1,y2,..,yn), the length of the vector $\overrightarrow{XY}$ is:

$$\| \overrightarrow{XY} \| = \sqrt{\sum_{i=1}^{n} (yi - xi)^2}$$

Algebraically speaking, a vector is understood as having the origin at the origin of the system O(0,..,0). This means that is enough to give the end point of the vector in order to describe an unique vector ($\overrightarrow{X}$); the orientation is assumed to be from the origin to the end point.

Now, having vector $\overrightarrow{X}$ and $\overrightarrow{Y}$ we can define the sum and difference of them as:

$\overrightarrow{X} \pm \overrightarrow{Y} = (x1 \pm y1, x2 \pm y2, .., xn \pm yn)$

$a * \overrightarrow{X} = (a * x1, a * x2, .., a * xn)$

the last equality represents the product between a scalar (a number) and a vector. Please notice that both the above operations have as result another vector !!

We define the transposed vector $\vec{X}'$ as $\begin{pmatrix} x1 \\ x2 \\ . \\ xn \end{pmatrix}$. If the vector is given vertically (column vector), its transposed will be horizontal (row or line vector).
When it comes to the product, there are two types of products between two vectors.

THE SCALAR(DOT-in US) PRODUCT:
The scalar product of two n-dimensional vectors is a scalar(a number)!!! NOT a vector!!
The scalar product can be computed only between two equal dimensional column and row vectors.
Suppose:

$$\vec{X} = (x1, x2,..xn)$$

$$\vec{Y} = \begin{pmatrix} y1 \\ y2 \\ . \\ yn \end{pmatrix}$$

$$\vec{X} \bullet \vec{Y} = \sum_{i=1}^{n} xi * yi$$

THE VECTOR(CROSS-in US) PRODUCT: Before we talk about the cross product we need to talk about matrices.

MATRICES:
A matrix is a table of equal dimensional (column or row) vectors.

Matrix $A = \begin{pmatrix} a11 & a12 & & a1n \\ a21 & a22 & & a2n \\ & & & \\ & & & \\ am1 & am2 & & amn \end{pmatrix}$ can be sad it is formed with column vectors

$$\begin{pmatrix} a11 \\ a21 \\ \\ am1 \end{pmatrix}\begin{pmatrix} a12 \\ a22 \\ \\ am2 \end{pmatrix}..\begin{pmatrix} a1n \\ a2n \\ \\ amn \end{pmatrix}$$

or with row vectors $(a11 \quad a12 \qquad a1n),(a21 \quad a22 \qquad a2n),..(am1 \quad am2 \qquad amn)$.

- aij means element of matrix a placed at the intersection of row i and column j .

Notice that the number of rows is not necessarily equal with the number of columns. Matrix A is said to be of dimension mxn (to be read m by n); m represents the number of rows, n the number of columns). If m=n, the matrix is called a square matrix.
One can add, subtract only matrices of the same dimension, and is done element by element (like in the case of vectors). One can also multiply a scalar with a matrix, and is done by multiplying the scalar with each of the elements of the matrix.
The transposed matrix (A') is obtained by changing the column vectors into row vectors (or the row vectors into column vectors). Example:

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 3 & 3 & 7 \\ 2.1 & 0 & 1 \end{pmatrix};.B = \begin{pmatrix} 1 & 3 & p \\ 0 & -2 & \frac{1}{3} \end{pmatrix}$$

$$A' = \begin{pmatrix} 1 & 3 & 2.1 \\ 4 & 3 & 0 \\ 7 & 7 & 1 \end{pmatrix};.B' = \begin{pmatrix} 1 & 0 \\ 3 & -2 \\ p & \frac{1}{3} \end{pmatrix}$$

You can notice that both A and A' have the same elements on the diagonal top-left to bottom-right (1 3 1). The diagonal is called the first diagonal and the sum of its elements is called the TRACE of the matrix A. A general formula is:

$$Trace(A) = \sum_{i=1}^{n} aii$$

By transposing a 2x3 matrix B becomes a 3x2 matrix B'. Instead of B' you may see B^T. These are the most common notations I have seen so far.

In order to compute the product between 2 matrices you need to take care of one important rule: **the number of columns of the first matrix should equal the number of rows of the second matrix.** Which means if A is mxn, B has to be nxp, in order to compute AxB. The result (AxB) is going to be another matrix of dimension mxp.
Example: (don't get scared!! ..Yet)

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 3 & 2 & -1 \end{pmatrix};.B = \begin{pmatrix} 3 & -2 & 0 & -1 \\ 0 & 1 & 5 & 0 \\ 1 & 1 & 7 & 2 \end{pmatrix}$$

$$C = A \times B = \begin{pmatrix} 1x3 + 0x0 + 4x1 & 1x(-2) + 0x1 + 4x1 & 1x0 + 0x5 + 4x7 & 1x(-1) + 0x0 + 4x2 \\ 3x3 + 2x0 + (-1)x1 & 3x(-2) + 2x1 + (-1)x1 & 3x0 + 2x5 + (-1)x7 & 3x(-1) + 2x0 + (-1)x2 \end{pmatrix}$$

You compute the value of cij by calculating the scalar product between the i-th row vector of matrix A and the j-th column vector of matrix B.

Now we can talk about the vector product between two vectors. Actually there is not much to talk about. Think to the vectors as matrices. First one is a matrix mx1 and second 1xp (the result is a matrix mxp) or 1xm and mx1 (case which gives you a 1x1 matrix or a scalar).

Like in the case of real numbers, there is a (mxn) matrix called **the null or zero matrix (having all elements =0)** denoted O such that A+/-O=A for any mxn matrix A, and a matrix I called **the identity or unit nxn matrix (having 1 on the first diagonal and 0 in rest)** such that AxI=IxA=A for any square nxn matrix A.

$$O=\begin{pmatrix}0 & . & 0\\ . & . & .\\ 0 & . & 0\end{pmatrix}$$

$$I=\begin{pmatrix}1 & 0 & . & 0\\ 0 & 1 & . & 0\\ . & . & . & .\\ 0 & 0 & . & 1\end{pmatrix}$$

It is VERY IMPORTANT to understand that due to the way of computing the product of two matrices, THE MATRIX MULTIPLICATION IS NOT COMMUTATIVE!!! That means that **AxB is not equal BxA (not like in the case of number multiplication).**
So you may say O and I play the same role 0 and 1 play in the case of numbers, respectively.
There are two more notions to be mentioned about matrices. One is **the determinant of a matrix (det(A))**. Only for square matrices one can compute their determinant. It is a number and since there are a lot of software up there to compute the determinant of a matrix for you, I will show you how to compute it for a 2x2 and 3x3 matrix.
For 2x2:
Det(A)=a11*a22-a12*a21
For 3x3:
Det(B)= b11*b22*b33+b12*b23*b31+b21*b32*b13
-b13*b22*b31-b12*b21*b33-b23*b32*b11
You can imagine how it will look for a 4x4 matrix…
The other notion is **the inverse of a matrix (A^{-1})**. A matrix has a inverse only if its determinant is different from 0 – notice that means the matrix has to be square too!
The inverse of a matrix A is a matrix A^{-1} such that Ax$A^{-1}=A^{-1}$xA=I . So is for matrix A like what the inverse of a number is for that number (a*a^{-1}=1).

Why do we need matrices?
One of the most important applications of matrices is in solving systems of equations. Suppose we have a BIG system to solve (n equations with n variables). It will take 'days' to solve by substitution, unless you already know a faster way or someone else is doing for you. If neither the case, pay attention:
Let's say we have the unknowns (variables) x1, x2, .., xn:

$$a11*x1+a12*x2+...a1n*xn=b1$$

$$a21*x1+a22*x2+...a2n*xn=b2$$

……

$$an1*x1+an2*x2+...ann*xn=bn$$

where all aij and bi are real numbers.

If you look carefully on each left side of the above equations we have the scalar product between the corresponding row vector of the matrix A=(aij) and the column vector of the variables $\begin{pmatrix} a11 & a12 & . & a1n \end{pmatrix} \bullet \begin{pmatrix} x1 \\ x2 \\ . \\ xn \end{pmatrix}$. So, that makes the entire left part of the system the product A*X where X=$\begin{pmatrix} x1 \\ x2 \\ . \\ xn \end{pmatrix}$. Now if we denote B=$\begin{pmatrix} b1 \\ b2 \\ . \\ bn \end{pmatrix}$, then we can rewrite the system as a matrix equation: A*X=B. So, what? Well this is all, because the solution is

$$X = B * A^{-1}$$

but only if A^{-1} exists (which means det(a) not 0)

and not A^{-1} *B !!! (remember the matrix multiplication is not commutative).

Isn’t it, this easier?