Integer Programming Introduction

Divisibility assumption of LP allows for fractions: Produce 7.8 units of a product, buy 12500.33 liters of oil, hire 12.123 people for full time.

Clearly some activities cannot be done in fractions and must be specified as integers.

A small example:

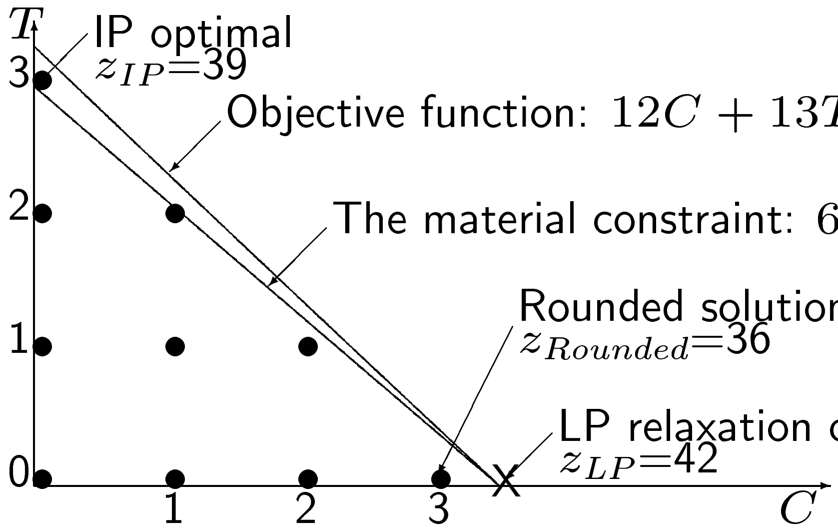
Consider producing chairs and tables using only 21 m^2 of wood. Each chair (table) requires 6 (7) m^2 of wood. Each chair is sold at \$12 and each table is sold at \$13.

Let C and T denote the number of tables and chairs produced.

$$\begin{array}{lllll} \text{Maximize :} & 12C & +13T & & \\ \text{Subject to} & & & & \\ & 6C & +7T & \leq 21 & (1) \\ & C & ,\ T & \geq 0 & (2) \\ & C & ,\ T & int & (3) \end{array}$$

Integer Programming: Rounding

Solve the LP graphically and round the LP solution:



Rounded solutions of LP relaxations are not necessarily optimal.

How about another approach; Evaluate all the integer solutions in the feasible region and pick the best? Practically impossible.

Two approaches are common: *Branch and Bound technique*, and *Cutting planes*.

A Knapsack Problem

Jean Luc (an MBA student) is going to study at most 40 hours/week in the next term and considering to take some of the following courses:

Operations Research	Supply Chain Management	Information Technology	Finance
9	7	5	4
Marketing	**Organizational Behaviour**	**Italian Cinema**	**Russian**
5	3	7	10

Completing each of these courses increases Jean Luc's chances of finding a job. But the contributions of courses towards this:

Operations Research	Supply Chain Management	Information Technology	Finance
0.10	0.14	0.06	0.09
Marketing	**Organizational Behaviour**	**Italian Cinema**	**Russian**
0.08	0.03	0.04	0.05

What courses Jean Luc should take to maximize his chances of finding a job?

A Knapsack Problem

Decision Variables

For each course, one by one, he answers the question whether a course is taken. Then he will know what he is taking.

For example, if OR is taken, the answer will be a "yes" for OR, otherwise a "no". "yes" $\equiv 1$ and "no" $\equiv 0$. Also let x_{OR} be the question. Then, $x_{OR} = 1$ implies a "yes" answer for OR.

$$x_j = \left\{ \begin{array}{ll} 1 & \text{if course } j \text{ is taken} \\ 0 & \text{otherwise} \end{array} \right\}.$$

Constraints

There is only one constraint: 40 hours are available for studies.

$$9x_{OR} + 7x_{SC} + 5x_{IT} + 4x_{Fi} + 5x_{Ma} + 3x_{OB} + 7x_{IC} + 10x_{Ru} \leq 40$$

A Knapsack Problem

Objective Function

Writing contributions of each course and summing:

Maximize $0.10x_{OR}+0.14x_{SC}+0.06x_{IT}+0.09x_{Fi}+0.08x_{Ma}+0.03x_{OB}+0.04x_{IC}+0.05x_{Ru}$

IP Formulation

Max $0.10x_{OR}+0.14x_{SC}+0.06x_{IT}+0.09x_{Fi}+0.08x_{Ma}+0.03x_{OB}+0.04x_{IC}+0.05x_{Ru}$
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$$9x_{OR}+7x_{SC}+5x_{IT}+4x_{Fi}+5x_{Ma}+3x_{OB}+7x_{IC}+10x_{Ru} \le 40$$
$$x_j \in \{0,1\}$$

With solution $x_{OR}=1$, $x_{SC}=1$, $x_{IT}=1$, $x_{Fi}=1$, $x_{Ma}=1$ and $x_{IC}=1$, which courses are taken, any slack time?

Binary Programs? Origins of the knapsack problem.

A Facility Location Problem

This problem is very similar to the transportation problem. Only difference is that we want to decide on warehouse locations as well as flows. There are m locations where we can open up warehouses. To open up a warehouse we pay a cost of F_i.

Decision Variables

Reminiscent from the transportation problem, flows from i to j:

$$x_{ij} = \text{Amount of flow from warehouse } i \text{ to retailer } j$$

In addition to flows, answer the question whether warehouse i is opened up. If we answer is "yes" to this question, warehouse i is opened.

$$y_i = \left\{ \begin{array}{ll} 1 & \text{if warehouse } i \text{ opened up} \\ 0 & \text{otherwise} \end{array} \right\}.$$

A Facility Location Problem

Constraints

Demand Constraints: $\sum_{i=1}^{m} x_{ij} \geq d_j \;\; j = 1..n$

Supply Constraints: If warehouse i is not opened up, we can not send any units out of it:

$$y_i = 0 \implies \sum_{j=1}^{n} x_{ij} = 0$$

On the other hand, if warehouse i is opened, we can send its supply b_i to reatilers:

$$y_i = 1 \implies \sum_{j=1}^{n} x_{ij} \leq b_i$$

When warehouse i is open its supply is b_i, otherwise it is 0. In general, its $y_i b_i$:

$$\sum_{j=1}^{n} x_{ij} \leq y_i b_i \quad i = 1..m \; .$$

A Facility Location Problem

Objective Function

The cost of sending materials from warehouses to retailers:

$$\sum_{i=1}^{j} \sum_{j=1}^{n} c_{ij} x_{ij}$$

We also pay F_i by opening warehouse i, and 0 otherwise. In either case, we pay $F_i y_i$. The total cost of warehouses is:

$$\sum_{i=1}^{m} F_i y_i$$

Summing up the flow and warehouse costs, we drive the total cost to be minimized:

$$\text{Minimize } \sum_{i=1}^{j} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} F_i y_i$$

A Facility Location Problem

IP Formulation

Minimize $\sum_{i=1}^{j} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} F_i y_i$

Subject to

$$\sum_{i=1}^{m} x_{ij} \geq d_j \quad j = 1..n$$
$$\sum_{j=1}^{n} x_{ij} \leq y_i b_i \quad i = 1..m$$
$$x_{ij} \geq 0 \quad i = 1..m, \; j = 1..n$$
$$y_i \in \{0, 1\} \quad i = 1..m$$

Mixed Integer Linear Program (MILP)?

Player Selection at Dallas Cowboys

Dallas Cowboys wants to sign 3 new players. There are five players under consideration. If player j demands the salary c_j, make up an IP to decide on which players should be signed to minimize the signing budget considering

1. Player 1, 2 and 4 can play the quarter back position and Cowboys want to bring in at least one new player for this position.

2. Each player has a strength measured in s_j and Cowboys want to add at least S units of strength to the team by hiring some of these 5 players.

3. Player j is likely to have a_j assaults per season and Cowboys imposes a quota of A assaults per season for these players.

Player Selection at Dallas Cowboys

Decision Variables

$$x_j = \left\{ \begin{array}{ll} 1 & \text{if player } j \text{ is signed} \\ 0 & \text{otherwise} \end{array} \right\}.$$

Constraints

At least one quarterback: $x_1 + x_2 + x_4 \geq 1$

Minimum strength: $\sum_{j=1}^{5} s_j x_j \geq S$

Assaults quota: $\sum_{j=1}^{5} a_j x_j \leq A$

Objective Function

$$\text{Minimize} \sum_{j=1}^{5} c_j x_j$$

Player Selection at Dallas Cowboys: If-then constraints

- If player 1 and 2 are bodies and will come to Dallas only together, add a constraint to guarantee that either they are signed together or they are not signed at all.

  $(x_1 = 1 \implies x_2 = 1)$ and $(x_1 = 0 \implies x_2 = 0)$.

  Thus add : $x_1 - x_2 = 0$

- If player 1 hates player 3 and will not come if player 3 comes, add an appropriate constraint that does not allow signing player 1 and 3 together.

  $(x_3 = 1 \implies x_1 = 0)$ and $(x_1 = 1 \implies x_3 = 0)$.

  Thus add: $x_1 + x_3 \leq 1$

- If player 1 is signed and player 3 is not, player 4 is not going to sign. Add an appropriate constraint that does not allow signing player 4 when 1 is signed but 3 is not.

  $(x_1 = 1 \, , \, x_3 = 0 \implies x_4 = 0)$

  Thus add: $x_4 \leq 1 - x_1 + x_3$.

Player Selection at Dallas Cowboys: If-then constraints

- If player 1 is signed and player 3 is not, player 4 and 5 are not going to sign. Add appropriate constraints that do not allow signing player 4 or 5 when 1 is signed but 3 is not.

  $(x_1 = 1 \,,\ x_3 = 0 \implies x_4 = 0)$ and $(x_1 = 1 \,,\ x_3 = 0 \implies x_5 = 0)$

  Thus add: $x_4 \leq 1 - x_1 + x_3$ and $x_5 \leq 1 - x_1 + x_3$

- If player 1 is signed and player 3 is not, one of player 4 and 5 must be signed. Add appropriate constraints that signs player 4 or 5 when 1 is signed but 3 is not.

  $((x_1 = 1 \,,\ x_3 = 0) \implies (x_4 = 1$ or $x_5 = 1))$

  Thus add: $x_4 + x_5 \geq x_1 - x_3$

Integer programming is a very versatile formulation tool, almost to the level that you can formulate daily speech!