Introduction

- What is Operations Research?
- A brief history of the subject.
- Applications.

What is Operations research?

- Modern OR can be defined as the application of scientific methods to improve the effectiveness of operations, decisions and management.

- OR is the science of decision making.

- What are the specific benefits?
  - Increase revenue or return on investment; increase market share.
  - Decrease cost or investment.
  - Gain greater utilization from limited equipment, facilities, money and personnel.
  - Assess the likely outcomes of decision alternatives and uncover better alternatives.
  - Manage and reduce risk.
  - Quantify and balance qualitative considerations.
  - Improve quality.
  - Provide a better basis for forecasting and planning.
  - Demonstrate feasibility and workability and assist with training.

Business functions using OR routinely:

- **Board room and senior executive offices:** Top-level, high-impact policy setting, mergers and acquisitions, major expansions, valuation of companies, facility location, redesign of the entire supply chain, quantifying risk.

- **Manufacturing and service operations:** Scheduling of all kinds, routing, inventory control, improvement of work flow and elimination of bottlenecks, transportation and telecommunications.

- **Finance:** Investments, lending and borrowing, financial planning, insurance financial projections and forecasts.

- **Marketing:** Planning, sales forecasting, market selection, market measurement, segmentation and targeting, product portfolio analysis, location of distribution facilities, advertising and product design.

Business functions using OR routinely:Cont.

- **Information Technology:** Decision support systems, online analytical processing, optimization components of enterprise resource planning (ERP) systems, e-commerce systems.

- **Human resources:** Analysis of retirement plans, compensation studies and compensation planning, reduction of travel expenses.

- **Engineering:** Design and testing, optimization problems of all kinds.

- **Supply chain management.**

- **Research and development.**

- **Data mining.**

OR History

- 1910s Inventory models
- 1920s Quality control by Shewart of Bell Labs
- 1940s Positioning of limited number of radars on eastern UK coast for early detection of Luftwaffe missions
- 1940s Statistical study of the depth of U Boats while fending off a destroyer bombing
- 1950s Small scale planning and logistics activities, e.g., Berlin Air Lift
- 1970s OR becomes commonplace with fast computers

Linear Programming History

- Study of linear algebra
  - Equations and solutions: 2000 BC by Egyptians and Babylonians, 100 BC by Chinese Mathematician Chiu-Chang Suan-Shu in *Nine Book of Arithmetic*, 300s by Diophantos of Alexanderia, 500s by Hindu Mathematician Āryabhata, 800s by Arab Mathematician Al-Khowarizmi in *Al-jebr w'almuqabala*.
  - Determinants: 1693 by Leibnz (3x3) in a letter to l'Hospital. 1748 by MacLaurin (4x4) in *Treatise of Algebra*, 1750 by Cramer in *Introduction à l'analyse des lignes courbes algébriques*, also 1767 by Bézout, 1776 by Vandermonde, 1776 by Laplace.
  - Solution by elemination, row operations: 1805 by Legendre, 1809 by Gauss, 1888 by Clasen and 1904 by Jordan in *Handbuch der Vermessungskunde*.

- Geometry in higher (> 3) dimensions: 1827 by Möbius, 1843 and 1846 by Cayley: *". . . sans recourir à aucune notion métaphysique à l'égard de la possibilité de l'espace à quatre dimensions."*

Linear Programming History

- Optimization: 1797 by Lagrange in *Théorie des fonctions analytiques*, 1798 by Fourier in *Mémoire sur la statique*, 1890s Farkas and Minkowski establish structural results.

- Marrying algebra and geometry (the simple case: line of an equation): 1933 by Dines and McCoy, 1935 by Weyl and 1936 by Motzkin.

- Birth of Simplex: 1940s by Dantzig, von Neumann, Kantorovich, Koopmans, 1951 by Gale, Kuhn and Tucker, **1951 by Dantzig**.
  See Dantzig's web page: http://www.stanford.edu/dept/eesor/people/faculty/dantzig/

What are some of the broad classes of methods which OR uses?

- *Simulation Methods:* the OR professional develops simulators that give clients (executives, managers, front-line workers) the ability to dry run different approaches, on a computer, to search for improvements and to test improvement ideas.

- *Optimization Methods:* enable clients to search among possible choices – often where thousands and millions of choices are feasible, or where comparing choices is complex – to locate the very best or near-best.

- *Pattern Recognition Methods:* help clients detect patterns and connections in data, useful in such applications as forecasting and data mining.

Model Formulation

Mathematical Models

Model: A structure which has been built purposefully to exhibit features and characteristics of some other object.

Why to build models?

1. Improved understanding and communication

2. Experimentation

3. Standardization for analysis

Model Formulation: cont.

Common features of mathematical models:

- Abstraction, are details overlooked?
- Computability, can the model be manipulated with ease?
- Inputs, data requirements
- Uncertainty, are inputs and relationships between them uncertain?
- Decision horizons, flexibility, risk considerations ...

A **deterministic** model: All data known with certainty.

Model Formulation: cont.

Comments on the results/analysis from deterministic models

- In some cases, data characteristics **difficult to quantify**. For example, customer satisfaction.

- Sometimes, the use of **deterministic models** might force us to choose specific values for characteristics when, in reality, only a range is available.

- Use the results obtained to gain **insights** for the problem. Modify results if necessary.

Model Components with an Example: Machine Plant

- A plant has 120 (with 3 lathes) hours of turning capacity/week. It has 36 hours of grinding capacity per week.

- Two people work halftime and one person works full time for bolt and nail production $\Longrightarrow$ Manpower capacity 80 hours per week.

- Grinding and manpower needs to produce 1000 bolts and 1000 nails.

Activity	# of hrs required per 1000 bolts	# of hrs required per 1000 nails
Turning	3	4
Grinding	2	1
Manpower	1	3

- The plant makes a profit of $13 from every 1000 bolts and $15 from every 1000 nails. The objective is to maximize weekly profits.

Model Components with an Example: Machine Plant: Cont.

1. The Decision Variables

Decision variables capture the level of activities that the model studies.

Machine Plant Example: A machine plant wants to optimize its bolt and nail production. Number of bolts (nails) a machining plant produces in a week, denoted by B (N), is a common decision variable.

2. The Objective Function

We wish to maximize or minimize a quantity such as cost, profit, risk, net present value, number of employees, customer satisfaction, etc. The quantity is known as *objective function*.

Suppose that the machining plant makes a profit of \$13 from every 1000 bolts and \$15 from every 1000 nails. Then the objective function for maximizing weekly profit :

$$\text{Objective Function : } \textit{Maximize } 0.013B + 0.015N.$$

Model Components with an Example: Machine Plant: Cont.

3. The Constraints

Constraints represent the limitations such as available capacity, daily working hours, raw material availability, etc.

Activity	hrs per 1000 B	hrs per 1000 N
Turning	3	4
Grinding	2	1
Manpower	1	3

Turning Constraint : $3B/1000 + 4N/1000 \leq 120.$

Similarly we can write down two other constraints one for grinding capacity and one for manpower capacity:

Grinding Constraint : $2B/1000 + N/1000 \leq 36.$

Manpower Constraint : $B/1000 + 3N/1000 \leq 80.$

Nonnegativity Constraints : $B \geq 0 \; , \; N \geq 0.$

Scaling decision variables and parameters?

Linear Programming

- What is linear programming?

  A linear programming model is an optimization model in which we do the following:

  - We attempt to maximize or minimize a *linear* objective function.
  - The values of the decision variables must satisfy a set of constraints. Each constraint must be a linear equation or linear inequality.
  - A *sign restriction* is associated with each variable. For example a variable x_i may be non-negative ($x_i \geq 0$) or it may be unrestricted in sign.

Linear Programming: Cont.

- What does the term "linear function" mean?

  Any function in the form $c_1x_1 + c_2x_2 + ... + c_nx_n$ where $x_1, x_2, ..., x_n$ are variables and $c_1, c_2, ..., c_n$ are constants.

  For example, $2x_1 + x_2$ is a linear function of x_1 and x_2, but $f(x_1, x_2) = x_1^2x_2$ and $f(x_1, x_2, x_3) = \frac{x_1x_3}{x_2}$ are not.

- What does "linear inequality" mean?

  For any number b, any inequality in the form $c_1x_1 + c_2x_2 + ... + c_nx_n \geq b$ or $c_1x_1 + c_2x_2 + ... + c_nx_n \leq b$.

  For example, $2x_1 + 3x_2 \leq 11$ and $3x_1 - 2x_1 \geq 7.6$ are linear inequalities.

Linear Programming Assumptions

1. *Proportionality*: Contribution of each activity to an objective function and constraints is **proportional** to the level of that activity.

   - For example, if 1000 bolts require 3 turning hours, 100 bolts require 0.3 hours and 2000 bolts require 6 hours.
   - This assumption fails when we have economies of scale.

2. *Additivity*: Individual contribution of different activities can be **summed** up to obtain an objective function and constraints.

   - For example, the total time required for turning is obtained by summing turning hours required for bolts and nails.

3. *Certainty*: Each parameter in the formulation is known for **sure**.

4. *Divisibility*: Decision variables can take integer as well as **fractional values**.

   - For example, $x_1 = 0.7, x_2 = 5.14, x_3 = 2$

A Production Scheduling Problem

- An auto company manufactures cars (C) and trucks (T). Each vehicle must be processed in the paint shop and body assembly shop.

- If the paint shop were only painting trucks, 40 per day could be painted.

- If the paint shop were only painting cars, 60 per day could be painted.

- If the body shop were only producing cars, it could process 50 per day.

- If the body shop were only producing trucks, it could process 50 per day.

- Each truck contributes $300 to profit, and each car contributes $200 to profit.

- **Goal:** Use linear programming to determine a daily production scheduling that will maximize the company's profit.

A Production Scheduling Problem: Cont.

1. The decision variables

The company must decide how many cars and trucks should be produced daily. This leads us to define the following decision variables:

$$T = \text{number of trucks produced daily}$$
$$C = \text{number of cars produced daily}$$

2. The objective function

The company's daily profit (in hundreds of dollars) is $300T + 200C$, so the objective function is

$$\text{Maximize } 300T + 200C$$

A Production Scheduling Problem: Cont.

3. Constraints

If T trucks are produced, a $\frac{T}{40}$ fraction of the day would be spent in painting them. If C trucks are produced, a $\frac{C}{60}$ fraction of the day would be spent in painting them. Finish the painting job in 1 day:

$$\frac{T}{40} + \frac{C}{60} \leq 1$$

Similarly for the body shop

$$\frac{T}{50} + \frac{C}{50} \leq 1$$

We also have $T \geq 0$ and $C \geq 0$.

$$\begin{aligned} \text{Maximize} \quad & 300T + 200C \\ \text{Subject to:} \quad & \\ & \tfrac{1}{40}T + \tfrac{1}{60}C \leq 1 \\ & \tfrac{1}{50}T + \tfrac{1}{50}C \leq 1 \\ & T \geq 0 \,,\; C \geq 0. \end{aligned}$$

A Production Planning Problem

Scheduling the monthly production levels of a certain product for a planning horizon of twelve months. Generate a production schedule that minimizes the total production and inventory-holding.

- The total demand for the product j is d_j.
- The cost of producing each unit is c_j.
- The inventory holding cost is h_j.
- The production capacity is m_j.

<u>Assumptions:</u>

1. There is no initial inventory.
2. Production in month j is immediately available.
3. Shortage of the product is not allowed.

A Production Planning Problem: cont.

1. The Decision Variables

- $x_j =$ the production level for month j, $j = 1, 2, \ldots, 12$.

2. The Objective Function

Consider the first month again.

- The production cost equals $c_1 x_1$.
- The inventory-holding cost equals $h_1(x_1 - d_1)$

For the second month, we have:

- The production cost equals $c_2 x_2$.
- The inventory-holding cost equals $h_2(x_1 - d_1 + x_2 - d_2)$. This follows from the fact that the starting inventory level for this month is $x_1 - d_1$, the production level for the second month is x_2, and the demand for the second month is d_2.

A Production Planning Problem: cont.

- The total production cost for the entire planning horizon equals

$$\sum_{j=1}^{12} c_j x_j \equiv c_1 x_1 + c_2 x_2 + \cdots + c_{12} x_{12} \,.$$

- The total inventory-holding cost for the entire planning horizon equals

$$\begin{aligned} \sum_{j=1}^{12} h_j \left[\sum_{k=1}^{j} (x_k - d_k) \right] &\equiv h_1 \left[\sum_{k=1}^{1} (x_k - d_k) \right] + h_2 \left[\sum_{k=1}^{2} (x_k - d_k) \right] + \ldots \\ &\quad + h_{12} \left[\sum_{k=1}^{12} (x_k - d_k) \right] \\ &= h_1 [x_1 - d_1] + h_2 [(x_1 - d_1) + (x_2 - d_2)] + \ldots \\ &\quad + h_{12} [(x_1 - d_1) + (x_2 - d_2) + \cdots + (x_{12} - d_{12})] \,. \end{aligned}$$

A Production Planning Problem: cont.

Our goal is to minimize the total production and inventory-holding costs

$$\text{Min} \quad \sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j \left[\sum_{k=1}^{j} (x_k - d_k) \right].$$

3. The Constraints

- The production capacity constraints: $x_j \le m_j$ for $j = 1..12$
- No shortage constraint: $\sum_{k=1}^{j} (x_k - d_k) \ge 0$ for j=1..12.
  - Consider the first month. The production x_1 should be at least equal to the demand. That is, $x_1 \ge d_1$ which is the same as $x_1 - d_1 \ge 0$.
  - Now, let us consider the second month. A quantity of $x_1 - d_1$ gets carried over from the first month to the second. If x_2 is the amount produced during the second month, then a total of $x_1 - d_1 + x_2$ is available in the second month. This should be at least the demand d_2. So, we have $x_1 - d_1 + x_2 \ge d_2$ which is the same as $x_1 - d_1 + x_2 - d_2 \ge 0$.

A Production Planning Problem: cont.

We obtain a set of 24 constraints.

- Nonnegativity constraints: $x_j \geq 0$ for $j = 1..12$

4. LP Formulation

$$\begin{aligned}
&\text{Minimize} \quad && \sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j \left[\sum_{k=1}^{j} (x_k - d_k) \right] \\
&\textit{Subject to:} && \\
& && x_j \leq m_j \quad for \;\; j = 1, 2, \ldots, 12 \\
& && \sum_{k=1}^{j} (x_k - d_k) \geq 0 \quad for \;\; j = 1, 2, \ldots, 12 \\
& && x_j \geq 0 \quad for \;\; j = 1, 2, \ldots, 12 .
\end{aligned}$$

- Why did we formulate our problem in this specific format?
- Is this the *only* way to formulate the problem?

An Alternative Formulation for Production Planning

For any given month j, the inner sum in the objective, $\sum_{k=1}^{j}(x_k - d_k)$, is the ending inventory level for month j. Introduce an additional set of decision variables to represent the ending inventory levels.

- $y_j =$ the ending inventory level for month j, $j = 1, 2, \ldots, 12$;

then, the objective function:

$$\text{Min} \quad \sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j y_j \ .$$

No-shortage constraints simplify to $y_j \geq 0$ for $j = 1..12$. However, need to introduce a new set of constraints to "link" the x_j's and the y_j's.

$$y_j = y_{j-1} + x_j - d_j \ \ for\ j = 1..12$$

An Alternative Formulation for Production Planning

We arrive at the following new formulation:

$$\text{Min} \quad \sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j y_j$$

$$\begin{array}{lllll} Subject\ to: & & & & \\ & x_j & \leq & m_j & for\ j = 1, 2, \ldots, 12 \\ & y_j & = & y_{j-1} + x_j - d_j & for\ j = 1, 2, \ldots, 12 \\ & x_j & \geq & 0 & for\ j = 1, 2, \ldots, 12 \\ & y_j & \geq & 0 & for\ j = 1, 2, \ldots, 12. \end{array}$$

A linear program with 24 decision variables, 24 functional constraints.

An Alternative Formulation for Production Planning

–How flexible is this formulation?

- Relax Assumption 1: Set y_0 to whatever given value.

- Relax Assumption 2: Suppose instead there is a production delay of one month. Then, we replace inventory holding constraint by

$$y_j = y_{j-1} + x_{j-1} - d_j \;\; for \; j = 1..12$$

  For the first month, the given value of y_0 must be no less than d_1; otherwise, the resulting LP will not have any solution.

- Relax Assumption 3: Define s_j as the inventory shortage at the end of month j and introduce a shortage penalty cost of, say, p_j per unit of shortage at the end of month j.

An Alternative Formulation for Production Planning Allowing Shortages

$$
\begin{array}{llcll}
\text{Min} & \multicolumn{3}{l}{\sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j y_j + \sum_{j=1}^{12} p_j s_j} & \\
\textit{Subject to:} & & & & \\
& x_j & \leq & m_j & \textit{for } j = 1, 2, \ldots, 12 \\
& y_j - s_j & = & y_{j-1} - s_{j-1} + x_j - d_j & \textit{for } j = 1, 2, \ldots, 12 \\
& x_j & \geq & 0 & \textit{for } j = 1, 2, \ldots, 12 \\
& y_j & \geq & 0 & \textit{for } j = 1, 2, \ldots, 12 \\
& s_j & \geq & 0 & \textit{for } j = 1, 2, \ldots, 12
\end{array}
$$

If I have 10 units in the inventory at the end of month 1, is it $y_1 = 10$, $s_1 = 0$, or $y_1 = 20$, $s_1 = 10$, or any combination such that $y_j - s_j = 10$?

Even worse, what if my LP gives me a solution where $y_1 = 20$, $s_1 = 10$. How to avoid this nonsense?

A Project Scheduling Formulation

Nathan and his roommates are to clean their apartment: 8 cleaning steps. The order among these steps is governed by **precedence relations**.

Predecessor	Step	Duration (mins)
N/A	A = Clean the Fridge	10
N/A	B = Wash the Dishes	25
N/A	C = Make up the Beds	15
A , B	D = Clean the Sink	7
C , D	E = Take the Dust	18
E	F = Vacuum the Carpet	12
D	G = Take the Garbage out	3
F , G	H = Tidy up the Apartment	14

We make the following assumptions :

1. Nathan has plenty roommates (WLOG, say 8 people) to be assigned to the cleaning steps.

2. The number of people assigned to a step does not affect the duration of that step.

A Project Scheduling Formulation

1. Decision Variables

We want to minimize the finishing time of the last step (= H, why?). The starting times of the cleaning steps are sufficient to characterize the finishing times of all (including the last) step.

- t_j = Start time of step j, $j \in \{A, B, C, D, E, F, G, H\}$.

We assume that the first activity starts at time 0.

2. Objective Function

We want to minimize the finishing time of the last step:

$$\text{Min } t_H + 14.$$

A Project Scheduling Formulation

3. Constraints

A step can only start after its predecessor is finished. For example, “sink can only be cleaned after the dishes are washed” or B precedes D is represented as:

$$\text{Constraint B} \rightarrow \text{D} : t_D \geq t_B + 25$$

Similarly “sink can only be cleaned after the fridge is cleaned” or A precedes D is written as:

$$\text{Constraint A} \rightarrow \text{D} : t_D \geq t_A + 10$$

For each remaining precedence relation, we write a constraint:

$$\text{Constraint C} \rightarrow \text{E} : t_E \geq t_C + 15 \quad \text{Constraint D} \rightarrow \text{E} : t_E \geq t_D + 7$$

$$\text{Constraint E} \rightarrow \text{F} : t_F \geq t_E + 18 \quad \text{Constraint D} \rightarrow \text{G} : t_G \geq t_D + 7$$

$$\text{Constraint F} \rightarrow \text{H} : t_H \geq t_F + 12 \quad \text{Constraint G} \rightarrow \text{H} : t_H \geq t_G + 3$$

In addition to these 8 constraints, we add nonnegativity constraints: $t_A, t_B, t_C \geq 0$.

A Project Scheduling Formulation

4. Remarks

1. Convince yourself that the nonnegativity constraints on D, E, F, G, H are not needed. These constraints are *implied* by functional constraints and the nonnegativity constraints on A, B and C.

2. Does the optimal solution change if we drop 14 from the objective function? How about the objective of Min $2t_H$, does the optimal solution change this time? Any generalizations?

3. How can we be sure about H being the last operation? Can we generalize the formulation so that we do not need to know which operation is the last?

4. Problems of this type are known as "activity scheduling" problems. Besides LP, a method called CPM (critical path method) can be used to obtain a solution.

An Investment Problem

Investor has $100 on Monday. At the start of **every** day, if he invests x dollars on that day and matches that initial investment with $x/2$ dollars the next day, then he will receive a total return of $2x$ dollars on the third day. The investor wishes to determine an investment schedule that maximizes his total cash on Saturday.

Simplifying assumptions:

1. If an initial investment is not matched on the subsequent day, the initial investment is lost.
2. Any return that is due on any given day can be reinvested immediately.
3. Cash carried forward from one day to the next does not accrue interest.
4. Borrowing money is not allowed.

Simple strategy: Invest $(2/3)100$ on Monday and $(1/3)100$ on Tuesday to get $(4/3)100$ on Wednesday. Invest $(2/3)(4/3)100$ on Wednesday and $(1/3)(4/3)100$ on Thursday to get $(4/3)(4/3)100$ on Friday. How good is this simple strategy?

Note that under the divisibility assumption, the set of possible strategies is a continuum.

An Investment Problem

1. The Decision Variables

- $x_j =$ the amount of new investment at the beginning of Day j, $j = 1, 2, 3, 4$,

There is no need to introduce x_5, why?

On Monday, we would invest x_1 dollars and carry a cash saving of $100 - x_1$ forward to Tuesday. On Tuesday, after executing a second installment of $x_1/2$ dollars, we would have $100 - x_1 - x_1/2$ dollars available for an allocation of a new investment and a new cash saving.

An Investment Problem

To simplify constraints, introduce additional sets of decision variables. At the start of a given day and ask: What actions do I need to take at this point?

1. Cough up half of the amount of new investment (if any) that started in the previous day.

2. Initiate a new investment.

3. Carry the remaining cash (if any) forward to the next day.

Observe that these actions cannot be committed unless we know how much saving is being carried forward from the previous day.

- $s_j =$ the amount of saving carried forward from Day j to Day $j+1$, $j = 1, 2, 3, 4, 5$.

$s_0 = 100$. 9 decision variables, four x_j's and five s_j's.

An Investment Problem

2. The Objective Function

On Saturday (Day 6), there are two income streams; one is the yield from the investment cycle that started on Thursday $2x_4$ and the other is the saving from Friday s_5.

$$\text{Max } 2x_4 + s_5.$$

3. The Constraints

Balance the cash flow at the beginning of each day. For Day 1, we have $s_0 = 100$ dollars available apportioned into a new investment and a saving.

$$s_0 = x_1 + s_1.$$

For Day 2, we have s_1 dollars available apportioned into $0.5x_1$, x_2, and s_2.

$$s_1 = 0.5x_1 + x_2 + s_2.$$

An Investment Problem

At the beginning of Day 3, we receive a yield of $2x_1$, which is immediately available for reinvestment. It is therefore added into s_2; and the total amount is then divided into three parts as in Day 2.

$$2x_1 + s_2 = 0.5x_2 + x_3 + s_3.$$

Continuation of this argument yields

$$2x_2 + s_3 = 0.5x_3 + x_4 + s_4 \text{ for Day 4.}$$

$$2x_3 + s_4 = 0.5x_4 + s_5 \text{ for Day 5.}$$

Clearly, the x_j's must be nonnegative. We never overspend, so the s_j's are required to be nonnegative as well.

An Investment Problem

4. LP Formulation

$$\begin{aligned} \text{Max} \quad & 2x_4 + s_5 \\ \text{Subject to:} \quad & \\ s_0 &= x_1 + s_1 \\ s_1 &= 0.5x_1 + x_2 + s_2 \\ 2x_1 + s_2 &= 0.5x_2 + x_3 + s_3 \\ 2x_2 + s_3 &= 0.5x_3 + x_4 + s_4 \\ 2x_3 + s_4 &= 0.5x_4 + s_5 \end{aligned}$$

$x_j \geq 0$ for $j = 1..4$ and $s_j \geq 0$ for $j = 1..5$.

9 decision variables, 5 equality constraints, and 9 nonnegativity constraints.

5. Remarks

1. It may be instructive to attempt to formulate this problem using the x_j's only. Give it a try; it would be quite messy.

2. With x_5, the fifth constraint would have come out as $2x_3 + s_4 = 0.5x_4 + x_5 + s_5$. Consider an investment strategy that prescribes, say, $x_5 = 5$ and $s_5 = 10$. Why this cannot be optimal, answer by offering a better solution?

An Investment Problem

3. Relax Assumption 2 with reinvestment delay of one day.: for example $2x_1$ derived from the new investment on Day 1 won't be available until Day 4. Delete the term $2x_1$ from the lhs of the third constraint and transfer this term to that of the fourth constraint: $s_2 = 0.5x_2 + x_3 + s_3$ and $2x_1 + s_3 = 0.5x_3 + x_4 + s_4$. Make a similar revisions to other constraints. See lecture notes.

4. Relax Assumption 3: Daily interest rate is 1%.

Max $\quad 2x_4 + 1.01s_5$

Subject to:

$$\begin{aligned} s_0 &= x_1 + s_1 \\ 1.01s_1 &= 0.5x_1 + x_2 + s_2 \\ 2x_1 + 1.01s_2 &= 0.5x_2 + x_3 + s_3 \\ 2x_2 + 1.01s_3 &= 0.5x_3 + x_4 + s_4 \\ 2x_3 + 1.01s_4 &= 0.5x_4 + s_5 \end{aligned}$$

$x_j \geq 0$ for $j = 1..4$ and $s_j \geq 0$ for $j = 1..5$,

5. Relax Assumption 4: If borrowing is allowed, we can simply remove the nonnegativity requirements for the s_j's. For realism, we should, introduce some limits on loans.

A Food Blending Problem

A salad dressing supplier to major DFW area restaurants is to decide mix of oils in its dressing. This dressing (market value of $400/ton) is manufactured by refining raw oils and blending them together. Oil prices are below:

	Olive	Corn	Animal
Oct	280	390	110
Nov	290	400	90
Dec	310	430	100

There are three tanks to store each type of oil separately, each tank has 300 tons of capacity. The supplier dedicates a separate refining unit to each of the olive, corn and animal oils with capacities 190, 270, 210 for Olive, Corn, Animal oil. It is unhealthy to store refined oil. Storage costs per ton per month are $10 for all types of oil.

The hardness of salad dressings is regulated and has to be within 3 and 6. Generally, hardness blends linearly. Hardness of raw oils given below.

Olive	Corn	Animal
3.1	2.4	7.2

We will formulate this blending problem to maximize supplier's profit.

A Food Blending Problem

1. Decision Variables

- Let OB_i be the olive oil bought in month i. Similarly define OU_i and OS_i as the olive oil used in month i and stored at the end of month i.

- Similarly define CB_i, CU_i, CS_i and AB_i, AU_i, AS_i

- Let D_i be the dressing produced and sold in month i.

2. Objective Function

We want to maximize the (profit = revenue - cost).

- Revenue is obtained by selling the dressing: $400(D_1 + D_2 + D_3)$.
  - Raw oil purchase costs: $280OB_1 + 390CB_1 + 110AB_1 + 290OB_2 + 400CB_2 + 90AB_2 + 310OB_3 + 430CB_3 + 100AB_3$
  - Storage costs: $10(OS_1 + CS_1 + AS_1 + OS_2 + CS_2 + AS_2 + OS_3 + CS_3 + AS_3)$.

A Food Blending Problem

3. Constraints

1. **Inventory balance constraints:**

   - $OS_i = OS_{i-1} + OB_i - OU_i$ for $i = 1, 2, 3$.
   - $CS_i = CS_{i-1} + CB_i - CU_i$ for $i = 1, 2, 3$.
   - $AS_i = AS_{i-1} + AB_i - AU_i$ for $i = 1, 2, 3$.

   No oils on stock initially $\Longrightarrow OS_0 = CS_0 = AS_0 = 0$.

2. **Storage tank capacity constraints:**

   - For olive, corn, animal oil storage tanks $OS_i \leq 300$, $CS_i \leq 300$, $AS_i \leq 300$ for $i = 1, 2, 3$.

3. **Refining capacity constraints:**

   - For olive, corn, animal oil refining $OU_i \leq 190$, $CU_i \leq 270$, $AU_i \leq 210$for $i = 1, 2, 3$.

A Food Blending Problem

4. **Weight conservation constraint:**
   $OU_i + CU_i + AU_i - D_i = 0 \;\; for \; i = 1, 2, 3.$

5. **Hardness constraints:**
   - $3 \leq \frac{3.1OU_i + 2.4CU_i + 7.2AU_i}{D_i} \leq 6.$
     Manipulating the inequality above, we obtain the following inequalities:
   - $3.1OU_i + 2.4CU_i + 7.2AU_i - 6D_i \leq 0.$
     $3.1OU_i + 2.4CU_i + 7.2AU_i - 3D_i \geq 0.$

6. **Nonnegativity constraints:**
   $OU_i \geq 0$, $CU_i \geq 0$, $AU_i \geq 0$, $OB_i \geq 0$, $CB_i \geq 0$, $AB_i \geq 0$, $OS_i \geq 0$, $CS_i \geq 0$, $AS_i \geq 0$ for $i = 1, 2, 3$.

A Data Envelopment Analysis Problem

An OPRE 6201 professor receives a lot of complaints from students about grades. Many students complain that their grade does not reflect the number of hours they put into the course or the background they had from previous courses. The professor gives a shot to study how effective students are converting inputs such as number of hours worked per week and background into outputs such as grades, learning and enjoyment. Grades are measured in the overall course grade between 0 and 100, the number of hours worked per week is measured in hours between 0 and 168. The professor makes up a scale from 1 to 100 to measure background, learning and enjoyment. A value of 100 for background means that the student is extremely well prepared. A value of 100 for learning corresponds to learning all the topics in the course (presumably this is correlated with the grade). A value of 100 for enjoyment means that the class is student's favorite activity so much that the student attends the class only to hear the professor cracking jokes. Eventually the professor comes up with a table summarizing all the inputs and outputs where each student is given a row:

Student	Output Grade	Output Learning	Output Enjoyment	Input Hours/week	Input Background
k	G_k	L_k	E_k	H_k	B_k

A Data Envelopment Analysis Problem

The question here is to determine which students used their inputs most efficiently to obtain outputs. For example between two students A and B, if A started with weaker background and worked less than B and still got a higher grade, learned more and enjoyed more, we say that A is more efficient than B. Vaguely speaking efficiency is a measure of outputs divided by inputs.

First step, take weighted sum of outputs to boil down all outputs down to a single number O_k. Do another weighted some (possibly with different weights) to get a single number I_k for measuring inputs.

$$Efficiency\ of\ student\ k = \frac{O_k}{I_k}$$

The professor, known for his leniency, lets each student to choose a different set of weights. A student who gets a low grade can assign 0 weight to grade in calculating his output. Or a person who had a very strong background can assign 0 weight to background top decrease input and look efficient.

Each student, well-trained in LP formulations, is set out to develop a formulation to find best weights for him/her.

A Data Envelopment Analysis Problem Formulation for the kth student

1. Decision Variables

Clearly, weights are decision variables.
We use w for the weights of outputs, v for the weihgts of inputs.
Let wg_k be the weight of grade for the kth student and define wl_k, we_k similarly.
Let vh_k be the weight for work hours and define vb_k similarly.

2. Objective Function

Each student maximizes his/her total weighted output O_k.

$$Max \quad O_k = G_k wg_k + L_k wl_k + E_k we_k$$

A Data Envelopment Analysis Problem Formulation for the kth student

3. Constraints

1. Weights cannot lead to efficencies greater than 1:

$$\frac{O_j}{I_j} = \frac{G_j wg_j + L_j wl_j + E_j we_j}{H_j vh_j + B_j vb_j} \leq 1 \quad for\ all\ students\ j$$

2. Limit the weights on inputs, otherwise will get an “unbonded solution”:

$$H_k vh_k + B_k vb_k = 1$$

A Data Envelopment Analysis Problem

4. Remarks

1. Convince yourself that the solution the formulation will not change even if we replace "$=$" in constraint 2 with "$\leq$".

2. Constraints in 1 prohibit a student to choose weights arbitrarily. Indeed, there are no weights student B can assign to get a better efficiency than student A. If the solution to our formulation for the kth student has objective value below 1, that student is not efficient. Furthermore, (using sensitivity analysis) we can find a linear combination of other students which is more efficient (i.e. uses less inputs to obtain more).

3. This formulation is known as Data Envelopment Analysis. It used in comparing efficiencies of different but comparable units, especially in the public sector: efficencies of public hospitals, public schools, several franchises of the same company, etc.

A Toy Problem with Piece Wise Linear Costs

A furniture manufacturer produces and sells TV stands at a price of $100. If 10 or fewer stands are manufactured per week, each stand costs $60. Manufacturer's regular capacity is 10 stands per week. Manufacturer can hire additional workforce to bring its capacity up to 20 stands per week. However, additional capacity is costly so any stand produced after the 10th costs $75. For example 12 stands cost $10 \cdot 60 + 2 \cdot 75 = 750$ dollars. Manufacturer has a weekly operating capital of $1200 so its weekly costs can not exceed this amount. Provide an LP formulation to maximize the manufacturer's profit.

A problem with Piece Wise Linear Costs

1. Decision Variables

Let x be the number of stands produced.
Can we express all the quantities we are interested in with x? For example let c be the production cost.

$$c = \left\{ \begin{array}{ll} 60x & \text{if } 0 \leq x \leq 10 \\ 75x - 150 & \text{if } x \geq 10 \end{array} \right\}.$$

This is not a linear constraint/equality. However, notice that

$$c = \max\{60x,\ 75x - 150\}$$

which implies

$$c \geq 60x \text{ and } c \geq 75x - 150$$

These are linear inequalities. Keep c as (decision?) variable.

A problem with Piece Wise Linear Costs

2. Objective Function

Maximize (profit=revenue-cost):

$$Max \;\; 100x - c$$

3. Constraints

From the definition of c:

$$c \geq 60x \text{ and } c \geq 75x - 150$$

Operating capital:

$$c \leq 1200$$

Nonnegativity and Capacity on production:

$$0 \leq x \leq 20$$

A problem with Piece Wise Linear Costs

$$
\begin{array}{llrrcrr}
\text{Max} & & 100x & -c & & & \\
\text{Subject to:} & & & & & & \\
& & 60x & -c & \leq & 0 & (1) \\
& & 75x & -c & \leq & 150 & (2) \\
& & & c & \leq & 1200 & (3) \\
& & x & & \leq & 20 & (4) \\
& & x, & c & \geq & 0 & .
\end{array}
$$

A Location Problem

It is desirable to locate service facilities physically close to where the demands are. Suppose that there are only 4 existing apartment complexes in a small town and the coordinates of the jth complex is given as (x_j, y_j) for $j = 1..3$. We want to locate a mall at a location (a, b) where a, b are to be decided upon. The distance between the mall and the jth complex is given by $|a - x_j| + |b - y_j|$. Provide an LP formulation to minimize the total **rectilinear distance** between the mall and the apartments.

A Location Problem

1. Decision Variables

We are deciding on the coordinates of the mall specified by (a, b): a, b are decision variables. Let d_j^x be the x component of the rectilinear distance between the mall and the complex j. Define d_j^y similarly for y components.

2. Objective Function

$$Min d_1^x + d_2^x + d_3^x + d_1^y + d_2^y + d_3^y$$

A Location Problem

3. Constraints

$$d_1^x = |a - x_1| \quad \Longrightarrow \quad d_1^x = max\{a - x_1, -(a - x_1)\}.$$

$$d_1^x = max\{a - x_1, -(a - x_1)\} \quad \Longrightarrow \quad d_1^x \geq a - x_1 \ and \ d_1^x \geq -(a - x_1).$$

Then in the optimal solution:

$$d_1^x \geq a - x_1 \ and \ d_1^x \geq -(a - x_1) \quad \overset{Min\ d_1^x + \dots}{\Longrightarrow} \quad d_1^x = a - x_1 \ or \ d_1^x = -(a - x_1)$$

Thus, $d_1^x = |a - x_1|$ is satisfied by the optimal solution. No more constraints on d_1^x.

A Location Problem

Min $d_1^x + d_2^x + d_3^x + d_1^y + d_2^y + d_3^y$

Subject to:

$$\begin{aligned} d_j^x &\geq a - x_j & j = 1,2,3 \\ d_j^x &\geq -(a - x_j) & j = 1,2,3 \\ d_j^y &\geq b - y_j & j = 1,2,3 \\ d_j^y &\geq -(b - y_j) & j = 1,2,3 \end{aligned}$$

A Location Problem

4. Remarks

1. Do we need nonnegativity constraints on d_j^x or d_j^y ?

2. Suppose that the first apartment complex has twice as many people as others. How do we modify the formulation?

3. When the objective function is the (population) weighted sum of distances, it represents a private sector objective. When p facilities are to be located, the problem is called the p-Median problem.

4. Suppose that we are locating a fire station (as opposed to a mall) and we want to minimize the maximum of the distances between the station and the apartments, i.e. min $\{$max $\{|a - x_j| + |b - y_j|\ ,\ j = 1..3\}\}$. How do we modify the formulation?

5. When the objective function is minimizing the distance to the furthest apartment complex, it represents a public sector objective. When p facilities are to be located, the problem is called the p-Center problem.

A Manpower Planning Problem

Lockheed Martin has won the bid for the manufacturing of new generation fighter jets. Bulding these jets will require skilled labor hours:

	2002	2003	2004	2005
Labor hrs in Millions	3.4	3.8	3.6	4.0

At the begining of 2002, there will be 1000 skilled workers at Lockheed Martin. Each worker works at most 2400 hours per year. In order to meet future labor demand new workers must be trained for a year. During training, each trainee must be supervised for 1000 hours by a skilled worker. Each skilled worker is paid $50 K and each trainee is paid $30 K per year. At the end of each year 10% of the skilled workers quit to join Boeing. formulate an LP to minimize the labor costs while meeting annual labor needs.

A Manpower Planning Problem

1. Decision variables

T_i: Number of trainees during year i.

S_i: Number of skilled workers in year i.

2. Objective function

Minimize labor costs:

$$Min \quad 50(S_{2002} + S_{2003} + S_{2004} + S_{2005}) + 30(T_{2002} + T_{2003} + T_{2004} + T_{2005})$$

A Manpower Planning Problem

3. Constraints

Worker balance equations:

$$S_{2002} = 1000$$
$$S_{2003} = 0.9S_{2002} + T_{2002}$$
$$S_{2004} = 0.9S_{2003} + T_{2003}$$
$$S_{2005} = 0.9S_{2004} + T_{2004}$$

Annual labor need constraints:

$$2400S_{2002} - 1000T_{2002} \geq 3.4M$$
$$2400S_{2003} - 1000T_{2003} \geq 3.8M$$
$$2400S_{2004} - 1000T_{2004} \geq 3.6M$$
$$2400S_{2005} - 1000T_{2005} \geq 4.0M$$

Nonnegativity constraints.

This is an **infeasible** problem.

Conclusion: We solved various examples

1. Machine Plant
2. Production Scheduling for Cars and Trucks
3. Production Planning with and without shortages
4. Project Scheduling with and without the knowledge of the last operation
5. Investment with and without interest rates
6. Food Blending
7. Data Envelopment Analysis
8. Piece-wise Linear Costs
9. Location, p-Median and p-Center
10. Manpower planning