

## 1 Inventories with Certainty: $\Re\epsilon\alpha\delta$ Chapter 10 of Chopra

### 2 Solved Examples

1. All-units quantity discounts: A popular shoe store sells 8000 pairs per year. The fixed cost of ordering shoes from the distribution center is \$15 and holding costs are taken as 25% of the shoe costs. The per unit purchase costs from the distribution center is given as

$$\left\{ \begin{array}{ll} C_1 = 60, & \text{if } 0 \leq Q \leq 50 \\ C_2 = 55, & \text{if } 50 \leq Q \leq 150 \\ C_3 = 50, & \text{if } 150 \leq Q \end{array} \right\}$$

where  $Q$  is the order size. Determine the optimal order quantity.

Solution: There are three ranges for lot sizes in this problem:  $(0, q_1 = 50)$ ,  $(q_1 = 50, q_2 = 150)$  and  $(q_2 = 150, \infty)$ . Holding costs in these ranges of shoe prices are given as  $h_1 = (0.25)60 = 15$ ,  $h_2 = (0.25)55 = 13.75$  and  $h_3 = (0.25)50 = 12.5$ . EOQ quantities in these ranges are

$$EOQ_1 = \sqrt{\frac{2(15)(8000)}{15}} = 126.5; \quad EOQ_2 = \sqrt{\frac{2(15)(8000)}{13.75}} = 132.1; \quad EOQ_3 = \sqrt{\frac{2(15)(8000)}{12.5}} = 138.6;$$

Only  $EOQ_2 = 132.1$  is in the appropriate range, i.e.  $q_1 \leq EOQ_2 \leq q_2$ , so it is a candidate solution. Since  $EOQ_1 > q_1$ , we take  $q_1 = 50$  as the candidate solution for the second range. Since  $EOQ_3 < q_2$ , we take  $q_2 = 150$  as the candidate solution for the third range.

It is clear that ordering  $Q = 50$  at the cost  $C_1 = 60$  is worse than ordering  $Q = 50$  at the cost  $C_2 = 55$ . Moreover ordering  $Q = 132.1$  is better than ordering  $Q = 50$  in the second range. Combining the last two statements, the costs at  $Q = 132.1$  and  $C_2 = 55$  is smaller than the costs at  $Q = 50$  and  $C_2 = 60$ . Therefore we can eliminate  $Q = 50$  from the consideration. Evaluating the remaining lot sizes

$$TC_2(Q = 132.1) = 8000(55) + 8000(15)/132.1 + (0.25)(55)(132.1)/2 = 441,800$$

$$TC_3(Q = 150) = 8000(50) + 8000(15)/150 + (0.25)(50)(150)/2 = 401,900$$

Then  $Q = 150$  is the optimal solution with a cost of 401,900.

2. Marginal units quantity discounts: Refer to the previous exercise, suppose the distribution center gives marginal unit discount and compute the optimal order quantity.

Solution:  $V_0 = 0$ ,  $V_1 = V_0 + (50)60 = 3000$  and  $V_2 = V_1 + 100(55) = 8500$  are cost of buying exactly 0,  $q_1 = 50$  and  $q_2 = 150$  units. Then EOQs are

$$EOQ_1 = \sqrt{\frac{2R(S + V_0 - q_0c_0)}{hC_1}} = \sqrt{\frac{2(8000)(15 + 0 - 0)}{15}} = 126.5;$$

$$EOQ_2 = \sqrt{\frac{2R(S + V_1 - q_1c_1)}{hC_2}} = \sqrt{\frac{2(8000)(15 + 3000 - (50)55)}{13.75}} = 555.3;$$

$$EOQ_3 = \sqrt{\frac{2R(S + V_2 - q_2c_2)}{hC_3}} = \sqrt{\frac{2(8000)(15 + 8500 - (150)50)}{12.5}} = 1139.8;$$

For the first and second ranges candidates are  $Q = 50$  and  $Q = 150$ . Then  $TC_1(Q)$  is decreasing over  $0 \leq Q \leq 50$  and similarly  $TC_2(Q)$  is decreasing over  $50 \leq Q \leq 150$ . For the third range  $Q = 1139.8$ . We do not need to compute  $TC_1(Q = 50)$  or  $TC_2(Q = 150)$  because they are larger than  $TC_3(Q = 1139.8)$  is decreasing. However we write these costs down for the sake of exercise.

$$TC_1(Q = 50) = (8000/50)15 + (0 + (50)60)0.25/2 + (8000/50)(0 + (50)60)$$

$$TC_2(Q = 150) = (8000/150)15 + (3000 + (100)55)0.25/2 + (8000/150)(3000 + (100)55)$$

$$\begin{aligned} TC_3(Q = 1139.8) &= (8000/1139.8)15 + (8500 + (1139.8 - 150)50)0.25/2 + (8000/1139.8)(8500 + (1139.8 - 150)50) \\ &= 421,266 \end{aligned}$$

Then  $Q = 1139.8$  is the optimal solution with a cost of 421,266.

### 3 Preliminaries from Probability

#### 3.1 Numerical events and random variables

Consider tossing a coin once. We do not know what the outcome will exactly be but we know that it is either Head (H) or Tail (T). The set of possible outcomes is called a sample space and is denoted by  $S$ . For a single toss,  $S = \{H, T\}$ . For two tosses in a row,  $S = \{HH, HT, TH, TT\}$ . A real valued function  $X$  from the sample space  $S$  to real variables is a random variable. For a single coin toss,  $X_1$  given as  $X_1(H) = 0$  and  $X_1(T) = 1$  is random variable.  $X_2(H) = 1$  and  $X_2(T) = 0$  is another random variable.  $X_1$  denotes the number of tails and  $X_2$  denotes the number of heads in a single toss.

Example: Consider tossing a coin twice. The sample space is  $S = \{HT, TH, HH, TT\}$ . Let  $X_1$  be the number of tails. Then  $X_1(HT) = 1$ ,  $X_1(TH) = 1$ ,  $X_1(HH) = 0$ ,  $X_1(TT) = 2$ . Let  $X_2$  be the number of heads. Then  $X_2(HT) = 1$ ,  $X_2(TH) = 1$ ,  $X_2(HH) = 2$ ,  $X_2(TT) = 0$ . Let  $X_3 = X_1 - X_2$ , then  $X_3(HT) = 0$ ,  $X_3(TH) = 0$ ,  $X_3(HH) = -2$ ,  $X_3(TT) = 2$ . How would you define  $X_3$  in English. Suppose that  $X_4$  denotes the earnings in a betting situation where we earn \$5 if H comes up and earn \$10 if T comes up, i.e.  $X_4(HT) = 15$ ,  $X_4(TH) = 15$ ,  $X_4(HH) = 10$ ,  $X_4(TT) = 20$ . If you let  $X_4^1$  and  $X_4^2$  be the earnings in the first and the second toss then  $X_4^1(HT) = 5$ ,  $X_4^1(TH) = 10$ ,  $X_4^1(HH) = 5$ ,  $X_4^1(TT) = 10$  and  $X_4^2(HT) = 10$ ,  $X_4^2(TH) = 5$ ,  $X_4^2(HH) = 5$ ,  $X_4^2(TT) = 10$ . Note that  $X_4 = X_4^1 + X_4^2$ . Clearly, one can define various random variables using the same sample space.

A *discrete random variable* can assume only a finite or countably infinite number of distinct values. All the random variables presented above are discrete because they take a finite number of values. Here is an example of a discrete random variable with countably infinite values: Suppose that we are tossing a coin until H comes up, then  $S = \{H, TH, TTH, \dots, T..TH, \dots\}$ . Let  $X$  be the number of tosses until H appears so  $X(H) = 1$ ,  $X(TH) = 2$ ,  $X(TTH) = 3$ , etc. Clearly  $X$  assumes all positive integers. There infinite number of integers but they are countable. Consequently,  $X$  is discrete. Some other examples of discrete random variables:

1. Number of oil reserves to be found in Texas in this year
2. Number of machines breaking down in a plant today
3. Number of BS students graduating this year

We represent probabilities associated with a random variable in a handy way by  $P(X = a)$  where  $a$  is a constant.  $P(X = a)$  is the probability that the random variable  $X$  would assume value  $a$ .  $P(X = a)$  is called the probability mass function (pmf for short).

Example: Consider tossing a coin twice. Let  $X$  be the number of Tails coming up. The sample space, the random variable and associated probabilities are:

$S$	$X$	$Probability$
$HT$	1	1/4
$TH$	1	1/4
$TT$	2	1/4
$HH$	0	1/4

$$P(X = a) = \begin{cases} 1/4 & \text{if } a = 0 \\ 1/2 & \text{if } a = 1 \\ 1/4 & \text{if } a = 2 \\ 0 & \text{if } a = 3 \end{cases}$$

Exercise: Suppose that 4 people including you and your friend line up at random. Find the pmf for  $X$ , the number of people standing between you and your friend. Solution:  $P(X = 0) = 3/6$ ,  $P(X = 1) = 2/6$ ,  $P(X = 2) = 1/6$ . Repeat this question with a total of 5 people. Solution:  $P(X = 0) = 4/10$ ,  $P(X = 1) = 3/10$ ,  $P(X = 2) = 2/10$  and  $P(X = 3) = 1/10$ .

Averages are generally used to summarize data. We often summarize a random variable with the expected value and variance:

$$E(X) = \sum_a a \cdot P(X = a) \qquad Var(X) = E(X - E(X))^2$$

Expected value has a physical meaning: If you put weights of  $P(X = a)$  at each point  $a$ , then  $E(X)$  is the center of the gravity. No such intuitive reasoning can be provided for variance, except for saying that it is the expected value of the square of the variation from the mean. Surely other measures than variance can be built to measure variation.

Example: Consider tossing a coin twice. Let  $X$  be the number of Tails coming up.

$$E(X) = 0 \cdot 1/4 + 1 \cdot 1/2 + 2 \cdot 1/4 = 1$$

$$Var(X) = (0 - 1)^2 \cdot 1/4 + (1 - 1)^2 \cdot 1/2 + (2 - 1)^2 \cdot 1/4 = 1/2$$

Expected value of functions of random variables can be computed using:

$$E(g(X)) = \sum_a g(a) \cdot P(X = a)$$

where  $g$  is any function of the random variable  $X$ . The next example illustrates this concept.

Example: Reconsider the 4 people lining up at random, including you and your friend. Suppose you want to pass a full glass of water to your friend and 1/4 of the water in the glass spills over at every pass. What is the expected percentage of water your friend receives. Let us construct  $g$  first, let  $g(X)$  be the percentage of the water left in the glass after  $X$  passes.  $g(X = 1) = 3/4$ ,  $g(X = 2) = (3/4)^2$  and  $g(X = 3) = (3/4)^3$ . Then,

$$E(g(X)) = (3/4)(3/6) + (3/4)^2(2/6) + (3/4)^3(1/6) = 0.63$$

63% of the water remains and 37% is spilt. It is not hard to imagine applications of these ideas to communication networks where data is distorted as it moves through the network.

Most commonly used discrete distributions are: Binomial and Poisson. The former is used to model number of successes in a certain number of trials where success probability of each trial is constant and is independent of

other trials. The latter is used to model the number of occurrences, say number of arrivals to a bank teller in a day. There are various other useful discrete densities such as geometric and multinomial that are used in other fields like quality control, reliability, etc.

A *continuous random variable* takes any value in an interval. Inches of rainfall in Dallas this month and the elapsed time until you have a flat tire are examples of continuous random variables. Let  $X$  denote any (discrete or continuous) random variable, the distribution function is  $F(x)$  and is defined by

$$F(x) = P(X \leq x) \text{ for } -\infty < x < \infty.$$

Discrete random variables have distribution functions that look like a step function whereas continuous variables have continuous distribution functions. In the case of a continuous random variable, when the derivative of  $F$  can be obtained, it is called the probability density function or pdf for short and denoted by  $f(x)$ :

$$f(x) = \frac{dF(x)}{dx}.$$

For a continuous random variable  $X$ ,  $P(X = a) = 0$  for any value of  $a$ . This is equivalent to the area of the height of pdf at  $a$ , certainly the area of a line is zero. Indeed if  $P(X = a) > 0$  then  $X$  is not continuous at  $a$ . For a continuous variable  $X$ :

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b) = \int_a^b f(x)dx = F(b) - F(a)$$

Expected value and variances of continuous random variables are computed analogously to discrete random variables. If  $X$  is a continuous random variable,

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx \quad \text{Var}(X) = E(X - E(X))^2 \quad E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)dx$$

where  $g$  is again a function of random variable  $X$ .

Example: Let us find the expected value of a random variable  $X$  given by its pdf  $f(x)$ :

$$f(x) = \begin{cases} 3x^2/2 + x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

You may first want to check that this density is valid, i.e. it sums up to one:

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 3x^2/2 + x dx = (1/2)x^3 + (1/2)x^2 \Big|_{x=0}^{x=1} = 1.$$

Now the expected value:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^1 x(3x^2/2 + x)dx = (3/8)x^4 + (1/3)x^3 \Big|_{x=0}^{x=1} = 17/24.$$

Two common continuous distributions are Uniform and Normal. We say that  $X$  has a uniform distribution over the interval  $(a, b)$  and write  $X \sim U(a, b)$  if the probability of  $X$  taking any value in the interval is constant, i.e. pdf is constant. To make sure that probabilities over  $(a, b)$  sum up to 1, we choose pdf as:

$$f(x) = \begin{cases} 1/(b - a) & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Also note that

$$F(x) = \begin{cases} 0 & \text{if } x < a \\ x/(b - a) & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases}$$

Uniform distribution arises in practice if the knowledge about the random variable is very limited to the level that we only know its range  $(a, b)$  and nothing else. In this case, we assume that each outcome over the interval is equally likely. Many companies use such an approach in demand forecasting; Future demand forecasts are modelled by a uniform distribution over  $(a, b)$  where  $a$  ( $b$ ) stands for the low (high) demand scenario. Although uniform distribution is simple to work with, it lacks a very important property that limits its justification. That is, sum of two uniform random variables is not a uniform random variable. That is why other distributions especially Normal finds an extensive use in the practice.

Before we study Normal distribution let us see why it is so popular:

- Most observations are expected to be around the mean. Extreme observations are very rare. Because of this, normal is called a light-tail distribution.
- Sum of independent normal random variables is another normal random variable.
- Technically speaking averages of all independent random variables (as the number of variables in the average grows) converge to normal random variable (this is known as central limit theorem). This property is extensively used in statistics.

Let us elucidate why we are interested in the sums of random variables. Suppose we are modelling the weekly demand for a supplier. One approach is to figure out the weekly demands of each customer of the supplier and sum them up. This is demand aggregation over customers. Another approach is to find daily demands for the supplier and sum them up over a week. This is temporal aggregation over days. Clearly one can aggregate the demand over both customers and days. Important point is that we often deal with aggregated numbers coming from different sources. Then assuming a normal distribution for sum/average of numbers is very justifiable as such sum/average converges to a normal distribution.

If  $X$  is a normal random variable with mean  $\mu$  and variance  $\sigma^2$ , its pdf is

$$f(x) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) \text{ for } -\infty < x < \infty$$

where  $\exp$  is the exponential function. We write  $X \sim N(\mu, \sigma^2)$ .

Using this pdf form we can argue that linear transformations of normal random variables are normal but with an appropriate scaling of parameters. Namely if  $\alpha, \beta$  are constants and  $X \sim N(\mu, \sigma^2)$  then  $\alpha X + \beta \sim N(\alpha\mu + \beta, \alpha^2\sigma^2)$ . This property, known as scaling of normal, becomes useful in computing normal probabilities :

$$P(a \leq X \leq b) = P\left(\frac{a - \mu}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{b - \mu}{\sigma}\right) = P\left(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma}\right) = \int_{\frac{a - \mu}{\sigma}}^{\frac{b - \mu}{\sigma}} \phi(z) dz$$

where  $Z$  is a standard normal variable, i.e.  $Z \sim N(0, 1)$  and  $\phi(z)$  is the pdf for the standard normal variable. We use  $\Phi$  for cdf of standard normal variable. Because of the above equality, we can use standard normal densities to compute probabilities for any normal random variable. This is the reason why books tabulate only standard normal probabilities.

It also follows that we can use scaling to compute cumulative density function,

$$F(x) = P(X \leq x) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = P\left(Z \leq \frac{x - \mu}{\sigma}\right) = \int_{-\infty}^{\frac{x - \mu}{\sigma}} \phi(z) dz = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

Because of this scaling normal probabilities for all normal variables can be counted via standard normal variable or its cdf  $\Phi$ , which is often tabulated at the end of books.

Another set of equalities useful for us in inventory deals with computation of expected stockouts. Suppose that we currently hold  $ROP$  units on hand and facing random demand  $X$ . Expected stockout is  $E(X - ROP)^+ := E \max(X - ROP, 0)$ . In general, expected shortages are found by numerically integrating

$$\int_{ROP}^{\infty} (x - ROP)f(x)dx.$$

It may be conceptually useful to rewrite expected stockouts as

$$\int_{ROP}^{\infty} (x - ROP)f(x)dx = \int_{ROP}^{\infty} \int_x^{ROP} du f(x)dx = \int_{ROP}^{\infty} \int_u^{\infty} f(x)dxdu = \int_{ROP}^{\infty} 1 - F(u)du$$

where we change the order of integration to obtain the second equality. However, these manipulations cannot save us from numerical integration. In the case of Normal demand, we can avoid integrations, see the Formulas on p.220 of Chopra.

As you may have already realized by now, manipulations with normal density is not easy. For example,  $F(x)$  cannot be put into a closed form expression. However, it is possible to obtain it with numerical integration and all modern software -including Excel- has built-in functions to compute these probabilities.

Computing probabilities for	$X \sim N(\mu, \sigma^2)$	
<i>Functions</i>	<i>Matlab</i>	<i>Excel</i>
pdf, $f(x)$	$normpdf(x, \mu, \sigma)$	$normdist(x, \mu, \sigma, 0)$
cdf, $F(x)$	$normcdf(x, \mu, \sigma)$	$normdist(x, \mu, \sigma, 1)$
inverse cdf, $F^{-1}(p)$	$norminv(p, \mu, \sigma)$	$norminv(p, \mu, \sigma)$

For a given  $p$  ( $0 \leq p \leq 1$ ), inverse cdf finds the outcome value  $x$  such that the probability of observing smaller outcomes than  $x$  is exactly  $p$ . Note that if  $p = F(x)$  then  $x = F^{-1}(p)$ . Note that Excel uses a single function to compute both pdf and cdf, use the last argument of normdist to specify whether you want pdf or cdf. Discussion here has been purposefully very sketchy. More information on probability can be found in [3], [4] and [2].

## 4 Inventories with Uncertainty: $\Re\epsilon\alpha\delta$ Chapter 11 of Chopra

### 5 Postponement

Postpone each product as much as possible, ideally until the random demand for that product is observed. Recall the discussion about how the variances of the forecast for a given period reduce as time passes, see chapter SC03. By postponing, it is possible to learn more information about demand which effectively reduces demand forecast variability. As we know from the above discussion, safety stocks are used to counter the variability in the demand. Thus, lower safety stocks are needed when the variability is low. Consequently, postponement is used to reduce safety stock holding costs.

If we can single out a single customization — generally the highest value adding — stage, we can split supply chain into two before the customization and after the customization. Inventories before the customization have two significant properties. First, before the customization, raw materials or semi-finished assemblies have low manufacturing or material costs. Consequently, inventory holding costs are lower for these items, so even when a high level of inventory is held, the cost may be relatively insignificant. Second, before the customization, raw material or semi-finished assemblies have more or less the same characteristics and functionality so they can be pooled into broader categories of inventories. Inventory pooling reduces the variability of the aggregated demand and hence the safety stock levels.

A word of warning is in order here. Reduction in demand variability is due to both the resolution of uncertainty and inventory pooling. Many works in the literature fail to identify the distinction between these two factors. They go ahead with a partial analysis of studying only inventory pooling effects on the aggregated demand. Such

a partial analysis could be defended by saying that resolution of uncertainty is not measured in almost all of the instances in practice. However, this defense points out the need for resolution of uncertainty data rather than the correctness of the partial analysis.

We use the term product family to refer to a group of products which are differentiated at the customization stage. Postponement saves significant amount of money under the following conditions.

1. Products are almost indistinguishable before customization. That is, the higher the component commonality before the customization, the more the variance reduction due to inventory pooling.
2. Customization stage adds a significant value to the product. This condition will make inventory holding cost rates significantly low before the customization and high after the customization. Therefore, postponement becomes more of a cost cutting strategy. For example LCD displays add a high value to laptops. If the stage where displays are assembled is the customization stage, savings due to postponement will be higher.
3. Demand is highly random and is fluctuating so forecasts are very variable. Therefore, safety stocks are significant and so will the savings in safety stock holding costs be, when postponement is implemented.
4. SC is flexible to allow for postponements to various stages/times. Uncertainty resolves fairly fast before the customization time but slows down after it. For example, in Figure 1 the uncertainty in product family A resolves fast before A's customization time and slowly after then. On the contrary, product family B's uncertainty mainly resolves after A's customization time. Consequently, benefits of postponement can best be reaped by delaying customization of family B beyond the customization time of family A. Shortly, customization times should be chosen so that after customization there is not much uncertainty to resolve, i.e. everything looks as if they are deterministic. When choosing customization time/stage as such is not possible, postponement's benefits will be limited.

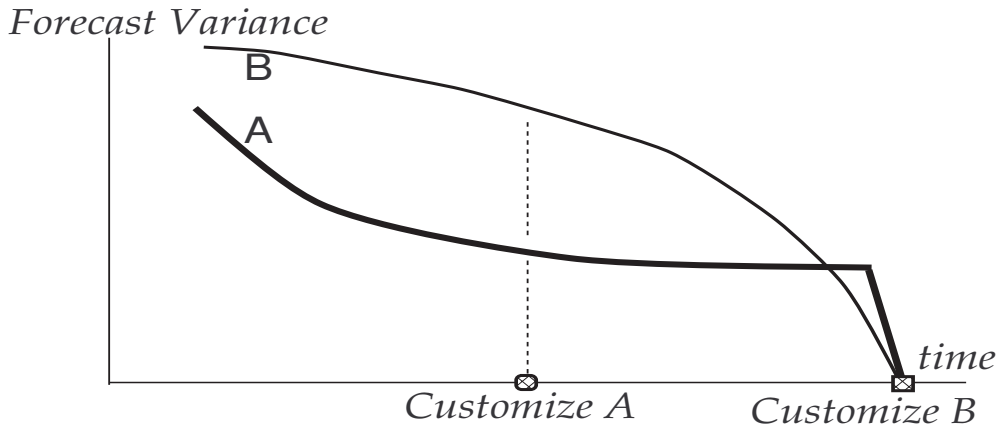


Figure 1: Resolution of uncertainty (decrease in the forecast variance) in product families A and B.

The flexibility we are referring to here is SC flexibility. Thus it includes flexibility of production lines, e.g. the extent to which we can modify the lines to postpone some operations. It also includes supplier flexibility, e.g. the extent to which supplier can quickly deliver orders. Since some operations are delayed purposefully with the postponement strategy, the risk of missing customer deadlines is higher. It may be necessary and also wise to incur some extra costs to obtain a better supplier service, such as frequent deliveries from the supplier or high product availability at the supplier. Postponement puts similar “burdens” on procurement operations, sales departments, personnel, etc. For example H&P uses its warehouses also as packing facilities to achieve delayed customization. This adds some extra responsibility to warehouse personnel and complicates the operations at the warehouses. Shortly, postponement stretches out SC, sometimes to the point of breaking. A functional IT infrastructure is also needed to satisfy the needs of a stretched SC.

5. Products inside a product family have negative correlations. That is, they are substitute products. Negative correlations further reduce the variance of the aggregated demand and lead to lower safety stocks.

Postponement, when implemented properly, has the potential to save inventory holding costs, obsolescence costs and to increase sales, forecast accuracy. On the other hand, modifying SC to implement postponement has its own costs. These costs range from redesigning products for postponement to inefficiencies introduced into SC to achieve component commonality and to training cost of the personnel. Reader should not get the impression that postponement always reduces total costs. Rather it is the proper analysis of the postponement, trading off cost savings against extra costs due to postponement, which has the potential to reduce total costs.

## 6 Product Availability: $\Re\epsilon\alpha\delta$ Chapter 12 of Chopra

### 7 Optimal Wholesale Pricing <sup>HD</sup>

Consider a supplier(S) and a retailer(R) subject to random demand  $D$  with pdf  $f$ , cdf  $F$  and  $\bar{F} := 1 - F$ . Let us define the following cost parameters:

- $c$ : Supplier's cost of supplying 1 unit. If supplier manufactures this unit,  $c$  is the marginal manufacturing cost. If the supplier simply buys the unit,  $c$  is the purchasing price for the supplier.
- $w$ : Wholesale price that supplier charges to retailer.
- $p$ : Market price that retailer charges to customers.

Naturally  $c \leq w \leq p$ . From supplier's perspective,  $c$  and  $p$  are constant. However, the supplier can determine  $w$ . We will study finding optimal  $w$  to maximize the supplier's profits. Note supplier's profits depend on how many units retailer purchases, call this  $y$ .

For every given value of  $w$  retailer reacts to set how many units it will buy from the supplier. For example, if the wholesale price is large, the retailer will buy less, i.e.  $y(w)$  decreases in  $w$ . Let us quantify this observation, first let  $\Pi^R(y, D)$  denote the profit of retailer if it buys  $y$  and the demand is  $D$ .

$$\Pi^R(y, D) = p(y \wedge D) - wy$$

$\Pi^R(y, D)$  is a random function, taking its expected value:

$$\begin{aligned} \Pi^R(y) &:= E(\Pi^R(y, D)) = p \int_0^\infty (y \wedge D) f(D) dD - wy = p \int_0^\infty \int_{x=0}^{y \wedge D} dx f(D) dD - wy \\ &= p \int_0^y \int_x^\infty f(D) dD dx - wy = p \int_0^y \bar{F}(D) dD - wy. \end{aligned}$$

It is easy to show that  $\Pi^R(y)$  is concave, so it is maximized at  $y^* = \bar{F}^{-1}(w/p)$ . In order to emphasize the dependence of the retailer's order size, we use the notation  $y^*(w) = \bar{F}^{-1}(w/p)$ .

We now study supplier's profit  $\Pi^S(w)$ ,

$$\Pi^S(w) = (w - c) \left( \arg \max_y \Pi^R(y) \right) = (w - c) y^*(w) = (w - c) \bar{F}^{-1}(w/p)$$

The first order condition for a critical  $w$  is:

$$\frac{d\Pi^S(w)}{dw} = \bar{F}^{-1}(w/p) + \frac{w - c}{p\bar{F}'(\bar{F}^{-1}(w/p))}$$

Now remember that  $\bar{F}' = -f$ ,  $y^* = \bar{F}^{-1}(w/p)$  and  $w = p\bar{F}(y^*)$ .

$$\frac{d\Pi^S(w)}{dw} = y^* - \frac{p\bar{F}(y^*) - c}{pf(y)}$$

Therefore the optimal order quantity for the retailer when the wholesale price maximizes supplier's profit,  $y^*$ , satisfies:

$$\bar{F}(y^*) \left( 1 - \frac{y^* f(y^*)}{\bar{F}(y^*)} \right) = \frac{c}{p} \quad (1)$$

Once this  $y^*$  is found than the optimal wholesale price is simply:

$$w = p\bar{F}(y^*) \quad (2)$$

As (2) is a simple equation, we focus on (1). For the existence and uniqueness of  $y^*$ , consider  $f(y)/\bar{F}(y)$  inside the paranthesis in (1). This ratio is nothing but the failure rate function of the demand density. For a moment suppose that this failure rate is increasing in  $y$ , than the entire term inside the paranthesis decreases in  $y$ . In addition  $\bar{F}(y)$  decreases in  $y$ . Then the left hand side of (1) is decreasing in  $y$  while the right hand side is constant. As  $y$  increases from 0 to  $\infty$ , the left hand side decreases from 1 to zero and hits  $c/p$  at a unique  $y$ . These observations yield the following proposition of [1].

**Proposition 1.** *If demand distribution has increasing failure rate, then the unique wholesale price maximizing supplier's revenue is found by solving (1) and (2).*

## 8 Exercises

- Let the total annual holding and ordering cost of an inventory system be  $TC(Q)$

$$TC(Q) = (R/Q)S + (Q/2)hC$$

which is minimized by  $Q^* = EOQ$ .

- Compute  $TC(2Q^*)/TC(Q^*)$ .
  - Compute  $TC(Q^*/2)/TC(Q^*)$ .
  - In (a) and (b) you computed the sensitivity of total costs to using a nonoptimal lot size. In the first case it was  $2Q^*$  and in the second  $Q^*/2$ . According to your results, is  $TC(Q)$  sensitive to  $Q$ ?
- Solve Chopra exercise 4 of Chapter 10 on p.292.
  - Solve Chopra exercise 5 of Chapter 10 on p.292.
  - Consider the online retailer DO which faces a customer demand of 10,000 boxes of vitamins per month. DO incurs a fixed order placement (with the manufacturer) cost of 100 dollars and incurs a holding cost rate of 20% over the purchase price of each per year. DO buys a box of vitamins at 3 dollars from the manufacturer. Each time DO places an order, the manufacturer has to process the order which costs a 250 to the manufacturer. The manufacturer also incurs a 20% holding cost rate and manufactures a box of vitamins for 2 dollars.
    - Find the EOQ quantity for DO. Compute DO's and the manufacturer's holding and ordering costs per year.
    - Now consider an all units discount scheme offered by the manufacturer to the DO: DO will pay the regular price of 3 dollars for purchases in quantities below 9165 and will pay 2.9978 dollars for purchases in larger quantities. Find the optimal order quantity for DO under the new pricing scheme.
    - Compute DO's holding, ordering and purchase costs when the order quantity in b) is used. Also compute the manufacturer's profit (= revenue made from DO - production costs - holding cost - ordering cost) when DO orders in the quantities computed in b).
    - Is DO or the manufacturer better off or worse off with the discounts? If both are better off where are the savings coming from?

5. List and compare the motives of Kraft and American Airlines for trade promotions/discounting.
6. Refer to §10.4 of Chopra. Suppose that promotion opportunity is available only now and  $Q^d$  is an integer multiple of  $EOQ$ . Consider ordering in the lot size  $Q^d$  only once when the promotion is available instead of ordering  $EOQ$  in  $Q^d/Q$  cycles.
- a) Let us use the inventory notation to express the cost savings achieved by switching from  $EOQ$  to  $Q^d$

$$\text{Savings in material input cost} = Q^d d$$

$$\text{Savings in ordering cost} = (Q^d - EOQ)S/EOQ$$

Explain these expressions using one sentence for each.

- b) What is *Savings in holding cost*? This number will be negative but let us be consistent and work with savings.
- c) Sum all the savings to construct total cost  $TC(Q^d)$ . Then check that

$$\frac{\partial TC(Q^d)}{\partial Q^d} = d + \frac{S}{EOQ} + \frac{(hC)EOQ}{2R} - \frac{Q^d(C-d)h}{R}$$

Set this derivative equal to zero and simplify by using the definition of  $EOQ$  to argue for  $S/EOQ = (hC)EOQ/(2R)$ .

- d) Find the optimal  $Q^d$  that maximizes total cost savings in (c) and show that  $Q^d \geq EOQ$ .
7. In front of the Arts and Humanities secretarial office coffee is brewed and offered to general public at a cost of 25 cents per cup. Coffee powder is bought in bags and each bag costs 2 dollars and contains enough coffee to prepare 10 cups. Suppose that the coffee demand over every two hours is normally distributed with expected value of 6 and variance of 1. The office is open for 8 hours every day and does not provide coffee outside the office hours.
- a) Assume that demand in every two hour period is independent of other period's demand, find the distribution of the coffee demand per day.
- b) Suppose the office uses 3 coffee bags every day to prepare coffee. First find out how many cups are prepared and then compute the probability of coffee stockout.
- c) If the office uses 3 coffee bags every day, find the safety stock level and compute the fill rate .
- d) Briefly explain why independence assumption of (a) might be flawed.
8. UTD bookstore is trying to determine how many Supply chain books to order for the next year. The book store buys the books at 50 dollars each and sells at 80 dollars. Any unsold book is bought back by Prentice Hall at 40 dollars. It is estimated that the class size will be some where between (and including) 21 and 32. In this range, the class size can take any integer value with equal probabilities.
- a) What is an appropriate distribution for book demand, specify the distribution with its parameters.
- b) Compute how many books should be ordered.
- c) Compute the marginal cost of ordering one more book beyond the number you found in (b).
- d) What is the total cost of understocking and overstocking with the number of books found in (b)?
- e) If you were the manager in charge of ordering books at the bookstore, what steps would you consider to reduce the costs in (d) ?
9. Whole Foods offers a good selection of cheeses. One of the cheeses it sells is Bulgarian Feta Cheese. Bulgarian Feta contains more fat than Greek Feta and is more tasty to eat for breakfast. Bulgarian cheeses are purchased in jars from a Bulgarian farm. Each jar costs 10 dollars. The replenishment lead time from Bulgaria is 6 months. During the replenishment lead time Whole Foods sell 100 jars of cheese. The standard deviation of the demand during the lead time is 25 jars. Whole Foods assumes that the demand is Normally distributed. Whole Foods uses an interest rate of 20% for its inventory holding cost computations. When Whole Foods out of the stock for the cheese, it incurs a stockout cost of 25 dollars per jar. Placing an order

costs about 50 dollars, independent of its size.

- a) Find optimal lot sizes and reorder point for Whole Foods.
- b) Find safety stock level.
- c) Find the proportion of order cycles in which stock outs occur.

10. Philips manufactures lighting products (various lamps) in Taipei, Taiwan and ships them to Balkan countries for sale. The weekly sales in these countries are independently and Normally distributed with mean demand and standard deviation as below:

Country	Mean	Standard deviation
Macedonia	100	40
Romania	300	80

Currently, Philips packs the lamps with manuals in each country's language in Taiwan. These packages are sent to DCs in each country to meet that country's demand. The transportation time from Taiwan to these countries are 6 weeks.

- a) How much safety stock is needed in each country for a CSL of 90%?
  - b) Suppose Philips establishes a regional DC in Romania to meet both Romanian and Macedonian demand. The manuals are quickly put into packages in this DC after the retailer orders in each country are observed. Thus, Philips produces to stock and the regional DC orders to stock while the country specific customization happens according to retailer orders. What is the distribution of the weekly demand for the regional DC? Compute the safety stock savings with regard to a) if the CSL of 90% is still maintained.
  - c) Repeat b) if Macedonian and Romanian demand have a correlation of 0.5.
11. Capacitated Newsvendor Problem: First download the associated Excel file from the web site. Consider a restaurant allocating its tables to customer groups A and B which ask for reservation by calling in advance. If a table is allocated to group but sufficient people do not show from that group, the table has to be given to walk-in customers. Per table revenue from walk-in customers is \$30 and is smaller than revenues of groups making in-advance reservations. Actually \$30 revenue does not even cover the cost of providing service for a table.
- a) According to the parameters in the Excel file, what is the cost of providing service for a table for groups A and B? What is the revenue per table expected from groups A and B? How many tables are there in the restaurant?
  - b) How much should the restaurant owner be willing to pay to add 1 more table?
  - c) If there were only 8 tables in the restaurant, how many of these should be allocated to groups A and B?
  - d) If there were 14 tables in the restaurant, how many of these should be allocated to groups A and B?
12. Play the Beer Game the real version for 30 periods. Save your results (retailer, distribution center, warehouse, factory) onto a single floppy. Submit them along with your homework.
13. Comparison of overstock and understock: For a better understanding and comparison of overstock and understock, let us compare demand  $D$  and order quantity  $Q$  in a single period (season) context. Suppose that  $Q$  is given and is not necessarily optimal for our context. Further suppose that the demand is nonnegative and is less than 100. By the definitions of understock and overstock, recall that

$$E(\text{understock}) = E(D - Q)^+ \quad \text{and} \quad E(\text{overstock}) = E(Q - D)^+.$$

- a) Further suppose that  $Q = 10$  units and consider the function  $y_1(D) = (D - 10)^+$ . Note that this is a function of one variable (i.e.  $D$ ) and is almost a line. Note that  $y_1(D)$  can also be written as

$$y_1(D) = \max\{D - 10, 0\}.$$

Express  $y_1(D)$  in your words. Draw  $y_1(D)$  as  $D$  varies (x-axis is  $D$ , y-axis is  $y_1(D)$ ).

- b) Keep  $Q = 10$  and define and  $y_2(D) = (10 - D)^+$  and  $y_3(D) = D - 10$ . Express  $y_2(D)$  and  $y_3(D)$  in your

words. Draw  $y_2(D)$  and  $y_3(D)$  onto the same coordinate system used in a). You may use pencils of different colors or different line styles.

c) Now looking at your drawings determine if

$$i. (D - 10)^+ - (10 - D)^+ = D - 10 \quad \text{or} \quad ii. (D - 10)^+ + (10 - D)^+ = D - 10.$$

is correct. Express the correct equality in your words.

d) Now take the expected value of all the terms in the correct equality to obtain either

$$i'. E(D - 10)^+ - E(10 - D)^+ = E(D - 10) \quad \text{or} \quad ii'. E(D - 10)^+ + E(10 - D)^+ = E(D - 10).$$

Express in your words the correct equality.

e) Explain why your conclusion in d) will not change if  $Q \neq 10$ . Then for any order quantity

$$i''. E(D - Q)^+ - E(Q - D)^+ = E(D - Q) \quad \text{or} \quad ii''. E(D - Q)^+ + E(Q - D)^+ = E(D - Q).$$

Express in your words the correct equality. With this example, you have just found an important equality involving understock and overstock. The equality can be used to compare these quantities.

14. Lead Time as a Real Option: Looking at the safety stock computations, we realize that safety stocks are very sensitive to the length of the lead times. This is because, safety stocks are used to cover for the variabilities during the lead times. If the length of the lead times can be controlled even slightly, the chances of a stockout or overstock during lead time can be decreased. Suppose that the lead times have two parts:

$$\text{Lead Time (LT)} = \text{Manufacturing Lead Time (MLT)} + \text{Delivery Lead Time (DLT)}$$

The orders are first manufactured in MLT time at a manufacturer and then delivered in DLT time to the retailer. DLT can be decreased (increased) by using a faster (slower) transportation mechanism or mode. Suppose that the retailer has the option of modifying the DLT when the manufacturing is completed at the manufacturer. Should the retailer check its inventory level when making this modification to the DLT, why? When would the retailer ask for a faster transportation mechanism? Can you give a real-life example or make up an example of this kind of DLT modifications? In general, options in supply chains are on “real” (tangible) aspects so they are called Real Options to differentiate from Financial Options.

15. When would manufacturers implement quick response? In a manufacturer and a retailer setting, consider quick response; manufacturer’s effort to decrease its lead times to quickly respond to retailer orders. Explain how key retailer inventory characteristics (inventory levels and order sizes) can change with quick response. You are advised to study the quick response example in the textbook. Now consider quick response from the manufacturer’s standpoint. Is it easy for a manufacturer to decrease its lead times? Can you give 1-2 examples of how such a decrease can be achieved? What does manufacturer gain from quick response? For example, does the manufacturer sell more or less to the single retailer? How would your answer change if the manufacturer is competing with other manufacturers to supply the retailer?
16. <sup>HD</sup> For each of the following demand distributions decide if there is a unique wholesale  $w$  price that maximizes the supplier’s profit  $\Pi^S(w)$ : Normal, exponential, uniform, gamma, logistic, weibull.

## References

- [1] M.A. Lariviere and E.L. Porteus (2001). *Selling to the Newsvendor: An Analysis of Price Only Contracts*. Working paper, Kellogg School of Management, Northwestern University.
- [2] W.C. Rinaman. (1993). *Foundations of Probability and Statistics* published by Saunders College Publishing. ISBN 0-03-071806-6.

- [3] S.M. Ross. (1993). *Introduction to Probability Models* 5th edition published by Academic Press, San Diego, CA. ISBN 0-12-598455-3.
- [4] D.D. Wackerly, W. Mendenhall and R.L. Scheaffer. *Mathematical Statistics with Applications* 5th edition published by Wadsworth Publishing Company, Belmont, CA. ISBN 0-534-20916-5.

## Appendix: Beer Game

### SC Costs

- Holding costs incurred at F, D, W, R:
  - 0.25 dollar per keg per day at F
  - 0.50 dollar per keg per day at D
  - 0.75 dollar per keg per day at W
  - 1 dollar per unit per day at R
- Backordering costs incurred only at R:
  - 10 dollars per unit per day at R
- Minimize sum of these over 30 days

### Playing role and style

- Play as R or W or D or F.
- Play as a team
- Without transferring demand information
- Play the training version first, then the real version

### Demand distribution

- Random demand discretized
- Truncated
- With mean 50 and standard deviation 20

### Order fulfillment

- Observe the incoming shipment from the upstream partner, add this to the inventory.
- Observe the incoming order from the downstream partner, take out this from the inventory.

If the result is nonnegative hold the inventory, otherwise backorder.

### Order of daily events at the Retailer(R)

1. Inform upstream partner of the size of order placed by you Info-LT days ago. In day 4, R informs W of R's order in day 2.
2. Upstream partner informs you of the size of incoming shipment sent by the upstream partner Mater-LT days ago. In day 4, W informs R of the shipment made by W in day 2. Enter.
3. Observe the customer demand.
4. Order fulfillment.
5. Outstanding orders advance 1 day.
6. Decision: Order from W. Enter.

**Order of daily events at the Warehouse(W)**

1. Downstream partner informs you of the size of the order it placed Info-LT days ago. W learns about what R ordered Info-LT days ago. In day 4, W receives R's incoming order placed by R in day 2. Enter.
2. Inform upstream partner of the size of order placed by you Info-LT days ago. In day 4, W informs D of W's order in day 2.
3. Upstream partner informs you of the size of incoming shipment sent by the upstream partner Mater-LT days ago. In day 4, D informs W of the shipment made by D in day 2. Enter.
4. Inform downstream partner about how many kegs it is receiving on this day.
5. Shipments advance 1 day.
6. Order fulfillment.
7. Outstanding orders advance 1 day.
8. Decision: Order from D. Enter.

**Order of daily events at the Distributor(D)**

1. Downstream partner informs you of the size of the order it placed Info-LT days ago. D learns about what W ordered Info-LT days ago. In day 4, D receives W's incoming order placed by W in day 2. Enter.
2. Inform upstream partner of the size of order placed by you Info-LT days ago. In day 4, D informs F of D's order in day 2.
3. Upstream partner informs you of the size of incoming shipment sent by the upstream partner Mater-LT days ago. In day 4, F informs D of the shipment made by F in day 2. Enter.
4. Inform downstream partner about how many kegs it is receiving on this day.
5. Shipments advance 1 day.
6. Order fulfillment.
7. Outstanding orders advance 1 day.
8. Decision: Order from F. Enter.

### **Order of daily events at the Factory(F)**

1. Downstream partner informs you of the size of the order it placed Info-LT days ago. F learns about what D ordered Info-LT days ago. In day 4, F receives D's incoming order placed by D in day 2. Enter.
2. Inform downstream partner about how many kegs it is receiving on this day.
3. Shipments advance 1 day.
4. Order fulfillment.
5. Factory brews scheduled production.
6. Decision: Schedule production to be received in 2 days. Enter.

### **Reporting**

- After you finished playing the game, you can graph several performance measures, get a summary of your actions. Click on "Show History" for this.
- Results can be saved to A drive by first clicking on "End Game" and then saying "Yes".
- You can copy results and paste into another file.