

1 All-unit quantity discount: The case of price dependent holding cost

If you get confused by reading the course slides, the notes below can help you. These notes are exact and rigorously written. This is another approach to explain discounts. If you do not like this approach, please go to the solved exercise directly.

In the all-unit quantity discount case, we have

$$\text{Price per unit} = \left\{ \begin{array}{ll} p_1 & \text{if } q_0 \leq Q < q_1 \\ p_2 & \text{if } q_1 \leq Q < q_2 \\ \dots & \dots \\ p_N & \text{if } q_{N-1} \leq Q < q_N \end{array} \right\} \quad (1)$$

where we set $q_0 = 0$ and $q_N = \infty$. Q denotes the purchase quantity (order size or lot size). Prices are p_n s, quantities where price breaks happen are q_n s. Prices are monotonically ordered by $p_{n-1} \geq p_n$ for $2 \leq n \leq N$. We coin the term *region n* for the interval $[q_{n-1}, q_n)$.

Given the average cost in (1), the total purchasing cost of Q units can be written as follows:

$$\text{Purchasing cost of } Q \text{ units} = \left\{ \begin{array}{ll} p_1 Q & \text{if } 0 \leq Q < q_1 \\ p_2 Q & \text{if } q_1 \leq Q < q_2 \\ \dots & \dots \\ p_N Q & \text{if } q_{N-1} \leq Q < q_N \end{array} \right\}$$

If $q_{n-1} \leq Q < q_n$, we use the EOQ cost formula to define the total costs in region n as

$$TC_n(Q) := \frac{h}{2}(p_n Q) + \frac{D}{Q}S + \frac{D}{Q}(p_n Q). \quad (2)$$

In (2), D is the constant demand rate and S is the fixed ordering cost. In the same expression, h is the holding cost rate per time per dollar. This cost h differs from H , which is per time per item. Indeed, $H = hp_n$. Note that holding cost per item per time depends on the price of the item. Since the price of the item is not determined at the beginning, we work with the h as opposed to H . In summary, the holding cost per item per time is hp_n if the item is purchased at the price p_n , that is with a purchase lot size Q where $q_{n-1} \leq Q < q_n$.

We remark that the cost in (2) is true only if $q_{n-1} \leq Q < q_n$. Otherwise, it is incorrect. But luckily a similar cost expression is correct in other regions. It suffices to use the correct price p_n in each region n . Then the total cost over all regions is obtained by patching all TC_n together, i.e.,

$$TC(Q) = \left\{ \begin{array}{ll} TC_1(Q) & \text{if } 0 \leq Q < q_1 \\ TC_2(Q) & \text{if } q_1 \leq Q < q_2 \\ \dots & \dots \\ TC_N(Q) & \text{if } q_{N-1} \leq Q < q_N \end{array} \right\}$$

In the marginal-units quantity discount problem, we want to solve

$$\min_{Q \geq 0} TC(Q).$$

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Initialize  $n := N$  and  $CandidatesComplete = False$ .
While  $n \geq 1$  and  $CandidatesComplete = False$  do
  Set  $EOQ_n = \sqrt{2DS/(hp_n)}$ .
  If  $q_{n-1} \leq EOQ_n \leq q_n$ 
    A1: Add  $EOQ_n$  to the set of candidate solutions;
    A2:  $CandidatesComplete = True$ ;
  else if  $EOQ_n < q_{n-1}$ 
    B: Add  $q_{n-1}$  to the set of candidate solutions;
  else if
    C:  $CandidatesComplete = True$ ;
  If  $CandidatesComplete = False$ 
     $n:=n-1$ ;
Evaluate the total cost at each candidate solution.
Pick the order quantity yielding the minimum cost as the optimal.

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Table 1: All-unit quantity discount ordering algorithm

We propose the ordering algorithm in Table 1 to find the optimal order quantity. Note that this algorithm does not generate any candidate solution after step **A2** and step **C** because the boolean flag *CandidatesComplete* becomes true and stops the *while loop*.

Note that we are stopping the generation of the candidates in step **A2** and **C**, say when $n = n_0$. This is to say that we cannot obtain a better solution by investigating the order quantities at higher prices, i.e., smaller $n < n_0$. This happens because of the following facts:

1. TC_{n_0} is decreasing at q_{n_0-1} if steps **A2** or **C** happen.
2. Since TC_{n_0} is convex, $TC_{n_0}(Q) \geq TC_{n_0}(q_{n_0-1})$ for every $Q \leq q_{n_0-1}$. Recall that if a convex function is decreasing at q_{n_0-1} , it is decreasing on the left-hand side of q_{n_0-1} .
3. Since $TC_n(Q) \geq TC_{n_0}(Q)$ for every Q and $n \leq n_0$. Recall that $TC_n(Q)$ is the cost at price p_n and $p_n \geq p_{n_0}$.
4. Combine the last two facts to obtain $TC_n(Q) \geq TC_{n_0}(q_{n_0-1})$ for every $Q \leq q_{n_0-1}$ and $n \leq n_0$. The costs on the left-hand side of q_{n_0-1} are all larger than $TC_{n_0}(q_{n_0-1})$.

2 Solved Example

All-units quantity discounts: A popular shoe store sells 8000 pairs per year. The fixed cost of ordering shoes from the distribution center is \$15 and holding costs are taken as 25% of the shoe costs. The per unit purchase costs from the distribution center is given as

$$\left\{ \begin{array}{ll} p_1 = 60, & \text{if } 0 \leq Q \leq 50 \\ p_2 = 55, & \text{if } 50 \leq Q \leq 150 \\ p_3 = 50, & \text{if } 150 \leq Q \end{array} \right\}$$

where Q is the order size. Determine the optimal order quantity.

Solution: There are three ranges for lot sizes in this problem: $(0, q_1 = 50)$, $(q_1 = 50, q_2 = 150)$ and $(q_2 = 150, \infty)$. Holding costs in these ranges of shoe prices are given as $h_1 = (0.25)60 = 15$, $h_2 = (0.25)55 = 13.75$

and $h_3 = (0.25)50 = 12.5$. EOQ quantities in these ranges are

$$EOQ_1 = \sqrt{\frac{2(15)(8000)}{15}} = 126.5; \quad EOQ_2 = \sqrt{\frac{2(15)(8000)}{13.75}} = 132.1; \quad EOQ_3 = \sqrt{\frac{2(15)(8000)}{12.5}} = 138.6;$$

Only $EOQ_2 = 132.1$ is in the appropriate range, i.e. $q_1 \leq EOQ_2 \leq q_2$, so it is a candidate solution. Since $EOQ_1 > q_1$, we take $q_1 = 50$ as the candidate solution for the second range. Since $EOQ_3 < q_2$, we take $q_2 = 150$ as the candidate solution for the third range.

It is clear that ordering $Q = 50$ at the cost $p_1 = 60$ is worse than ordering $Q = 50$ at the cost $p_2 = 55$. Moreover ordering $Q = 132.1$ is better than ordering $Q = 50$ in the second range. Combining the last two statements, the costs at $Q = 132.1$ and $p_2 = 55$ is smaller than the costs at $Q = 50$ and $p_2 = 60$. Therefore we can eliminate $Q = 50$ from the consideration.

If you do not want to go through this argument note that $q_1 = 50 < EOQ_2 = 132.1 < 150 = q_2$. Then the step **A2** of the algorithm in Table 1 is reached when $n_0 = 2$ and we stop the generation of the candidates. Actually, we do not even compute EOQ_1 in the algorithm because we know that it cannot yield a better cost than EOQ_2 . That is why the *while loop* of the algorithm is terminated in step **A2**.

Evaluating the remaining candidate lot sizes

$$TC_2(Q = 132.1) = 8000(55) + 8000(15)/132.1 + (0.25)(55)(132.1)/2 = 441,800$$

$$TC_3(Q = 150) = 8000(50) + 8000(15)/150 + (0.25)(50)(150)/2 = 401,900$$

Then $Q = 150$ is the optimal solution with a cost of 401,900.

3 Exercise Questions

1. UTD Cafeteria is going to purchase plastic sets of fork, knife, spoon for the people who buy “to go” lunch. For orders less than 500 sets, the cafeteria is charged 30 cents per set. For orders more than 500 but fewer than 1000, the cafeteria is charged 25 cents per set. For orders above 1000, Cafeteria is charged 20 cents per set. Suppose that ordering cost is \$8 per order, holding cost is 6 cents per year and annual plastic set demand is constant at 900.

- Set up the problem: What are the costs and demands (S, H, D)? Also decide whether the quantity discount affects the holding cost per set per time.
- Compute the EOQ quantity.
- Find the optimal number of sets to purchase.

2. Refer to the previous problem however now assume that holding costs are based on 20% interest rate. Then, the holding cost per set per year is 20% of the price of the set.

- Set up the problem: What are the costs and demands (S, H, D)? Also decide whether the quantity discount affects the holding cost per set per time.
- Compute the EOQ quantities.
- Find the optimal number of sets to purchase.

3. Suppose you are inviting 10 friends to your Thanksgiving dinner, each of your friends consumes soft drinks (measured in milliliters) independently according to the normal distribution $N(\mu = 600, \sigma^2 = 100^2)$.

- What is the distribution of the total soft drink demanded by your friends? What is its mean and variance?
- If you buy 22 cans of soft drinks (each has 300 milliliters) for the dinner, what service level do you achieve? Here you are to compute the probability that the your drinks suffice for your guests, i.e. $Prob(Demand \leq 300 \times 22) = ?$

c) With 22 cans, what would be your safety stock?

4. Refer to the previous problem.

a) What is the expected amount of soft drink shortage during the dinner? Now we are after $E(\max\{0, Demand - 300 \times 22\})$. You need to use Table 11.3, but first figure out the expected demand and the standard deviation of the demand to compute

$$z = \frac{(300 \times 22) - \text{Expected demand}}{\text{Standard deviation of the demand}}$$

b) How does your estimate of the shortage change if each person drinks according to $N(\mu = 600, \sigma^2 = 200^2)$?

5. In front of the Arts and Humanities secretarial office coffee is brewed and offered to general public at a cost of 25 cents per cup. Coffee powder is bought in bags and each bag costs 2 dollars and contains enough coffee to prepare 10 cups. Suppose that the coffee demand over every two hours is normally distributed with expected value of 6 and variance of 1. The office is open for 8 hours every day and does not provide coffee outside the office hours.

a) Assume that demand in every two hour period is independent of other period's demand, find the distribution of the coffee demand per day.

b) Suppose the office uses 3 coffee bags every day to prepare coffee. First find out how many cups are prepared and then compute the probability of coffee stockout.

c) If the office uses 3 coffee bags every day, find the safety stock level and compute the fill rate.

d) Briefly explain why independence assumption of (a) might be flawed.

6. Aggregate planning involves ignoring some details in the planning process. For example, different but similar types of products are considered as one product, or different units of capacity (machine and manpower) is considered as one unit of capacity. What is the rationale behind aggregate planning? Why do not we plan every operation as detailed as possible? Briefly discuss.

7. A table is assembled from two subassemblies: a top and 4 legs. Legs are considered raw materials for our purposes. Tops are produced in-house and they have a lead time of 2 days. Suppose the table assembly takes 1 day if less than 100 tables are to be assembled, otherwise it takes 2 days. Legs are purchased from a supplier who can deliver legs in any quantity with a 1 day of lead time. Further suppose that we have to deliver 80 tables on the fourth day and 180 tables on the fifth day, currently there are 20 finished tables and 60 legs in the inventory. We are expecting a delivery of 40 legs on the second day.

a) Draw the BOM for a table.

b) Prepare MRP charts for tables and tops for the next 5 days.

8. Read the article "The ABCs of ERP pp.595-601 and answer Questions 2 and 3 at the end of the article.