STOCHASTIC PROCESSES

Let $t \in T$ = time
$\omega \in S$ = outcome, element of the sample space

$X(t, \omega)$ = **stochastic process**

Discrete T $\Rightarrow$ discrete time process
Connected T $\Rightarrow$ continuous time process

$X \in \mathcal{X}$ = states of the process

Discrete $\mathcal{X}$ $\Rightarrow$ discrete state process, chain
Connected $\mathcal{X}$ $\Rightarrow$ continuous state process

For any $t \Rightarrow X_t(\omega) =$ random variable
For any $\omega \Rightarrow X_\omega(t) =$ function of t
(path, trajectory, realization)

Examples

Temperature

Stock value

Number of jobs in a queue

Number of internet connections

Football score

Poisson process

Binomial process

Brownian motion

Markov processes

$X(t)$ is a **Markov process** if
for any $t_1 < ... < t_n < t$,

$$\boldsymbol{P}\{X(t) \in A \mid X(t_1) = x_1, \dots, X(t_n) = x_n\}$$

$$= \boldsymbol{P}\{X(t) \in A \mid X(t_n) = x_n\}$$

That is,

$$\boldsymbol{P}\{\text{ future } \mid \text{ past, present }\}$$

$$= \boldsymbol{P}\{\text{ future } \mid \text{ present }\}$$

Markov dependence:
"Future depends on the past only through the present"

Counting processes

They *count* events.

Therefore, $X \in \{0, 1, 2, 3, ...\}$
and $X(t)$ is non-decreasing.

Examples:

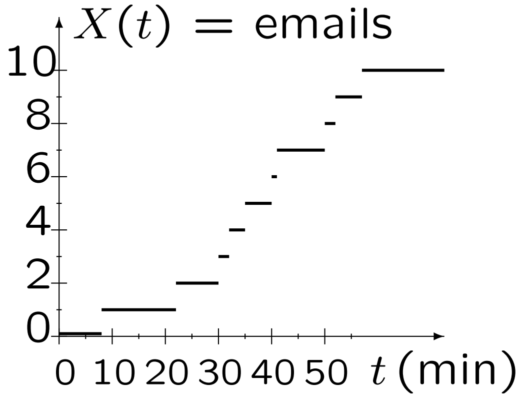
Binomial process

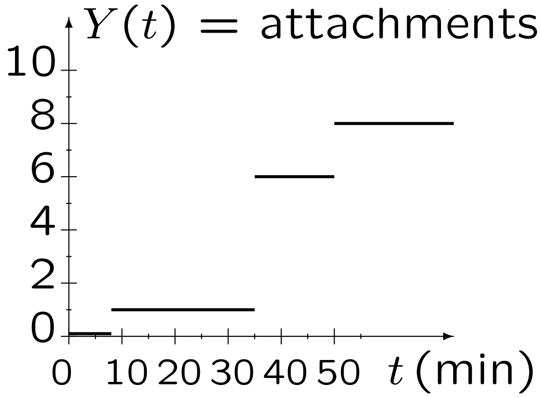
Poisson process

Example

$X(t)$ = number of arrived emails by the time t

$Y(t)$ = number of attachments by the time t





E-mails are transmitted at $t = 8$, 22, 30, 32, 35, 40, 41, 50, 52, and 57 min. Only 3 e-mails contained attachments. One attachment was sent at $t = 8$, five more at $t = 35$, and two more attachments at $t = 50$.

Binomial process

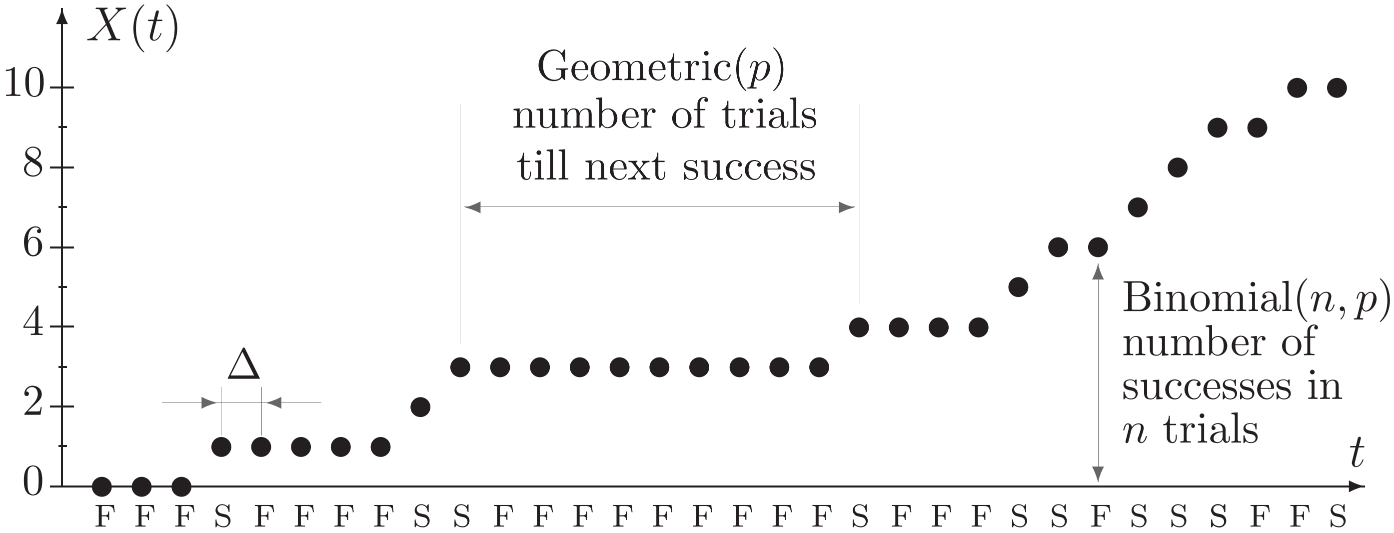
Binomial process $X(t)$ is the number of successes by the time t in a sequence of independent Bernoulli trials.

This process is
- discrete-time
- discrete-space
- counting
- Markov

Discrete time - one frame per trial

Random variables

$$\begin{aligned} X(t) &= \text{number of arrivals by the time } t \\ Y &= \text{number of frames between arrivals} \\ T &= \text{time between arrivals} \end{aligned}$$



Parameters

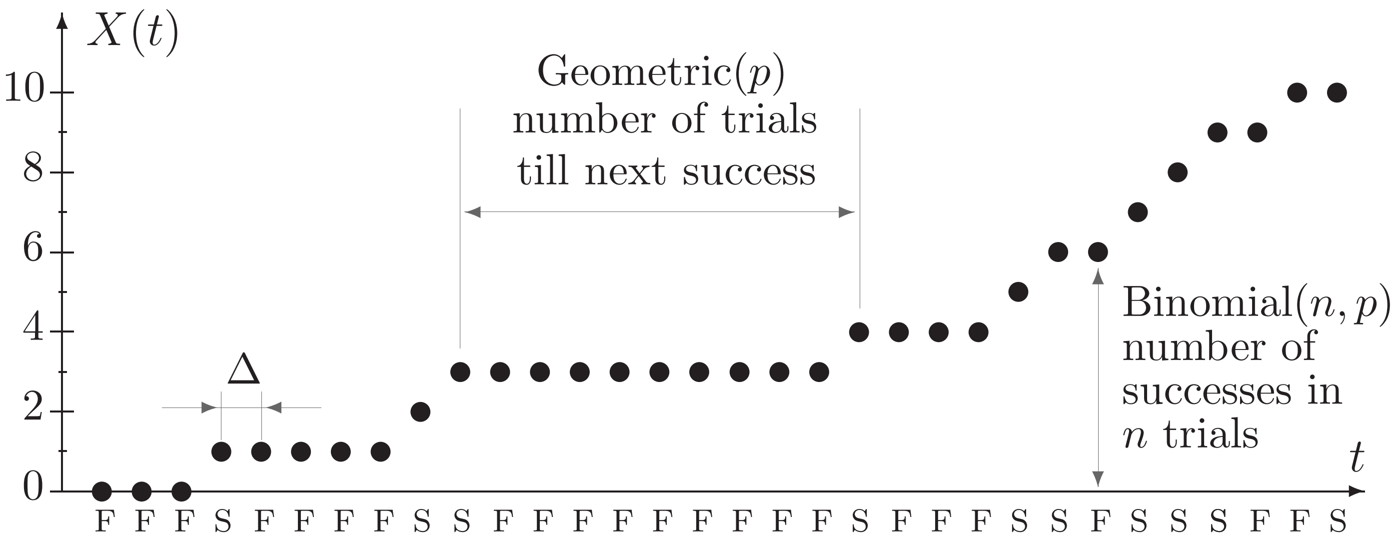
λ = arrival rate (successes per minute)

Δ = duration of 1 frame (minutes per frame)

p = prob. of success (successes per frame)

$$p = \lambda\Delta, \quad \lambda = p/\Delta, \quad \Delta = p/\lambda$$

Binomial process

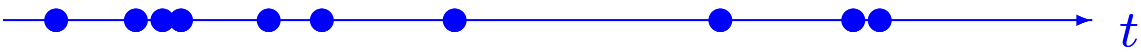


Distributions

$X(t) = Binomial(n,p)$	$\mathbf{E}X(t) = np$	$\mathrm{Var}X(t) = np(1-p)$
$Y = Geometric(p)$	$\mathbf{E}Y = \dfrac{1}{p}$	$\mathrm{Var}Y = \dfrac{1-p}{p^2}$
$T = Y\Delta$	$\mathbf{E}T = \dfrac{\Delta}{p} = \dfrac{1}{\lambda}$	$\mathrm{Var}Y = \Delta^2 \dfrac{1-p}{p^2}$

$n = t/\Delta$ is the number of frames during time t

Poisson process



$X(t)$ = continuous-time process counting *"rare events"*, with properties:

- $\boldsymbol{P}\{X(t+h) - X(t) = 1\}$
  $= \boldsymbol{P}\{$ 1 event in [t,t+h] $\}$
  $= \lambda h + o(h), \quad$ as $\quad h \to 0$

- $\boldsymbol{P}\{X(t+h) - X(t) > 1\}$
  $= \boldsymbol{P}\{$ more than 1 event in [t,t+h] $\}$
  $= o(h), \quad$ as $\quad h \to 0$

- For $t_1 < t_2 < t_3 < t_4$, the *increments* $X(t_2) - X(t_1)$ and $X(t_4) - X(t_3)$ are **independent**

Poisson process is continuous-time, discrete-state, Markov.

It is stationary if λ is constant.
where $\lambda = \mathbf{E}(\#\ events/min)$

Binomial process (discrete time; frame $= \Delta$)	$\longrightarrow$ $\Delta \to 0$ $p \to 0$ $\lambda = const$	**Poisson process** (continuous time)

Distributions

Let $X(t)$ = Binomial process
= number of events during time t
= number of events during $n = t/\Delta$ frames

$$X(t) = Binomial\left(n = \frac{t}{\Delta}, p\right) \to Poisson(\lambda t)$$

as $\Delta \to 0$,
hence $p \to 0$, $n = t/\Delta \to \infty$, $np = tp/\Delta = \lambda t$

$$\mathbf{E}X(t) = \lambda t, \qquad \text{Var}X(t) = \lambda t$$

Interarrival times

Interarrival time = $\min\{t, X(t) \geq 1\}$

Counting process	Interarrival times
Binomial Poisson	Δ·Geometric(p) Exponential(λ)

Let $T_1, T_2, \ldots$ = successive interarrival times

$$\begin{array}{ccc} \boldsymbol{P}\{T_1 + \ldots + T_n > t\} & = & \boldsymbol{P}\{X(t) < n\} \\ \uparrow & & \uparrow \\ Gamma(n, \lambda) & & Poisson(\lambda t) \\ \downarrow & & \downarrow \\ \boldsymbol{P}\{T_1 + \ldots + T_n \leq t\} & = & \boldsymbol{P}\{X(t) \geq n\} \end{array}$$

(Gamma-Poisson formula)