EXPECTATION AND MOMENTS

Expectation of a random variable

Let X = random variable.

$\mathbf{E}(X)$ = its *expectation*, the average value, the mean

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X is random. It takes different values with probabilities $P(x)$.

$\mathbf{E}(X)$ is constant, non-random.

Example 1: Bernoulli(p), $p = 1/2$.

$$X = \begin{cases} 0 & \text{with probability } 1/2 \\ 1 & \text{with probability } 1/2 \end{cases} \Rightarrow \mathbf{E}(X) = 1/2$$

Example 2: Bernoulli(p), $p = 1/3$.

$$X = \begin{cases} 0 & \text{with probability } 2/3 \\ 1 & \text{with probability } 1/3 \end{cases} \Rightarrow \mathbf{E}(X) = 1/3$$

Definition

Discrete case: $\mu = \mathbf{E}(X) = \sum_x xP(x)$

(center of mass)

Continuous case: $\mu = \mathbf{E}(X) = \int xf(x)dx$

Expectation of a function $Y = g(X)$

Discrete case: $\mathbf{E}g(X) = \sum_x g(x)P(x)$

Continuous case: $\mathbf{E}g(X) = \int g(x)f(x)dx$

Variance of a random variable

Example. Consider two financial deals.

$$\begin{array}{llcll} \#1. & P(480) & = & P(520) & = & 0.5 \\ \#2. & P(0) & = & P(1000) & = & 0.5 \end{array}$$

Same $\mathbf{E}(X) = 500$.

In #1, values of X are close to $\mathbf{E}(X)$.
Low variability.

In #2, values of X are far from $\mathbf{E}(X)$.
High variability.

Market term: high volatility

Definition

Variance of X = $\text{Var}(X) = \mathbf{E}\{X - \mathbf{E}(X)\}^2$

Discrete case: $\text{Var}(X) = \sum_x (x - \mu)^2 P(x)$

Continuous case: $\text{Var}(X) = \int (x - \mu)^2 f(x) dx$

Standard deviation $\sigma = \sqrt{\text{Var}(X)}$.

X, $\mu = \mathbf{E}(X)$, σ are measured in units
$\sigma^2 = \text{Var}(X)$ is measured in *squared units*

Variance of the profit = 1 mln. squared dollars
Variance of the enrollment = 1000 squared students

Properties

$\mathbf{E}(aX + b) = a\,\mathbf{E}(X) + b$ - always

$\mathbf{E}(X + Y) = \mathbf{E}(X) + \mathbf{E}(Y)$ - always

$\mathbf{E}(XY) = \mathbf{E}(X)\,\mathbf{E}(Y)$ - for independent X, Y

$\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$ - always

$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$
- for independent X, Y

In general,
$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X, Y)$
- always

Covariance of X and Y

$$\text{Cov}(X,Y) = \mathbf{E}\{X - \mathbf{E}(X)\}\{Y - \mathbf{E}(Y)\}$$

Properties:

$\text{Cov}(X,X) = \text{Var}(X)$

$\text{Cov}(X,Y) = 0$ for independent X, Y

Independent $\Rightarrow$ uncorrelated, but
Uncorrelated $\not\Rightarrow$ independent, in general

X	$\mathbf{E}(X)$	$\mathsf{Var}(X)$
$Bernoulli(p)$	p	$p(1-p)$
$Binomial(n,p)$	np	$np(1-p)$
$Geometric(p)$	$1/p$	$(1-p)/p^2$
$Neg.Binomial(r,p)$	r/p	$r(1-p)/p^2$
$Poisson(\lambda)$	λ	λ
$Uniform(a,b)$	$(a+b)/2$	$(b-a)^2/12$
$Normal(\mu,\sigma)$	μ	σ^2
$Exponential(\lambda)$	$1/\lambda$	$1/\lambda^2$
$Gamma(r,\lambda)$	r/λ	r/λ^2
X	$\mathbf{E}(X)$	$\mathsf{Var}(X)$

Central Limit Theorem

Let $X_1, \ldots, X_n$ = random variables from **any** distribution with $\mu = \mathbf{E}(X_i)$ and $\sigma^2 = \text{Var}(X_i)$

As $n \to \infty$,

$$\frac{(X_1 + \cdots + X_n) - n\mu}{\sigma\sqrt{n}} \longrightarrow \text{Normal}(0, 1)$$

That is,

$$\boldsymbol{P}\left\{\frac{(X_1 + \cdots + X_n) - n\mu}{\sigma\sqrt{n}} < x\right\} \longrightarrow F_{Normal(0,1)}(x)$$

Examples:

$$\begin{array}{lcll} \text{Binomial}(n,p) & \approx & \text{Normal}(\mu,\sigma) & \text{for large } n \\ \text{Neg.Bin.}(r,p) & \approx & \text{Normal}(\mu,\sigma) & \text{for large } r \\ \text{Gamma}(\alpha,\lambda) & \approx & \text{Normal}(\mu,\sigma) & \text{for large } \alpha \end{array}$$

where $\mu = \mathbf{E}(X)$, $\sigma^2 = \text{Var}(X)$