

Research plan

“Bifurcation of Spiking Oscillations in Resonate-and-Fire Neurons”

The goal is to initiate the development of a framework where nonsmooth bifurcation theory is applied in order to prove the existence and stability of spiking limit cycles in Izhekevich’s neuron model ([5])

$$\begin{aligned} \dot{v} &= f(v) - u + I, & \text{reset: } v(t+0) &= v_R, \ u(t+0) = u_R(u(t-0)), \text{ if } v(t) = v_{th}, \\ \dot{u} &= av - bu, & \text{pulse excitation: } v(t+0) &= v(t-0) + \delta, \text{ if } t = t_* + kT, \ k \in \mathbb{N}, \end{aligned} \quad (1)$$

where I is a constant current, T and t_* are periods and phase of pulse input.

1. Border-splitting bifurcation of spiking limit cycles from a point:

Autonomous case $\delta = 0$ (scenario 1 of the approved project). Considering $(v_0, u_0) \in \{(v, u) : f(v) - u + I = 0\}$, $v_{th} = v_0 + \varepsilon$, $v_R = v_0 - \varepsilon$, $u_R(u) = u - \varepsilon d$, where $\varepsilon > 0$ is a small parameter, we used [7, Corollary 2.4] to link limit cycles of (1) near (v_0, u_0) (see Fig. 1) to the fixed points of the Poincaré map

$$P_\varepsilon(y) = -(y - y_0 - \varepsilon d) + \alpha(y - y_0 - \varepsilon d)^2 - \beta \frac{\varepsilon d}{y - y_0 - \varepsilon d} + \text{remainder}.$$

Possible paper 1: Extend to the case $v_R = v_0 - c\varepsilon$. Requires amending the proof of [7, Lemma 2.2]. Can there occur a bifurcation of two asymptotically stable limit cycles (two fixed points of P_ε)?

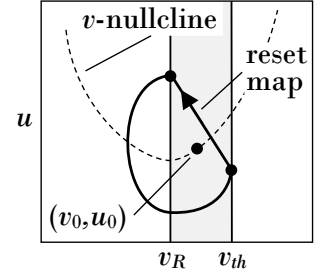


Figure 1: Spiking limit cycle that occurred from (v_0, u_0) as ε increased through zero.

2. Grazing bifurcation of spiking limit cycles from a family of cycles (scenarios 2 and 3 of the approved project). The research here concerned the following simplified version of (1)

$$\begin{aligned} \dot{v} &= -u + \varepsilon g(\omega t), & v(t) &\rightarrow v(t) - \varepsilon h, \ u(t) \rightarrow u(t) + \sqrt{\varepsilon} \cdot l, \text{ if } v(t) = 1. \\ \dot{u} &= v + \varepsilon bu, \end{aligned} \quad (2)$$

a) Autonomous case: $g = 0$. The PI proved earlier that the grazing cycle (Fig. 2a) always produces an asymptotically stable spiking limit cycle (Fig. 2b) as ε increases through zero, provided that

$$l^2 < b\pi < 2l^2. \quad (3)$$

b) Non-autonomous case: $g(t) = t - [2] + 1/2$ (ramp function). Numeric simulations (Fig. 3) show that the conclusion of a) still holds, i.e., for small values of $\varepsilon > 0$, the spiking oscillations of (2) still develop near the grazing cycle constructed for $\varepsilon = 0$. This observation can possibly explain (suggested by Prof. Hasselmo) why spikes occur at the pick of membrane potential oscillations (Domnisoru et al [1]) rather than occurring at a fixed voltage threshold.

In the absence of impacts in (2), the occurrence of asymptotically stable periodic oscillations (resonances) comes from Melnikov subharmonic bifurcation theory [3], which identifies the cycle of the family of Fig. 2a that produces an attractive periodic solution when ε crosses zero. Numerical simulations of Fig. 3 suggest it is no longer the case when the impacts are present (the periodic solution always occurs from the grazing cycle). We still have to carry out an analytic proof which

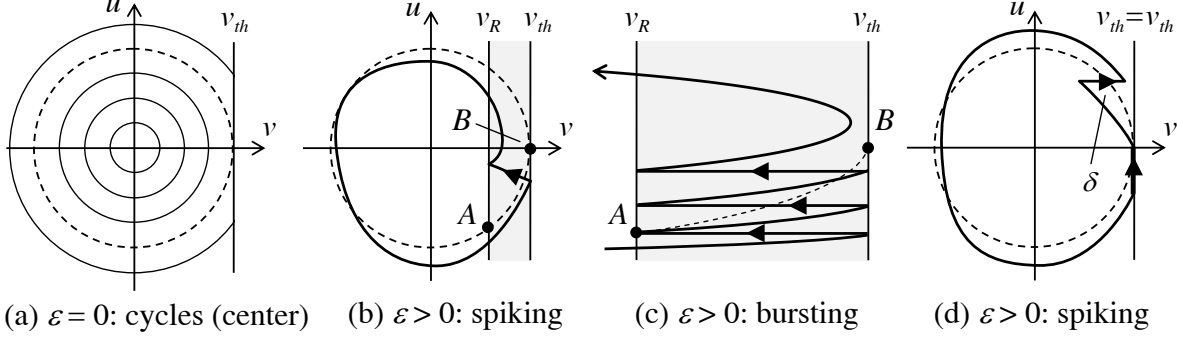


Figure 2: Scenario 2 for different $\tilde{f}, \tilde{b}, \tilde{\omega}, \tilde{\delta}, \tilde{\Delta}$. Arrows "►" stay for instantaneous jumps. The dash cycle transforms to spiking solution (bold curve) when ε increases through zero.

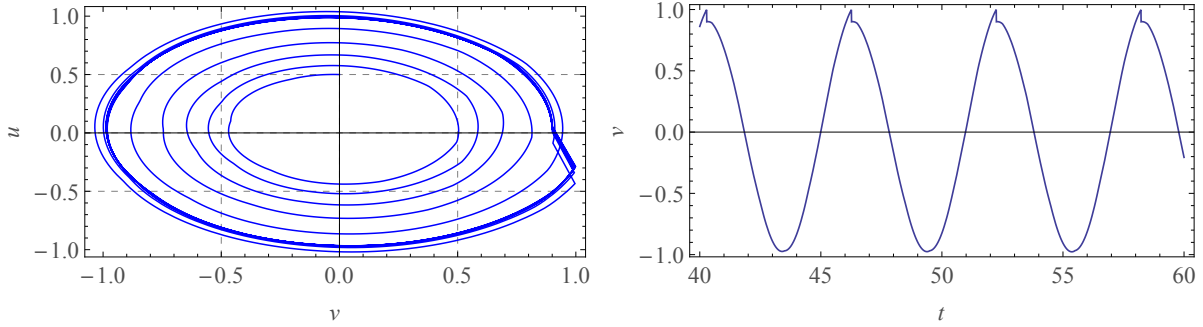


Figure 3: Simulations of (2) with $g(t) = t - [2] + 1/2$, $\varepsilon = 0.1$, $b = 1/2$, $\omega = h = l = 1$. Left: convergence from the inside ($v(0) = 0$, $u(0) = 0.5$). Right: the graph of $v(t)$.

will clarify the extent to which our numeric observation holds (in other words we have to extend the result of 2a to the non-autonomous case).

Possible paper 2: Extend (3) for the case where (2) contains an $\varepsilon f(v)$ term in the first equation. Carry out simulations of b) by adding sinusoidal oscillations to the ramp function $g(t)$. Add a theoretic justification of 2b) by combining a non-autonomous version of the result 2a with

<http://www.utdallas.edu/~makarenkov/time-periodic-impulses.pdf>,

including explaining why the ramp function $g(t)$ doesn't create resonances of arbitrary amplitudes. Is there possible a co-existence of grazing-induced spiking oscillations and those induced by a resonance? Extend the results to the full model (1) and account for a T -periodic impulsive input (see Fig. 2d).

Possible paper 3: Extend the results of part 2 to cover the case where the grazing limit cycle leads to bursting oscillations of Fig. 2c as ε crosses 0.

New research projects arose from discussions in Hasselmo lab

3. What determines length of clustering of spikes in stellate cells? (possible paper 4)

Izhikevich explained [5, §8.4.4] alternating clusters of spiking and subthreshold oscillations by the white noise driven switching between the attractor A_{rest} of the resting state and the attractor A_{spike} of the spiking limit cycle of (1). Using numeric simulations, we plan to link the length of the clusters of spiking oscillations (relative to the length of subthreshold oscillations) to the area

of A_{spike} (relative to the area of A_{rest}). The dependence of the areas of A_{spike} and A_{rest} on the parameters of the system near suitable bifurcation values can be then investigated analytically. In particular, it looks feasible to estimate the rate of the dependence of the area of the interior of the spiking limit cycle from part 2a on ε analytically (and to give conditions to ensure that increasing ε implies decreasing area). Earlier work of Hasselmo's Lab on clustering includes simulation of the effects of H current and AHP current on tendency for clustering (Fransen et al. [2]).

4. Describe the sag potential and how it can be used to measure the intrinsic recovery variable parameters to model the neurons. We found a particular mathematical question that partially responds to problem 4: Consider a saddle equilibrium of Izhikevich's model (1) and give a formula to compute the rate of the exponential convergence to this equilibrium along the stable manifold.

5. The role of impacts in the stability of coupled neurons (this project was also discussed with N. Kopell from BU Department of Mathematics). The goal of the project is to obtain spiking limit cycle in coupled neurons of [4, 6] through a bifurcation from a suitable grazing cycle and compare stability of such a spiking limit cycle with the stability of a spiking limit cycle obtained through a Hopf bifurcation.

References Cited

- [1] C. Domnisoru, A. A. Kinkhabwala, D. W. Tank, Membrane potential dynamics of grid cells, *Nature* 495 (2013) 199–204.
- [2] E. Fransen, A. A. Alonso, C. T. Dickson, J. Magistretti, M. E. Hasselmo, Ionic Mechanisms in the Generation of Subthreshold Oscillations and Action Potential Clustering in Entorhinal Layer II Stellate Neurons, *HIPPOCAMPUS* 14 (2004) 368–384.
- [3] J. Guckenheimer, P. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. *Applied Mathematical Sciences*, 42. Springer-Verlag, New York, 1990. 459 pp.
- [4] M. E. Hasselmo, C. F. Shay, Grid cell firing patterns may arise from feedback interaction between intrinsic rebound spiking and transverse traveling waves with multiple heading angles, *Frontiers in Systems Neuroscience* 8 (2014) 1–24.
- [5] E. M. Izhikevich, *Dynamical systems in neuroscience: the geometry of excitability and bursting*. Computational Neuroscience. MIT Press, Cambridge, MA, 2007, 441 pp.
- [6] M. M. McCarthy, N. Kopell, The Effect of Propofol Anesthesia on Rebound Spiking, *SIAM J. Appl. Dyn. Syst.* 11 (2012), no. 4, 1674–1697.
- [7] O. Makarenkov, Bifurcation of limit cycles from a fold-fold singularity in planar switched systems, *SIAM J. Appl. Dyn. Syst.* 16 (2017), no. 3, 1340–1371.