Bifurcation of limit cycles from a fold-fold singularity in a glacial cycles model
(joint work of Oleg Makarenkov and Esther Widiasih)

To prove the existence of an attractive limit cycle in the model

$$\begin{array}{rcl}\dot{w} & = & -\tau(w-F(\eta)),\\ \dot{\eta} & = & \rho(w+Ls_2(1-\alpha_0)p_2(\eta))-\rho(T_c^{\pm}+\varepsilon\bar{T}_c^{\pm}),\\ \dot{\xi} & = & \epsilon(b^{\pm}(\eta-\xi)-a(1-\eta)),\end{array}\tag{1}$$

we introduce a small parameter $\varepsilon>0$ as

$$\begin{array}{llllll} b^-=b_0+\varepsilon\bar{b}_0, & b^+=b_1+\varepsilon\bar{b}_1, & Paper: & b=1.75, & b_0=1.5, & b_1=5,\\ T_c^-=T^-+\varepsilon\overline{T}^-, & T_c^+=T^++\varepsilon\overline{T}^+, & Paper: & & T_c^-=-5.5, & T_c^+=-10.\end{array}$$

Esther's paper suggests that minus or plus in (1) take place according to whether $(b+a)\eta-a-\xi b$ is negative or positive. We therefore, need to study the occurence of limit cycles in the system

$$\begin{array}{rcl}\dot{x} & = & f(x,y),\\ \dot{y} & = & g(x,y)+g^i(\varepsilon),\\ \dot{z} & = & h^i(y,z,\varepsilon),\end{array}\quad q=\left\{\begin{array}{ll}-1, & \text{if } H(y,z)<0,\\ +1, & \text{if } H(y,z)>0,\end{array}\right.\quad H(y,z)=(b+a)y-a-bz.\tag{2}$$

$$l_0\in\{(x,y,z):H(y,z)=0\}=:L.$$

We plan to prove bifurcation of a stable limit cycle from the point l_0.

In what follows, we denote by $t\mapsto\left(X^i(t,\bar{x},\bar{y},\bar{z},\varepsilon),Y^i(t,\bar{x},\bar{y},\bar{z},\varepsilon),Z^i(t,\bar{x},\bar{y},\bar{z},\varepsilon)\right)^T$ the general solution of the i-th subsystem of (2) with the initial condition $x(0)=\bar{x}$, $y(0)=\bar{y}$, $z(0)=\bar{z}$. Throughout the paper we will allow ourselves to identify vector lines and vector columns where it doesn't lead to a confusion and makes expressions less bulky.

The following notations are required to formulate the theorem. They will also be used throughout the proof.

$$\begin{array}{rcl}\bar{\alpha}^i & = & -2\dfrac{(a+b)g_x(x_0,y_0)}{D^i},\\ \bar{\gamma}^i & = & -2\dfrac{(a+b)(g^i)'(0)-bh^i_\varepsilon(y_0,z_0,0)}{D^i},\\ D^i & = & bg_y(x_0,y_0)-\dfrac{b^2}{a+b}h^i_y(y_0,z_0,0)-bh^i_z(y_0,z_0,0)\end{array}$$

$$\zeta(z)=\frac{a+bz}{a+b},\quad \zeta'(0)=\frac{b}{a+b}$$

$$
\begin{aligned}
\bar{\bar{k}}^i &= \frac{a+b}{6}\left(g'_x(x_0,y_0)f'_y(x_0,y_0)+g''_{yy}(x_0,y_0)(g(x_0,y_0)+g^i(0))+(g'_y(x_0,y_0))^2\right)-\\
&-\frac{b}{6}\left(h^{i\prime\prime}_{yy}(y_0,z_0,0)+\frac{a+b}{b}h^{i\prime\prime}_{yz}(y_0,z_0,0)+\frac{a+b}{b}h^{i\prime\prime}_{zy}(y_0,z_0,0)+\frac{(a+b)^2}{b^2}h^{i\prime\prime}_{zz}(y_0,z_0,0)\right)-\\
&-\frac{b}{6}\left(h^{i\prime}_y(y_0,z_0,0)g'_y(x_0,y_0)+h^{i\prime}_z(y_0,z_0,0)h^{i\prime}_y(y_0,z_0,0)+\frac{a+b}{b}h^{i\prime}_z(y_0,z_0,0)^2\right)\\
\tilde{k}^i &= \frac{a+b}{2}\left(g'_x(x_0,y_0)f'_y(x_0,y_0)+g''_{yy}(x_0,y_0)(g(x_0,y_0)+\tilde{g}^i(0))+g'_y(x_0,y_0)g'_y(x_0,y_0)\right)\zeta'(0)-\\
&-\frac{b}{2}\left(h^{i\prime\prime}_{yy}(y_0,z_0,0)+\frac{a+b}{b}h^{i\prime\prime}_{zy}(y_0,z_0,0)\right)(g(x_0,y_0)+g^i(0))\zeta'(0)-\\
&-\frac{b}{2}\left(h^{i\prime}_y(y_0,z_0,0)g'_y(x_0,y_0)+h^{i\prime}_z(y_0,z_0,0)h^{i\prime}_y(y_0,z_0,0)\right)\zeta'(0)-\\
&-\frac{b}{2}\left(h^{i\prime\prime}_{yz}(y_0,z_0,0)+\frac{a+b}{b}h^{i\prime\prime}_{zz}(y_0,z_0,0)\right)(g(x_0,y_0)+g^i(0))-\frac{b}{2}(h^{i\prime}_z(y_0,z_0,0))^2,\\
\hat{k}^i &= \frac{a+b}{2}g''_{yy}(x_0,y_0)\zeta'(0)^2-b\left(\frac{1}{2}h^{i\prime\prime}_{yy}(y_0,z_0,0)\zeta'(0)^2+h^{i\prime\prime}_{yz}(y_0,z_0,0)\zeta^i(0)+\frac{1}{2}h^{i\prime\prime}_{zz}(y_0,z_0,0)\right)\\
\bar{\eta}^i &= -4\frac{1}{D^i}\left(4\frac{b}{a+b}\bar{\bar{k}}^i\frac{1}{h^i(y_0,z_0,0)}-\tilde{k}^i\frac{2}{h^i(y_0,z_0,0)}+\hat{k}^i\right),\\
B^i &= \frac{1}{2}f'_y(x_0,y_0)\frac{b}{a+b}\bar{\eta}^i+4\frac{b}{a+b}\left(\frac{1}{6}f'_x(x_0,y_0)g'_y(x_0,y_0)+\right.\\
&\left.+\frac{1}{6}f''_{yy}(x_0,y_0)(g(x_0,y_0)+g^i(0))+\frac{1}{6}f'_y(x_0,y_0)g'_y(x_0,y_0)\right)\frac{1}{h^i(y_0,z_0,0)}+\\
&-\left(f_x(x_0,y_0)f_y(x_0,y_0)+f_y(x_0,y_0)g_y(x_0,y_0)\right)\frac{b}{a+b}\cdot\frac{1}{h^i(y_0,z_0,0)}-\frac{1}{2}f_{yy}(x_0,y_0)\frac{b^2}{(a+b)^2},\\
k &= -\frac{\bar{\eta}^- - \bar{\eta}^+}{\bar{\alpha}^- - \bar{\alpha}^+},\\
m &= -\frac{\bar{\gamma}^- - \bar{\gamma}^+}{\bar{\alpha}^- - \bar{\alpha}^+},\\
A^i &= f_x(x_0,y_0)+\frac{1}{2}f_y(x_0,y_0)(g(x_0,y_0)+g^i(0)),\\
K &= 2\frac{kA^+ + B^+}{h^+(y_0,z_0,0)}-2\frac{kA^- + B^-}{h^-(y_0,z_0,0)}\\
C^i &= \frac{1}{2}f_y(x_0,y_0)\frac{b}{a+b}\bar{\gamma}^i.\\
M &= (mA^- + C^-)\beta^- - (mA^+ + C^+)\beta^+.
\end{aligned}
$$

Note, $\nabla H(y,z)$ doesn't depend on (y,z). Specifically,

$$\nabla H(y,z) = (a+b,-b) =: \nabla H$$

Theorem 1 *Let*

$$(x_0,y_0,z_0)\in L \tag{3}$$

be such a point that

$$f(x_0,y_0)=0 \quad \text{(border collision in } x \text{ variable)} \tag{4}$$

and

$$\nabla H \begin{pmatrix} g(x_0,y_0)+g^i(0) \\ h^i(y_0,z_0,0) \end{pmatrix} = 0, \quad i \in \{-1,+1\} \quad \text{(fold-fold in } (y,z) \text{ variables).} \tag{5}$$

Assume that

$$\bar{\alpha}^- - \bar{\alpha}^+ \neq 0, \tag{6}$$

$$KM < 0 \quad \text{(bifurcation of 2 fixed points occurs).} \tag{7}$$

Finally, assume that

$$h^+(y_0,z_0,0)\cdot(\bar{\eta}^+ - \bar{\eta}^-) < 0, \qquad \text{(stability of 1/2 fixed points)} \tag{8}$$

$$h^+(y_0,z_0,0)h^-(y_0,z_0,0) < 0, \qquad \text{(positivity of time map of 1/2 fixed points)} \tag{9}$$

$$bh^+(y_0,z_0,0)\cdot\left(g_y(x_0,y_0) - \frac{b}{a+b}h_y^+(y_0,z_0,0) - h_z^+(y_0,z_0,0)\right) < 0, \tag{10}$$

$$bh^+(y_0,z_0,0)\cdot\left(g_y(x_0,y_0) - \frac{b}{a+b}h_y^-(y_0,z_0,0) - h_z^-(y_0,z_0,0)\right) < 0. \tag{11}$$

(conditions for the limit cycle to be real rather than virtual) (12)

Then, for all $\varepsilon > 0$ sufficiently small, system (2) admits a unique attractive limit cycle in a small neighborhood of (x_0,y_0,z_0) that shrinks to (x_0,y_0,z_0) as $\varepsilon \to 0$. The period of the cycle equals

$$T = \frac{2}{|h^-(y_0,z_0,0)|}\sqrt{-\varepsilon M/K} + \frac{2}{|h^+(y_0,z_0,0)|}\sqrt{-\varepsilon M/K} + O(\varepsilon). \tag{13}$$

Proof. Step 1: *Expanding the time map $T_\varepsilon^i(x,z)$.* We will find $T_\varepsilon^i(x,z)$ as a solution of the equation

$$\begin{aligned} &H(Y^i(T,x,\zeta(z),z,\varepsilon),Z^i(T,x,\zeta(z),z,\varepsilon)) = \\ &= (a+b)Y^i(T,x,\zeta(z),z,\varepsilon) - a - bZ^i(T,x,\zeta(z),z,\varepsilon) = 0. \end{aligned} \tag{14}$$

To do this, we expand $Y^i(T,x,\zeta(z),z,\varepsilon)$ and $Z^i(T,x,\zeta(z),z,\varepsilon)$ in Taylor's series near $(T,x,z,\varepsilon) = (0,x_0,z_0,0)$. We write down an expansion for Z^i only as Y^i expands analogously.

$$\begin{aligned} Z(T,x,\zeta(z),z,\varepsilon) &= Z(0,x_0,\zeta(z_0),z_0,0) + Z^{i\prime}_{t}(l_{00})T + Z^{i\prime}_{y}(l_{00})\zeta'(0)(z-z_0) + Z^{i\prime}_{z}(l_{00})(z-z_0) + \\ &+ \left[\frac{1}{2}Z^{i\prime\prime}_{tt}(l_{00})T + Z^{i\prime\prime}_{tx}(l_{00})(x-x_0) + \left(Z^{i\prime\prime}_{ty}(l_{00})\zeta'(0) + Z^{i\prime\prime}_{tz}(l_{00})\right)(z-z_0) + Z^{i\prime\prime}_{t\varepsilon}(l_{00})\varepsilon\right]T + \\ &+ \left[\frac{1}{6}Z^{i\prime\prime\prime}_{ttt}(l_{00})T^2 + \frac{1}{2}\left(Z^{i\prime\prime\prime}_{tty}(l_{00})\zeta'(0) + Z^{i\prime\prime\prime}_{ttz}(l_{00})\right)T(z-z_0) + \right. \\ &\quad \left. + \left(\frac{1}{2}Z^{i\prime\prime\prime}_{tyy}(l_{00})\left(\zeta'(0)\right)^2 + Z^{i\prime\prime\prime}_{tyz}(l_{00})\zeta'(0) + \frac{1}{2}Z^{i\prime\prime\prime}_{tzz}(l_{00})\right)(z-z_0)^2\right]T + \text{remaining terms.} \end{aligned}$$

Using these expansions along with assumptions (4) and (5) we can rewrite (14) as

$$\begin{aligned} &\bar{m}^iT + \tilde{m}^i(x-x_0) + \hat{m}^i(z-z_0) + m^i\varepsilon + \\ &\qquad + \bar{k}^iT^2 + \tilde{k}^iT(z-z_0) + \hat{k}^i(z-z_0)^2 + \text{remaining terms} = 0, \end{aligned}$$

where

$$
\begin{aligned}
\bar{m}^i &= \frac{1}{2}(a+b){Y^i}''_{tt}(l_{00}) - \frac{1}{2}b{Z^i}''_{tt}(l_{00}), \\
\tilde{m}^i &= (a+b){Y^i}''_{tx}(l_{00}) - b{Z^i}''_{tx}(l_{00}), \\
\hat{m}^i &= (a+b)({Y^i}''_{ty}(l_{00})\zeta'(0) + {Y^i}''_{tz}(l_{00})) - b({Z^i}''_{ty}(l_{00})\zeta'(0) + {Z^i}''_{tz}(l_{00})), \\
m^i &= (a+b){Y^i}''_{t\varepsilon}(l_{00}) - b{Z^i}''_{t\varepsilon}(l_{00}), \\
\bar{k}^i &= \frac{1}{6}(a+b){Y^i}'''_{ttt}(l_{00}) - \frac{1}{6}b{Z^i}'''_{ttt}(l_{00}), \\
\tilde{k}^i &= \frac{1}{2}(a+b)({Y^i}'''_{tty}(l_{00})\zeta'(0) + {Y^i}'''_{ttz}(l_{00})) - \frac{1}{2}b({Z^i}'''_{tty}(l_{00})\zeta'(0) + {Z^i}'''_{ttz}(l_{00})), \\
\hat{k}^i &= (a+b)\left(\frac{1}{2}{Y^i}'''_{tyy}(l_{00})(\zeta'(0))^2 + {Y^i}'''_{tyz}(l_{00}) + \frac{1}{2}{Y^i}'''_{tzz}(l_{00})\right) - \\
&\quad -b\left(\frac{1}{2}{Z^i}'''_{tyy}(l_{00})(\zeta'(0))^2 + {Z^i}'''_{tyz}(l_{00}) + \frac{1}{2}{Z^i}'''_{tzz}(l_{00})\right).
\end{aligned}
$$

Therefore, we can compute T^i_ε as

$$
T^i_\varepsilon(x,z) = \alpha^i(x-x_0) + \beta^i(z-z_0) + \gamma^i\varepsilon + \eta^i(z-z_0)^2 + \text{remaining terms}, \tag{15}
$$

where

$$
\alpha^i = -\frac{\tilde{m}^i}{\bar{m}^i}, \quad \beta^i = -\frac{\hat{m}^i}{\bar{m}^i}, \quad \gamma^i = -\frac{m^i}{\bar{m}^i}, \quad \eta^i = -\frac{\bar{k}^i(\beta^i)^2 + \tilde{k}^i\beta^i + \hat{k}^i}{\bar{m}^i}.
$$

Step 2: *The Poincare map P_ε and its fixed points.* Expanding $X^i(T,x,\zeta(z),z,\varepsilon)$ about $(T,x,z,\varepsilon) = (0,x_0,z_0,0)$ we get

$$
\begin{aligned}
&X^i(T,x,\zeta(z),z,\varepsilon) = x_0 + {X^i}'_t(l_{00})T + {X^i}'_x(l_{00})(x-x_0) + \\
&+\left[\frac{1}{2}{X^i}''_{tt}(l_{00})T + {X^i}''_{tx}(l_{00})(x-x_0) + {X^i}''_{ty}(l_{00})\zeta'(0)(z-z_0) + {X^i}''_{tz}(l_{00})(z-z_0) + {X^i}''_{t\varepsilon}(l_{00})\varepsilon\right]T + \\
&+\left[\frac{1}{6}X'''_{ttt}(l_{00})T^2 + \frac{1}{2}{X^i}'''_{tty}(l_{00})T\zeta'(0)(z-z_0) + \frac{1}{2}{X^i}'''_{ttz}(l_{00})T(z-z_0) + \right. \\
&+\left.\left(\frac{1}{2}X'''_{tyy}(l_{00})\zeta'(0)^2 + X'''_{tyz}(l_{00})\zeta'(0) + \frac{1}{2}X'''_{tzz}(l_{00})\right)(z-z_0)^2\right]T + \text{remaining terms},
\end{aligned}
$$

and similarly for $Z^i(T,x,\zeta(z),z,\varepsilon)$. Therefore, for the map

$$
P^i_\varepsilon(x,z) = \begin{pmatrix} X^i(T^i_\varepsilon(x,z),x,\zeta(z),z,\varepsilon) \\ Z^i(T^i_\varepsilon(x,z),x,\zeta(z),z,\varepsilon) \end{pmatrix}
$$

we have

$$P^i_\varepsilon(x,z) = \left(\begin{array}{c} x + \Big[\frac{1}{2}X^{i\prime\prime}_{tt}(l_{00})T^i(x,z,\varepsilon) + X^{i\prime\prime}_{tx}(l_{00})(x-x_0) + X^{i\prime\prime}_{ty}(l_{00})\zeta'(0)(z-z_0) + \\ + \frac{1}{6}X^{i\prime\prime\prime}_{ttt}(l_{00})T^i(x,z,\varepsilon)^2 + \frac{1}{2}X^{i\prime\prime\prime}_{tty}(l_{00})T^i(x,z,\varepsilon)\zeta'(0)(z-z_0) + \\ + \frac{1}{2}X^{i\prime\prime\prime}_{tyy}(l_{00})\zeta'(0)^2(z-z_0)^2\Big]\, T^i(x,z,\varepsilon) \\ z + Z^{i\prime}_{t}(l_{00})T^i(x,z,\varepsilon) + \Big[\frac{1}{2}Z^{i\prime\prime}_{tt}(l_{00})T^i_\varepsilon(x,z) + Z^{i\prime\prime}_{ty}(l_{00})\zeta'(0)(z-z_0) + \\ + Z^{i\prime\prime}_{tz}(l_{00})(z-z_0)\Big]\, T^i(x,y,\varepsilon) \end{array}\right) +$$

$$+ \text{ remaining terms.}$$

Now we cancel some terms by observing that (5) yields

$$\beta^i = -\frac{2}{h^i(y_0,z_0,0)}, \tag{16}$$

from where

$$\begin{aligned} \frac{1}{2}X^{i\prime\prime}_{tt}(l_{00})\beta^i + X^{i\prime\prime}_{ty}(l_{00})\zeta'(0) &= 0, \\ \frac{1}{2}Z^{i\prime\prime}_{tt}(l_{00})\beta^i + Z^{i\prime\prime}_{ty}(l_{00})\zeta'(0) + Z^{i\prime\prime}_{tz}(l_{00}) &= 0, \end{aligned}$$

so that P^i_ε can be rewritten in the form

$$\begin{aligned} P^i_\varepsilon(x,z) &= \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} A^i(x-x_0) + B^i(z-z_0)^2 + C^i\varepsilon \\ h^i(y_0,z_0,0) \end{pmatrix} \cdot \\ &\cdot \left((\alpha^i,\beta^i)\begin{pmatrix} x-x_0 \\ z-z_0 \end{pmatrix} + \eta^i(z-z_0)^2 + \gamma^i\varepsilon\right) + \text{remaining terms,} \end{aligned} \tag{17}$$

where

$$\begin{aligned} A^i &= \frac{1}{2}X^i_{tt}(l_{00}) + X^{i\prime\prime}_{tx}(l_{00}), \\ B^i &= \frac{1}{2}X^{i\prime\prime}_{tt}(l_{00})\eta^i + \frac{1}{6}X^{i\prime\prime\prime}_{ttt}(l_{00})(\beta^i)^2 + \frac{1}{2}X^{i\prime\prime\prime}_{tty}(l_{00})\zeta'(0)\beta^i + \frac{1}{2}X^{i\prime\prime\prime}_{tyy}(l_{00})\zeta'(0)^2, \\ C^i &= \frac{1}{2}X^{i\prime\prime}_{tt}(l_{00})\gamma^i. \end{aligned}$$

Since the way P^i_ε is introduced implies that $P^i_\varepsilon = \left(P^i_\varepsilon\right)^{-1}$ (i.e. P^i_ε is an involution) we can find fixed points of the map

$$P_\varepsilon(u) = P^-_\varepsilon(P^+_\varepsilon(u))$$

by solving the equation

$$P^-_\varepsilon(u) = P^+_\varepsilon(u). \tag{18}$$

We will first solve the second equation of (18) and find $x(z,\varepsilon)$. This will be found uniquely. Then we will plug the result into the first equation of (18) and find $z(\varepsilon)$. The later will have two solutions which correspond to the two points where the cycle intersects the cross-section L.

Letting
$$\Phi(x,z,\varepsilon)=\left[P_\varepsilon^-(x,z)-P_\varepsilon^+(x,z)\right]_2,$$
we compute

$$\begin{array}{rcl}\Phi'_x(x_0,z_0,0) & = & \alpha^-h^-(y_0,z_0,0)-\alpha^+h^+(y_0,z_0,0),\\ \Phi'_z(x_0,z_0,0) & = & \beta^-h^-(y_0,z_0,0)-\beta^+h^+(y_0,z_0,0)=0,\\ \Phi''_{zz}(x_0,z_0,0) & = & 2\eta^-h^-(y_0,z_0,0)-2\eta^+h^+(y_0,z_0,0),\\ \Phi'_\varepsilon(x_0,z_0,0) & = & \gamma^-h^-(y_0,z_0,0)-\gamma^+h^+(y_0,z_0,0),\end{array}$$

Using assumption (6) we apply the Implicit Function Theorem and solve $\Phi(x,z,\varepsilon)=0$ in x when (z,ε) is near $(z_0,0)$. The Implicit Function Theorem gives

$$(x'_z,x'_\varepsilon)(z_0,0)=-\frac{1}{\Phi'_x(x_0,z_0,0)}\left(0,\Phi'_\varepsilon(x_0,z_0,0)\right),\quad x''_{zz}(z_0,0)=-\frac{1}{\Phi'_x(x_0,z_0,0)}\Phi''_{zz}(x_0,z_0,0).$$

Therefore,
$$x(z,\varepsilon)=x_0+k(z-z_0)^2+m\varepsilon+\text{remaining terms}. \tag{19}$$

Plugging expression (19) into the first line of (18) we obtain the following equation for $z-z_0$

$$J(z-z_0)^4+K(z-z_0)^3+L\varepsilon(z-z_0)^2+M\varepsilon(z-z_0)+N\varepsilon^2+\text{remaining terms}=0.$$

The change of the variables
$$z-z_0=\varepsilon^{1/2}p$$
yields
$$J\varepsilon^2p^4+K\varepsilon^{3/2}p^3+L\varepsilon^2p^2+M\varepsilon^{3/2}p+N\varepsilon^2+\text{remaining terms}=0,$$
or
$$J\varepsilon^{1/2}p^4+Kp^3+L\varepsilon^{1/2}p^2+Mp+N\varepsilon^{1/2}+\text{remaining terms}=0,$$
where we will need formulas for only some of J, K, L, M, and N, that we give a bit later. Assumption (7) ensures that this quadratic polynomial admits two solutions $\underline{z}-z_0$ and $\overline{z}-z_0$. Combining formulas for solutions of the quadratic polynomial with with formula (19), we conclude that the Poincare map P_ε possesses the following two fixed points around (x_0,z_0) for $\varepsilon>0$ sufficiently small

$$\begin{array}{l}\begin{pmatrix}\underline{x}(\varepsilon)\\ \underline{z}(\varepsilon)\end{pmatrix}=\begin{pmatrix}x_0\\ z_0\end{pmatrix}+\begin{pmatrix}\varepsilon(-kM/K+m)\\ -\varepsilon^{1/2}\sqrt{-M/K}\end{pmatrix}+\text{remaining terms},\\ \begin{pmatrix}\overline{x}(\varepsilon)\\ \overline{z}(\varepsilon)\end{pmatrix}=\begin{pmatrix}x_0\\ z_0\end{pmatrix}+\begin{pmatrix}\varepsilon(-kM/K+m)\\ \varepsilon^{1/2}\sqrt{-M/K}\end{pmatrix}+\text{remaining terms},\end{array}\tag{20}$$

where

$$\begin{array}{rcl}K & = & (kA^-+B^-)\beta^--(kA^++B^+)\beta^+,\\ M & = & (mA^-+C^-)\beta^--(mA^++C^+)\beta^+.\end{array}$$

Step 3: *Stability of fixed points* $(\underline{x}(\varepsilon), \underline{z}(\varepsilon))$ *and* $(\overline{x}(\varepsilon), \overline{z}(\varepsilon))$. The fixed point $(\underline{x}(\varepsilon), \underline{z}(\varepsilon))$ is stable, if the eigenvalues of the matrix

$$(P_\varepsilon)'(\underline{x}(\varepsilon), \underline{z}(\varepsilon)) = (P_\varepsilon^-)'(\underline{\underline{x}}(\varepsilon), \underline{\underline{z}}(\varepsilon)) \circ (P_\varepsilon^+)'(\underline{x}(\varepsilon), \underline{z}(\varepsilon)), \qquad (\underline{\underline{x}}(\varepsilon), \underline{\underline{z}}(\varepsilon))^T = P_\varepsilon^+(\underline{x}(\varepsilon), \underline{z}(\varepsilon)) \quad (21)$$

are strictly inside the unit circle. Formula (17) yields

$$\begin{pmatrix} \underline{\underline{x}}(\varepsilon) \\ \underline{\underline{z}}(\varepsilon) \end{pmatrix} = \begin{pmatrix} \underline{x}(\varepsilon) \\ \underline{z}(\varepsilon) - \varepsilon^{1/2}\beta^+ h^+(y_0, z_0, 0)\sqrt{-M/K} \end{pmatrix} + \text{remaining terms}, \quad (22)$$

and since

$$\begin{aligned}(P_\varepsilon^i)'(x,z) &= I + \begin{pmatrix} 0 \\ h^i(y_0,z_0,0) \end{pmatrix} \left(\alpha^i, \beta^i + 2\eta^i(z - z_0)\right) + \\ &+ \begin{pmatrix} A^i & 2B^i(z-z_0) \\ 0 & 0 \end{pmatrix} \left((\alpha^i, \beta^i) \begin{pmatrix} x - x_0 \\ z - z_0 \end{pmatrix} + \eta^i (z-z_0)^2 + \gamma^i \varepsilon \right) + \\ &+ \begin{pmatrix} A^i(x - x_0) + B^i(z-z_0)^2 + C^i\varepsilon \\ 0 \end{pmatrix} \left(\alpha^i, \beta^i + 2\eta^i(z-z_0)\right) + \text{remaining terms,}\end{aligned}$$

the composition (21) takes the form

$$\begin{aligned}(P_\varepsilon)'(\underline{x}(\varepsilon), \underline{z}(\varepsilon)) &= (\Psi^- - \varepsilon^{1/2}(1 + \beta^+ h^+(y_0,z_0,0))\Phi^-)\left(\Psi^+ - \varepsilon^{1/2}\Phi^+\right) + O(\varepsilon^{3/2}) = \\ &= (\Psi^- + \varepsilon^{1/2}\Phi^-)\left(\Psi^+ - \varepsilon^{1/2}\Phi^+\right) + O(\varepsilon^{3/2}),\end{aligned}$$

where

$$\begin{aligned}\Psi^i &= I + \begin{pmatrix} 0 & 0 \\ \alpha^i & \beta^i \end{pmatrix} h^i(y_0,z_0,0) = \begin{pmatrix} 1 & 0 \\ \alpha^i h^i(y_0,z_0,0) & -1 \end{pmatrix}, \\ \Phi^i &= \begin{pmatrix} A^i\beta^i & 0 \\ 0 & 2h^i(y_0,z_0,0)\eta^i \end{pmatrix} \sqrt{-M/K},\end{aligned}$$

or, equivalently,

$$\begin{aligned}(P_\varepsilon)'(\underline{x}(\varepsilon), \underline{z}(\varepsilon)) &= \Psi^-\Psi^+ + \varepsilon^{1/2}(\Phi^-\Psi^+ - \Psi^-\Phi^+) + O(\varepsilon) = \\ &= \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} + \varepsilon^{1/2} \begin{pmatrix} A^-\beta^- - A^+\beta^+ & 0 \\ * & -2h^-(y_0,z_0,0)\eta^- + 2h^+(y_0,z_0,0)\eta^+ \end{pmatrix},\end{aligned}$$

where *-symbols stay for some (different) constant, which values don't influence the conclusions. Therefore, the fixed point $(\underline{x}(\varepsilon), \underline{z}(\varepsilon))$ is stable, if

$$h^+(y_0,z_0,0)A^+ - h^-(y_0,z_0,0)A^- < 0, \qquad h^+(y_0,z_0,0)\eta^+ - h^-(y_0,z_0,0)\eta^- < 0.$$

Analogously, the fixed point $(\overline{x}(\varepsilon), \overline{z}(\varepsilon))$ is stable, if

$$h^+(y_0,z_0,0)A^+ - h^-(y_0,z_0,0)A^- > 0, \qquad h^+(y_0,z_0,0)\eta^+ - h^-(y_0,z_0,0)\eta^- > 0.$$

Assumption (8) ensures that one of these two sets of inequalities holds.

Step 4: *Verifying that P_ε^- and P_ε^+ map the points of L from the past to the future.*

Case 1: The points from the neighborhood of $(\underline{x}(\varepsilon), \underline{z}(\varepsilon))$ and $(\underline{\underline{x}}(\varepsilon), \underline{\underline{z}}(\varepsilon))$, i.e. the case where $h^+(y_0, z_0, 0) > 0$. We must check that

$$T^+(\underline{x}(\varepsilon), \underline{z}(\varepsilon) > 0 \quad \text{and} \quad T^-(\underline{\underline{x}}(\varepsilon), \underline{\underline{z}}(\varepsilon)) > 0. \tag{23}$$

Using (15), (16), (20), and (22), we have

$$\begin{aligned}
T^+(\underline{x}(\varepsilon), \underline{z}(\varepsilon)) &= \varepsilon^{1/2} \frac{2}{h^+(y_0, z_0, 0)} \sqrt{-M/K} + O(\varepsilon), \\
T^-(\underline{\underline{x}}(\varepsilon), \underline{\underline{z}}(\varepsilon)) &= -\varepsilon^{1/2} \frac{2}{h^-(y_0, z_0, 0)} \sqrt{-M/K} + O(\varepsilon).
\end{aligned}$$

Therefore the positivity properties (23) follow from (9).

Case 2: The points from the neighborhood of $(\overline{x}(\varepsilon), \overline{z}(\varepsilon))$ and $(\overline{\overline{x}}(\varepsilon), \overline{\overline{z}}(\varepsilon))$, i.e. the case where $h^-(y_0, z_0, 0) > 0$. By analogy with Case 1, one can use formulas (15), (16), (20), and an analogue of (22) in order to verify that

$$\begin{aligned}
T^+(\overline{x}(\varepsilon), \overline{z}(\varepsilon)) &= -\varepsilon^{1/2} \frac{2}{h^+(y_0, z_0, 0)} \sqrt{-M/K} + O(\varepsilon) > 0, \\
T^-(\overline{\overline{x}}(\varepsilon), \overline{\overline{z}}(\varepsilon)) &= \varepsilon^{1/2} \frac{2}{h^-(y_0, z_0, 0)} \sqrt{-M/K} + O(\varepsilon) > 0.
\end{aligned}$$

under assumption (9).

Step 5: *Verifying that P_ε^- and P_ε^+ act in the subspaces $\{(x, y, z) : H(y, z) \leq 0\}$ and $\{(x, y, z) : H(y, z) \geq 0\}$ respectively.*

Case 1: The trajectory with the initial condition at $(\underline{x}(\varepsilon), \zeta(\underline{z}(\varepsilon)), \underline{z}(\varepsilon))$, i.e. the case where $h^+(y_0, z_0, 0) > 0$.

a) The map P_ε^+. Note, a vector $v \in \mathbb{R}^3$ with the origin $(x, y, z) \in L$ points towards $\{(x, y, z) : H(y, z) \geq 0\}$, if $(0, \nabla H(y, z))v > 0$. Therefore, the vector field of the "+"-subsystem of (2) points to $\{(x, y, z) : H(y, z) > 0\}$ at $(\underline{x}(\varepsilon), \underline{\zeta}(z(\varepsilon)), \underline{z}(\varepsilon))$, if

$$\nabla H(\zeta(\underline{z}(\varepsilon)), \underline{z}(\varepsilon)) \begin{pmatrix} g(\underline{x}(\varepsilon), \zeta(\underline{z}(\varepsilon))) + g^+(\varepsilon) \\ h^+(\zeta(\underline{z}(\varepsilon)), \underline{z}(\varepsilon), \varepsilon) \end{pmatrix} > 0, \tag{24}$$

which follows from (10).

b) The map P_ε^-. Here we have to check that the vector field of the "-"-subsystem of (2) points towards $\{(x, y, z) : H(y, z) < 0\}$ at $(\underline{\underline{x}}(\varepsilon), \zeta(\underline{\underline{z}}(\varepsilon)), \underline{\underline{z}}(\varepsilon))$. Equivalently, we have to establish that

$$\nabla H(\zeta(\underline{\underline{z}}(\varepsilon)), \underline{\underline{z}}(\varepsilon)) \begin{pmatrix} g(\underline{\underline{x}}(\varepsilon), \zeta(\underline{\underline{z}}(\varepsilon))) + g^-(\varepsilon) \\ h^-(\zeta(\underline{\underline{z}}(\varepsilon)), \underline{\underline{z}}(\varepsilon), \varepsilon) \end{pmatrix} < 0, \tag{25}$$

which follows from (11).

Case 2: The trajectory with the initial condition at $(\overline{x}(\varepsilon), \zeta(\overline{z}(\varepsilon)), \overline{z}(\varepsilon))$, i.e. the case where $h^+(y_0, z_0, 0) < 0$. Considering $(\overline{x}(\varepsilon), \overline{z}(\varepsilon))$ in place of $(\underline{x}(\varepsilon), \underline{z}(\varepsilon))$ will just flip the sign in the respective expressions (24)-(25), which validity will still follow from (10)-(11) because the sign of $h^+(y_0, z_0, 0)$ flips as well. Therefore, the vector field of the "+"-subsystem of (2) points to $\{(x, y, z) : H(y, z) > 0\}$ at $(\overline{x}(\varepsilon), \zeta(\overline{z}(\varepsilon)), \overline{z}(\varepsilon))$ and the vector field of the "-"-subsystem of (2) points to $\{(x, y, z) : H(y, z) < 0\}$ at $(\overline{\overline{x}}(\varepsilon), \zeta(\overline{\overline{z}}(\varepsilon)), \overline{\overline{z}}(\varepsilon))$, if conditions (10)-(11) hold.

Auxiliary relations:

$$
\begin{aligned}
\bar{k}^i &= \bar{\bar{k}}^i \cdot (g(x_0, y_0) + g^i(0)),\\
\eta^i &= \frac{\bar{\eta}^i}{h^i(y_0, z_0, 0)},\\
\alpha^i &= \frac{\bar{\alpha}^i}{h^i(y_0, z_0, 0)},\\
\gamma^i &= \frac{\bar{\gamma}^i}{h^i(y_0, z_0, 0)},\\
C^i &= \frac{1}{2} f_y(x_0, y_0)(g(x_0, y_0) + g^i(0))\gamma^i.
\end{aligned}
$$

$$
\begin{aligned}
B^i \;=\; & \frac{1}{2} f'_y(x_0, y_0)(g(x_0, y_0) + g^i(0))\eta^i + \Bigg(\frac{1}{6} f'_x(x_0, y_0) g'_y(x_0, y_0)(g(x_0, y_0) + g^i(0)) + \\
& + \frac{1}{6} f'{}'_{yy}(x_0, y_0)(g(x_0, y_0) + g^i(0))^2 + \frac{1}{6} f'_y(x_0, y_0) g'_y(x_0, y_0)(g(x_0, y_0) + g^i(0))\Bigg) (\beta^i)^2 + \\
& + \left(\frac{1}{2} f_x(x_0, y_0) f_y(x_0, y_0) + \frac{1}{2} f_{yy}(x_0, y_0)(g(x_0, y_0) + g^i(0)) + \frac{1}{2} f_y(x_0, y_0) g_y(x_0, y_0)\right) \frac{b}{a+b}\beta^i + \\
& + \frac{1}{2} f'{}'_{yy}(x_0, y_0) \left(\frac{b}{a+b}\right)^2,
\end{aligned}
$$

The proof of the theorem is complete.

Simulations. We implement simulations with the following parameters

$$b = 1.75,\ a = 1.05,\ b_0 = 1.5,\ b_1 = 5,\ \epsilon = 0.03,\ T_c^- = -5,\ \bar{T}^- = 1,\ \bar{T}^+ = 0,\ \bar{b}_0 = 0,\ \bar{b}_1 = 0.$$

Using Mathematica software we conclude that condition (4) holds for system (1), if either

$$w_0 = 531.915,\ \eta_0 = -4.52983,\ \xi_0 = -7.84772,\ T_c^+ = -7.17737,$$

or

$$w_0 = -2.49476,\ \eta_0 = 0.65411,\ \xi_0 = 0.446575,\ T_c^+ = -5.13619,$$

or

$$w_0 = -5.44116,\ \eta_0 = 0.563218,\ \xi_0 = 0.301148,\ T_c^+ = -5.17198.$$

Considering the case of $T_c^+ = -5.13619$ and $\varepsilon = 0.01$, formula (13) yields $T = 0.61744$, while simulations show $T \approx 0.82$. Taking $\varepsilon = 0.000001$, formula (13) returns $T = 0.0061744$ and simulations show $T \approx 0.006$.