## 1 Introduction

A switched system is a system of differential equations with discontinuous or impulsive right-hand-terms. Automatic switchers are often used in control because of simplicity of implementation. One can think of a water tank. The simplest way to operate the tank would be to switch the water on when $h$ crosses $h_{1}$ and to switch the water off when $h$ crosses $h_{2}$ ([1, 2]), see Fig. 1.1(Left). Same type of so-called relay switching operates anti-lock braking systems [3, 4], electric power converters [22, 23], therapy alternation in medicine [6, 7], harvesting control in population dynamics [5], etc. Some discontinuities are integral parts of physical processes. Consider, for example, the simplest bipedal robot of Fig. 1.1(Right) standing on a gentle slope. Here we have two legs connected with a joint in the top whose masses $m$ are lumped in the middle of each leg. Perhaps surprisingly, if we push this double pendulum down the slope successfully, it will walk alternating support legs periodically [19, 20] (see movie [21). As a result, we will have a motion where each leg collides with the ground periodically. These collisions bring impulsive discontinuity in the differential equations of the motion. Similar discontinuities (called resets) are found in neuroscience [8, 9, 10]. We will consider some of these and other models in details during the course. A comprehensive list of applications of switched systems can be found in my survey [11].


Figure 1.1: Left: water tank with switching control of water level. Right: bipedal passive walker.

To summarize, 1) discontinuities appear in physical devices because of simplicity of practical implementation, 2) discontinuities and impulses are intrinsic natural features of some physical processes. One more reason as for why discontinuous differential equations are required in applied sciences is because 3) they can ensure much better stability (finite-time stability or finite-time stabilization) compared to smooth differential equations. We devote a dedicated section to justify the latter statement.

## 2 Finite-time stabilization

Given a system of ordinary differential equations

$$
\begin{equation*}
\dot{x}=\phi(t, x), \quad \phi \in C^{1}\left(\mathbb{R} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right) \tag{2.1}
\end{equation*}
$$

and a reference trajectory $\bar{x}(t)$, the stabilization problem is to find a function $\psi: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ (control) such that for every solution $x$ of the perturbed system

$$
\begin{equation*}
\dot{x}=\phi(t, x)+\psi(t, x) \tag{2.2}
\end{equation*}
$$

there exists $\bar{t}$ such that $x(t)=\bar{x}(t)$ for all $t \geq \bar{t}$. In the next section we see that the problem cannot be solved in the class $g \in C^{1}\left(\mathbb{R} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right)$ and the most part of this chapter is devoted to stabilization of the systems of differential equations

$$
\begin{equation*}
\dot{x}=f(t, x) \tag{2.3}
\end{equation*}
$$

with $f \notin C^{1}\left(\mathbb{R} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right)$, where we have 2.2$)$ in mind, as a particular case.

### 2.1 Insufficiency of smooth control for finite-time stabilization

The above stated problem cannot be solved in the class of $\psi \in C^{1}\left(\mathbb{R} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right)$. Indeed, assume the contrary. Then, given any two solutions $x_{a}$ and $x_{b}$ of 2.2 with $x_{a}(0) \neq x_{b}(0)$, there will be $\bar{t}$ such that

$$
x_{a}(t)=\bar{x}(t) \quad \text { and } \quad x_{b}(t)=\bar{x}(t) \quad \text { for all } t \geq \bar{t}
$$

This implies that the initial value problem $x(\bar{t})=\bar{x}(\bar{t})$ for (2.2) lacks backward uniqueness, which cannot happen to differential equations with $C^{1}$ right-hand-sides.

### 2.2 Finite-time stabilization in the water tank example: the problem of defining a solution

We use a water tank again to illustrate the finite-time stabilization concept. At Fig. 2.1(Left)



Figure 2.1: Left: water tank with just one water level control sensor $\bar{h}$. Right: two sample trajectories of $h(t)$ that start at $h(0)<\bar{h}$ and $h(0)>\bar{h}$ respectively.
the two sensors of Fig. 1.1(Left) are replaced by a single one $\bar{h}$. Assume that the incoming water is on for $h<\bar{h}$ and is off for $h>\bar{h}$. Assume further that given an initial water level $h(0)=h_{0}$, the level of the water changes according to the formula

$$
\begin{array}{ll}
h_{0}>\bar{h}: & h(t)=h_{0}-t, \quad \text { as long as } \quad h(t)>\bar{h}, \\
h_{0}<\bar{h}: & h(t)=h_{0}+3 t, \quad \text { as long as } \quad h(t)<\bar{h} .
\end{array}
$$

Starting from any initial condition $h_{0}$, the function $h(t)$ reaches $\bar{h}$ in finite time or we say that the water level stabilizes in finite time (see Fig. 2.1(Right)). The differential equation for $h$ can be written through the sign-function

$$
\operatorname{sign}(x)= \begin{cases}1, & \text { if } x>0 \\ 0, & \text { if } x=0 \\ -1, & \text { if } x<0\end{cases}
$$

as

$$
\begin{equation*}
\dot{h}=1-2 \operatorname{sign}(h-\bar{h}), \tag{2.4}
\end{equation*}
$$

whose right-hand-side is discontinuous at $h=\bar{h}$. This is what we paid for the phase portrait of (2.4) to look like Fig. 2.1 (Right), where each trajectory reaches $\bar{h}$ in finite time. A complication that comes with discontinuity in (2.4) (price for discontinuity) is that given $h_{0}<0$ the solution

$$
h(t)= \begin{cases}h_{0}+t, & \text { for all } 0 \leq t<\bar{h}-h_{0},  \tag{2.5}\\ \bar{h}, & \text { for all } t \geq \bar{h}-h_{0}\end{cases}
$$

(which we get by drawing the phase space of (2.4) in Fig. 2.1(Right)) is not differentiable at $t=\bar{h}-h_{0}$ and doesn't satisfy (2.4) for $t>\bar{h}-h_{0}$. Indeed, we get $\dot{h}=0$ and $\operatorname{sign}(h-\bar{h})=0$ for $t>\bar{h}-h_{0}$, that brings (2.4) to the wrong equality

$$
\begin{equation*}
0=1 \tag{2.6}
\end{equation*}
$$

We will call (2.5) a Filippov solution of (2.4) because, as we just concluded, $h(t)$ doesn't satisfy (2.4 formally when $t \geq \bar{h}-h_{0}$. Next section introduces the Filippov solution concept for a general system of time-dependent differential equations with piecewise continuous right-hand-sides. One can e.g. think of a function $g(t)$ replacing 1 in (2.4) that would model a tank where the carrying capacity of the outcoming water pipe changes in time.

### 2.3 Filippov definition of solution

Definition 1 (particular case of [12, p.49]) A function $f: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is called piecewise continuous, if it is continuous everywhere except on countably many co-dimension one smooth surfaces, whose number is at most finite in any bounded subset of $\{t\} \times \mathbb{R}^{n}$ for almost any fixed $t \in \mathbb{R}$, and if all discontinuities of $f(t, x)$ are of the first kind.

A hyperplane of $\mathbb{R}^{n}$ given by all $x \in \mathbb{R}^{n}$ that satisfy $a_{1} x_{1}+\ldots+a_{n} x_{n}=b$, where $a_{1}, \ldots, a_{n}, b$ are constants, is the main example of a co-dimension one smooth surface that we will use in this course.
Define the convexification $K[f]$ of $f$ as

$$
\begin{equation*}
K[f](t, x)=\bigcap_{\delta>0} \bigcap_{\mu(J)=0} \overline{\operatorname{co}} f\left(t, B_{\delta}(x) \backslash J\right), \tag{2.7}
\end{equation*}
$$

where $B_{r}(x)$ is the ball of $\mathbb{R}^{n}$ of radius $r$ and centered at $x$, and $\overline{\operatorname{co}} B$ is the closure of the smallest convex set that contain $B$ ([12, bottom of p. 61]), also known as convex hull. Consult [13, Definition 7.43] for the definition of a set $J$ of zero measure $(\mu(J)=0)$.

The next concept uses the definition of absolute continuity of a function $x:\left[t_{0}, t_{1}\right] \mapsto \mathbb{R}^{n}$. These are continuous functions which are differentiable almost everywhere (i.e. differentiable everywhere except on a set of zero measure) with the following important property: if $x$ is absolutely continuous on $\left[t_{0}, t_{1}\right]$, then $\int_{t_{0}}^{t} x^{\prime}(s) d s$ exists (in the sense of Lebesgue) and $x(t)=$ $x\left(t_{0}\right)+\int_{t_{0}}^{t} x^{\prime}(s) d s$ for $t \in\left[t_{0}, t_{1}\right]$. The best reference to learn about absolutely continuous functions is [14, §33.2].

Definition 2 ([12, p. 85]) An absolutely continuous function $x: I \rightarrow \mathbb{R}^{n}$ is a solution of (2.3) in the sense of Filippov (or Filippov solution), if

$$
\begin{equation*}
\dot{x}(t) \in K[f](t, x(t)) \quad \text { for almost all } t \in I \tag{2.8}
\end{equation*}
$$

Here $I=\left[t_{0}, t_{1}\right]$ or $I=\left[t_{0}, \infty\right)$.
Same definition is phrased differently in [12, p. 50 (item a)], where a justification of Filippov definition is briefly discussed. A justification of Filippov definition through a regularization approach is proposed in [15]. The authors of [15] (and their followers) show that Filippov solution is what one gets if first smooths the discontinuities of the right-hand-sides by allowing for some transition dynamics on the boundary layer and then pass to the limit as the boundary layer shrinks into a hyperplane.

Proposition $1{ }^{1}$ If $x(t)$ is a piecewise differentiable function that satisfies (2.3) everywhere on $[0, \infty)$ except in isolated moments $t_{1}, t_{2}, \ldots$ and $f$ is continuous at $(t, x(t))$ for all $t \geq 0$, $t \neq t_{i}, i \in \mathbb{N}$, then $x(t)$ is a solution of (2.3) in the sense of Filippov.

Proposition 1 says that we don't need Filippov theory, if none of solutions of (2.3) pass through discontinuities (points of $\mathbb{R}^{n+1}$ where $f(t, x)$ is discontinuous) infinite number of times in finite time. If none of solutions pass through discontinuities infinite number of times in finite time, then we, in particular, cannot have solutions that slide along discontinuities.

Proposition 2 If $x(t)$ is a Filippov solution of (2.3) on an interval I then, for all $t \in I$,

1) $\dot{x}_{i}(t)$ exists and $\dot{x}_{i}(t)=f_{i}(t, x(t))$, if $f_{i}$ is continuous in a neighborhood of $(t, x(t))$,
2) $\dot{x}(t)$ (when exists) doesn't necessary equal $f\left(t, x(t)\right.$ ), if $f$ is discontinuous at $(t, x(t))^{2}$.
[^0]The proof needs only one of the following two properties of the convexification function, but we formulate them both at once for further use:

$$
\begin{align*}
K[f](t, x) & \subset\left(\begin{array}{l}
K\left[f_{1}\right](t, x) \\
\cdots \\
K\left[f_{n}\right](t, x)
\end{array}\right), \quad \text { for all } t \in \mathbb{R}, x \in \mathbb{R}^{n},  \tag{2.9a}\\
K[f+g](t, x) & \subset K[f](t, x)+K[g](t, x), \quad \text { for all } t \in \mathbb{R}, x \in \mathbb{R}^{n},{ }^{3} L^{4} \tag{2.9b}
\end{align*}
$$

Proof of Proposition 2, 1) Let $f_{i}$ be continuous at a neighborhood $U$ of $\left(t_{*}, x\left(t_{*}\right)\right)$ for some $i$. Then $K\left[f_{i}\right](t, x)=\left\{f_{i}(t, x)\right\}$ for all $(t, x) \in U$ and by (2.9a) one gets

$$
\dot{x}_{i}(t)=f_{i}(t, x(t)) \quad \text { for almost all } t \text { such that }(t, x(t)) \in U .
$$

Therefore, $x_{i}$ is differentiable in those $t$ for which $(t, x(t)) \in U$ and $\dot{x}_{i}(t)=f_{i}(t, x(t))$ in all such $t$.
2) The counter-example that justifies the second statement was given in $\S(2.2$, see $(2.6)$. Alternatively, one can consult the classical time-dependent counter-example in [16, discussion of (1.1) at p. 2] (dry friction oscillator).

Theorem $1{ }^{5}$ If a piecewise continuous $f: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ satisfies

$$
\begin{equation*}
\|f(t, x)\| \leq c(t)(1+\|x\|) \quad \text { for all } t \in \mathbb{R}, x \in \mathbb{R}^{n} \tag{2.10}
\end{equation*}
$$

where $c$ is a continuous function, then given any $x_{0} \in \mathbb{R}^{n}$, the initial value problem $x(0)=x_{0}$ for (2.3) has a solution $x$ on $[0, \infty)$ in the sense of Filippov.

### 2.4 Finite-time stabilization by sign-functions of large amplitudes

We have seen in section 2.1 that a $C^{1}$ smooth $g$ cannot solve the finite-time stabilization problem. Then section 2.2 provided an elementary example where finite-time stabilization of the water tank is due to the presence of the sign-function in the respective mathematical model. In this section we show that a finite-time stabilization can be always achieved via a vector sign-function $g$. Indeed, consider

$$
\dot{x}=\phi(t, x)-\left(\begin{array}{l}
b_{1} \operatorname{sign}\left(x_{1}-\bar{x}_{1}\right)  \tag{2.11}\\
\cdots \\
b_{n} \operatorname{sign}\left(x_{n}-\bar{x}_{n}\right)
\end{array}\right)=: f(t, x),
$$

[^1]where $\phi \in C^{1}\left(\mathbb{R} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right)$ and $\bar{x}, b \in \mathbb{R}^{n}$. Our goal is to prove the following statement.
Proposition 3 Assume that
$$
\left|\phi_{i}(t, x)\right| \leq M_{i}, \quad t \in \mathbb{R}, x \in \mathbb{R}^{n}, \quad i=1, \ldots, n,
$$
for some $M_{i} \in\left[0, b_{i}\right), i=1, \ldots, n$. Then, given any bounded $U \subset \mathbb{R}^{n}$ and
$$
\bar{t}=\max _{i=1, \ldots, n} \sup _{u \in U} \frac{\left|u_{i}-\bar{x}_{i}\right|}{b_{i}-M_{i}},
$$
one has
$$
x(t)=\bar{x}, \quad t \in[\bar{t}, \infty),
$$
for any Filippov solution $x$ of (2.11) with the initial condition $x(0) \in U$.
We don't need to compute the convexification $K[f]$ of $f$ (which denotes the right-hand-side of (2.11) in order to prove the proposition. The proof will only rely on the fact that the Filippov solution $x$ of $(2.11)$ satisfies (2.11) at those $t$, where $\left(x_{1}(t)-\bar{x}_{1}\right) \cdot \ldots \cdot\left(x_{1}(t)-\bar{x}_{1}\right) \neq 0$ (see Proposition 22). However, we will construct $K[\phi]$ to practice (once) with the algebraic side of the Filippov definition.

### 2.4.1 Convexification of $f$ given by (2.11)

### 2.4.1.1 The formula for $K[\operatorname{sign}](s)$

Since sign-function acts from $\mathbb{R}$, the ball $B_{\delta}(s)$ is now the interval $(s-\delta, s+\delta)$. Fix some $s<0$ and consider the values of $\delta>0$ sufficiently small so that $0 \notin(s-\delta, s+\delta)$. Since $\operatorname{sign}(\tau)=-1$ for any $\tau \in(s-\delta, s+\delta)$, then the image of sign on $(s-\delta, s+\delta)$ consists of just one element -1 , i.e.

$$
\operatorname{sign}((s-\delta, s+\delta))=\{-1\}
$$

Here $\{-1\}$ is a subset of $\mathbb{R}$ that consists of just one element $-1^{6}$. No matter which set $J$ of zero measure one excludes from $(s-\delta, s+\delta)$, the sign-function will map $(s-\delta, s+\delta) \backslash J$ into -1 , i.e.

$$
\operatorname{sign}((s-\delta, s+\delta) \backslash J)=\{-1\} \quad \text { for any } \delta>0 \text { sufficiently small and any } J \text { of measure } 0 .
$$

By taking the intersection of the latter equality over all $\delta>0$ and $\mu(J)=0$ one, therefore, gets

$$
\bigcap_{\delta>0} \bigcap_{\mu(J)=0} \overline{\operatorname{co}} \operatorname{sign}((s-\delta, s+\delta) \backslash J)=\{-1\},
$$

where we also used $\overline{\mathrm{Co}}\{-1\}=\{-1\}$. Analogous arguments apply when $s>0$.

[^2]Consider now $s=0$. We have

$$
\operatorname{sign}((-\delta, \delta))=\{-1\} \cup\{1\}=\{-1,1\} \quad \Longrightarrow \quad \operatorname{sign}((-\delta, \delta) \backslash J)=\{-1,1\}
$$

for any $\delta>0$ and any $J \subset \mathbb{R}$ of zero measure. Since the minimal convex set that contains -1 and 1 is the interval $[-1,1]$, we have $\overline{c o}\{-1,1\}=[-1,1]$. Consequently,

$$
\bigcap_{\delta>0} \bigcap_{\mu(J)=0} \overline{\operatorname{co}} \operatorname{sign}((-\delta, \delta) \backslash J)=[-1,1] .
$$

In summary,

$$
K[\operatorname{sign}](s)=\operatorname{Sign}(s), \quad \text { where } \quad \operatorname{Sign}(s)= \begin{cases}1, & s>0  \tag{2.12}\\ {[-1,1],} & s=0 \\ -1, & s<0\end{cases}
$$

### 2.4.1.2 Convexification of $f$

Since

1) $K[\phi+\psi](t, x)=\{\phi(t, x)\}+K[\psi](t, x)$, for any continuous $\phi: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ and any piecewise continuous $\psi: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$,
2) $K[h](t, x)=\left(\begin{array}{l}K\left[h_{1}\right]\left(t, x_{1}\right) \\ \ldots \\ K\left[h_{n}\right]\left(t, h_{n}\right)\end{array}\right)$, for piecewise continuous $h(t, x)=\left(\begin{array}{l}h_{1}\left(t, x_{1}\right) \\ \ldots \\ h_{n}\left(t, x_{n}\right)\end{array}\right)$,
then ${ }^{7}$

$$
K[f](t, x)=\phi(t, x)-\left(\begin{array}{l}
b_{1} \operatorname{Sign}\left(x_{1}-\bar{x}_{1}\right) \\
\ldots \\
b_{n} \operatorname{Sign}\left(x_{n}-\bar{x}_{n}\right)
\end{array}\right)
$$

### 2.4.1.3 Filippov solution of (2.11) revised

To conclude this subsection, a Filippov solution of (2.11) on $\left[t_{0}, \infty\right)$ with the initial condition $x\left(t_{0}\right)=x_{0}$ is an absolutely continuous function $x$ that satisfies

$$
\dot{x}(t) \in \phi(t, x(t))-\left(\begin{array}{l}
b_{1} \operatorname{Sign}\left(x_{1}(t)-\bar{x}_{1}\right)  \tag{2.13}\\
\ldots \\
b_{n} \operatorname{Sign}\left(x_{n}(t)-\bar{x}_{n}\right)
\end{array}\right), \quad \text { for almost all } t \in\left[t_{0}, \infty\right),
$$

and $x(0)=x_{0}$. Here Sign is the set-valued sign-function given by 2.12 .

[^3]
### 2.4.2 Proof of Proposition 3

Fix a bounded $U \subset \mathbb{R}^{n}$ and consider a Filippov solution $x$ of 2.11 with the initial condition $x(0) \in U$. Let $T_{i}=\frac{\left|x_{i}(0)-\bar{x}_{i}\right|}{b_{i}-M_{i}}$.
Step 1. Here we show that there exist $\bar{t}_{i} \in\left[0, T_{i}\right]$ such that $x_{i}\left(\bar{t}_{i}\right)=\bar{x}_{i}, i=1, \ldots, n$. Assume the contrary, i.e. assume that there exists $i$ such that

$$
\begin{equation*}
x_{i}(t) \neq \bar{x}_{i}, \quad \text { for all } t \in\left[0, T_{i}\right] . \tag{2.14}
\end{equation*}
$$

Therefore, by Proposition 2, the derivative $\dot{x}_{i}$ exists on $\left[0, T_{i}\right]$ and

$$
\dot{x}_{i}(t)=\left\{\begin{array}{ll}
\phi_{i}(t, x(t))-b_{i}, & \text { if } x_{i}(0)>\bar{x}_{i}, \\
\phi_{i}(t, x(t))+b_{i}, & \text { if } x_{i}(0)<\bar{x}_{i},
\end{array} \quad \text { for all } t \in\left[0, T_{i}\right] .\right.
$$

$\underline{\text { Case } x_{i}(0)>\bar{x}_{i}}$. classroom proof) Let $\overline{\bar{x}}_{i}=\min _{t \in\left[0, T_{i}\right]} x_{i}(t)$. Since $x_{i}$ is continuous on $\left[0, T_{i}\right]$,


Figure 2.2: The dashed region is where the solution $x$ can exist based on 2.15 and 2.16. This rules out the existence of the solution on the bold interval.
then $\overline{\bar{x}}_{i}=x_{i}(\tau)$ for some $\tau \in\left[0, T_{i}\right]$, which gives

$$
\begin{equation*}
x_{i}(t) \geq \overline{\bar{x}}_{i}>\bar{x}_{i}, \quad \text { for all } t \in\left[0, T_{i}\right] . \tag{2.15}
\end{equation*}
$$

On the other hand, by the Mean Value Theorem [18, p. 285], there exists $c \in\left[0, T_{i}\right]$ such that

$$
\begin{equation*}
x_{i}(t)=x_{i}(0)+x_{i}(c) t \leq x_{i}(0)+\left(M_{i}-b_{i}\right) t, \quad \text { for all } t \in\left[0, T_{i}\right] . \tag{2.16}
\end{equation*}
$$

Combining the two estimates above, one gets

$$
\bar{x}_{i}<\overline{\bar{x}}_{i} \leq x_{i}(0)+\left(M_{i}-b_{i}\right) t, \quad \text { for all } t \in\left[0, T_{i}\right],
$$

see Fig. 2.2. When $t=T_{i}$ this inequality reduces to

$$
\bar{x}_{i}<\overline{\bar{x}}_{i} \leq \bar{x}_{i},
$$

which can never hold.

Case $x_{i}(0)<\bar{x}_{i}$. (shorter, but less geometric proof) By the Mean Value Theorem there exists $\overline{c \in\left[0, T_{i}\right] \text { such that }}$

$$
\begin{aligned}
x_{i}\left(T_{i}\right) & =x_{i}(0)+\dot{x}_{i}(c) T_{i}= \\
& =x_{i}(0)+\left(\phi_{i}(c, x(c))+b_{i}\right) \frac{\left|x_{i}(0)-\bar{x}_{i}\right|}{b_{i}-M_{i}} \geq \\
& \geq x_{i}(0)+\left(-M_{i}+b_{i}\right) \frac{-\left(x_{i}(0)-\bar{x}_{i}\right)}{b_{i}-M_{i}}= \\
& =x_{i}(0)-\left(x_{i}(0)-\bar{x}_{i}\right)= \\
& =\bar{x}_{i},
\end{aligned}
$$

i.e. $x_{i}(0)<\bar{x}_{i} \leq x_{i}\left(T_{i}\right)$, and the Intermediate Value Theorem gives $\tau \in\left[0, T_{i}\right]$ such that $x_{i}(\tau)=\bar{x}_{i}$ again. This case leads to a contradiction with (2.14) too. The proof of Step 1 is complete.

Step 2. Let us prove that

$$
x_{i}(t)=\bar{x}_{i} \quad \text { for all } t>\bar{t}_{i} .
$$

Assume the contrary, i.e. assume that there exists $i$ and $s_{i}>t_{i}$ such that $x_{i}\left(s_{i}\right) \neq \bar{x}_{i}$. Case $x_{i}\left(s_{i}\right)>\bar{x}_{i}$ : Since $x_{i}$ is continuous, the number

$$
\tau_{i}=\max \left\{t \in\left[\bar{t}_{i}, s_{i}\right]: x_{i}(t)=\bar{x}_{i}\right\}
$$

exists. By proposition 2, $x_{i}$ is differentiable on $\left(\tau_{i}, s_{i}\right]$ and

$$
\begin{equation*}
\dot{x}_{i}(t)=\phi_{i}(t, x(t))-b_{i} \leq M-b_{i}<0 \quad \text { for all } t \in\left(\tau_{i}, s_{i}\right] . \tag{2.17}
\end{equation*}
$$

On the other hand, the Mean Value Theorem ensures the existence of $c \in\left(\tau_{i}, s_{i}\right)$ such that

$$
\dot{x}_{i}(c)=\frac{x_{i}\left(s_{i}\right)-x_{i}\left(\tau_{i}\right)}{s_{i}-\tau_{i}}=\frac{x_{i}\left(s_{i}\right)-\bar{x}_{i}}{s_{i}-\tau_{i}}>0,
$$

which contradicts 2.17 ). This contradiction proofs that $x_{i}\left(s_{i}\right)>\bar{x}_{i}$ cannot happen.
Case $x_{i}\left(s_{i}\right)<\bar{x}_{i}$ : Exercise.
To complete the proof of proposition 3 it remains to observe that $T_{i} \leq \bar{t}$ as defined in the statement of the proposition.

### 2.4.3 Graphic illustration of solutions of 2.11 .

The graph of the solution $x$ is illustrated at Fig. 2.3. The part of the solution $x$ when it slides along a hyperplane is called a sliding mode.


Figure 2.3: The solution $x$ reaches a hyperplane $H_{i}=\left\{x \in \mathbb{R}^{n}: x_{i}=0\right\}$ at some $t_{i}$ (i.e. $x\left(t_{i}\right)=A \in H_{j}$ ) and then slides along $H_{i}$ forever. While sliding along $H_{i}$ the trajectory reaches another hyperplane $H_{j}=$ $\left\{x \in \mathbb{R}^{n}: x_{j}=0\right\}$ at some $t_{j}$ (i.e. $x\left(t_{j}\right)=B \in H_{j}$ ) and then slide along $H_{i} \cap H_{j}$ forever. After the solution $x$ reaches all the $n$ hyperplanes $H_{1}, \ldots, H_{n}$, one has $x(t)=\bar{x}$ forever.

### 2.5 Lyapunov theory of finite-time stability

In this section we study finite-time stability of the origin in system (2.3) under the assumption that $f$ is continuous everywhere outside the hyperplanes

$$
\begin{equation*}
S=\bigcup_{i=1}^{n}\left\{x \in \mathbb{R}^{n}: x_{i}=0\right\} \tag{2.18}
\end{equation*}
$$

### 2.5.1 Comparison lemma

Consider an ordinary differential equation along with the respective differential inequality

$$
\begin{align*}
\dot{x}(t) & =g(t, x(t))  \tag{2.19}\\
\dot{v}(t) & \leq g(t, v(t)) \tag{2.20}
\end{align*}
$$

The next result is also known as a differential inequalities approach, see [24, §1.4] and [25, Ch. III, §4] (the word "comparison" is borrowed from [26]). We extend the standard result by allowing absolutely continuous solutions in the inequality.

Lemma 1 Consider $g \in C^{0}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Let $x \in C^{1}([0, T], \mathbb{R})$ be a solution of (2.19) on $[0, T]$. Assume that $x(t)$ is the only solution of (2.19) that originates from $x(0)$. Let an absolutely continuous $v:[0, T] \rightarrow \mathbb{R}$ satisfies (2.20) for almost all $t \in[0, T]$. Then

$$
v(0) \leq x(0) \Longrightarrow v(t) \leq x(t) \quad \text { for all } t \in[0, T]
$$

Proof. $\sqrt[8]{8}$ Let $x_{\varepsilon}$ be a solution of the initial value problem

$$
\begin{align*}
\dot{x}_{\varepsilon}(t) & =g(t, x(t))+\varepsilon  \tag{2.21}\\
x_{\varepsilon}(0) & =x(0)
\end{align*}
$$

[^4]defined on $[0, T]$. The theorem on continuous dependence of solutions on the parameters ensures the existence of such a solution on $[0, T]$ (see [28, Theorem 4.1]) ${ }^{9}$ and the convergence
$$
x_{\varepsilon}(t) \rightarrow x(t) \quad \text { as } \quad \varepsilon \rightarrow 0 \quad \text { uniformly on }[0, T] .
$$

We will, therefore, prove that

$$
\begin{equation*}
v(t) \leq x_{\varepsilon}(t) \quad \text { on }[0, T] \quad \text { for any } \varepsilon>0 \tag{2.22}
\end{equation*}
$$

and then pass to the limit as $\varepsilon \rightarrow 0$. Assume that 2.22 doesn't hold. This means that there exists $\varepsilon>0$ and $\overline{\bar{t}} \in(0, T]$ such that

$$
\begin{equation*}
v(\overline{\bar{t}})>x_{\varepsilon}(\overline{\bar{t}}) \tag{2.23}
\end{equation*}
$$

Similar to the maneuver we used in the proof of Proposition 3, consider

$$
\tau=\max \left\{t \in[0, \bar{t}]: v(t) \leq x_{\varepsilon}(t)\right\}
$$

Inequality (2.23) implies that

$$
\begin{equation*}
v(\tau+\Delta t)>x_{\varepsilon}(\tau+\Delta t), \quad \text { for all } \Delta t>0 \text { sufficiently small } \tag{2.24}
\end{equation*}
$$

and we are going to show that $(2.24)$ can never happen, see Fig. 2.4. Indeed, by the Mean


Figure 2.4: Graph of $x_{\varepsilon}$ (solid curve) and a part of the graph of $v$ (dotted curve).
Value Theorem, we have

$$
x_{\varepsilon}(\tau+\Delta t)=x_{\varepsilon}(\tau)+\dot{x}_{\varepsilon}\left(c_{1}\right) \Delta t \xlongequal{\text { by }[2.21]} x_{\varepsilon}(\tau)+\left(g\left(c_{1}, x_{\varepsilon}\left(c_{1}\right)\right)+\varepsilon\right) \Delta t
$$

where $c_{1} \in[\tau, \tau+\Delta t]$. Since $v \notin C^{1}((\tau, \tau+\Delta t), \mathbb{R})$, a different approach is required to expand $v(\tau+\Delta t)$. The Mean Value Theorem for Integrals helps instead [18, p. 452] (and we also use (2.20) when writing the inequality):

$$
v(\tau+\Delta t) \xlongequal{\text { see p. 田 }} v(\tau)+\int_{\tau}^{\tau+\Delta t} \dot{v}(t) d t \leq v(\tau)+\int_{\tau}^{\tau+\Delta t} g(t, v(t)) d t=v(\tau)+g\left(c_{2}, v\left(c_{2}\right)\right) \Delta t,
$$

[^5]where $c_{2} \in[\tau, \tau+\Delta t]$. Substituting the expansions for $x_{\varepsilon}(\tau+\Delta t)$ and $v(\tau+\Delta t)$ into (2.24), one gets
$$
v(\tau)+g\left(c_{2}, v\left(c_{2}\right)\right) \Delta t>x_{\varepsilon}(\tau)+\left(g\left(c_{1}, x_{\varepsilon}\left(c_{1}\right)\right)+\varepsilon\right) \Delta t
$$

By the definition of $\tau, x_{\varepsilon}(\tau)=v(\tau)$ and we get

$$
g\left(c_{2}, v\left(c_{2}\right)\right)>g\left(c_{1}, x_{\varepsilon}\left(c_{1}\right)\right)+\varepsilon
$$

where $c_{1}, c_{2} \in[\tau, \tau+\Delta t]$. By passing to the limit as $\Delta t \rightarrow 0$, one finally gets

$$
0 \geq \varepsilon
$$

which is a contradiction when $\varepsilon$ is chosen positive. Inequality $(2.22)$, therefore, holds for all $\varepsilon>0$ and we get the required result by passing to the limit in (2.22) as $\varepsilon>0$ approaches 0 .

Note: The proof of this lemma demonstrates how the Mean Value Theorem for Integrals replaces the Mean Value Theorem for continuously differentiable functions when the function under consideration is only absolutely continuous.
2.5.2 Finite-time reachability of 0 for any absolutely continuous solution of $\dot{v} \leq-k v^{\alpha}$, with $k>0, \alpha \in[0,1)$, and $v(0)>0$

Since, for any positive initial condition $x(0)$, the solution of the differential equation

$$
\begin{equation*}
\dot{x}=-k x^{\alpha} \tag{2.25}
\end{equation*}
$$

is given by the formula ${ }^{10}$

$$
x(t)=\left((-\alpha+1)\left(-k t+\frac{x(0)^{-\alpha+1}}{-\alpha+1}\right)\right)^{\frac{1}{-\alpha+1}}
$$

we have the following corollary from lemma 1
Corollary $1{ }^{11}$ If an absolutely continuous function $v:[0, \infty) \rightarrow \mathbb{R}$ satisfies

$$
\dot{v} \leq-k v^{\alpha}, \quad \text { for almost all } t \geq 0
$$

where $k>0$ and $\alpha \in[0,1)$, then there exists a positive $\bar{t} \leq \frac{v(0)^{-\alpha+1}}{(-\alpha+1) k}$ such that $v(\bar{t})=0$.

[^6]
### 2.5.3 Piecewise continuous systems with only trivial sliding solutions

In what follows, $S$ is the union of hyperplanes (2.18), where $f$ may have discontinuities.
Definition 3 (hyperplane crossing in dimension n) Let $f: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be continuous outside $(\mathbb{R} \backslash J) \times S$, where $J$ is a countable set. Consider $(t, x) \in \mathbb{R} \times S$. Let $I_{0} \subset\{1,2, \ldots, n\}$ be the set of indexes of the vanishing components of $x$, i.e.

$$
x_{i}=0 \text { for all } i \in I_{0}, \quad x_{i} \neq 0 \text { for all } i \notin I_{0} .
$$

For $i \in I_{0}$, we say that the vector field $f$ crosses the hyperplane $x_{i}=0$ at the point $(t, x)$, if there exists $|a|=1$ such that

$$
\operatorname{sign}\left(\begin{array}{l}
\lim _{\substack{b_{0}(\tau-t) \rightarrow 0^{+} \\
b_{j} \xi_{j} \rightarrow 0^{+}, j \in I_{0} \\
\xi_{j}=x_{j}, j \notin I_{0}}} f_{i}(\tau, \xi)
\end{array}\right)=a
$$

regardless of the choice of $\left|b_{j}\right|=1, j \in\{0\} \cup I_{0}$. In this case we also say that $(t, x)$ is a point of hyperplane crossing.

Definition 5 and definition 6 below are the versions of Definition 3 in dimension 2 and 3 respectively (where the limiting relation simplifies significantly).

Definition 4 (crossing in dimension $n$ ) In the settings of Definition 3, if the vector-field $f$ crosses each of the hyperplanes $x_{i}=0, i \in I_{0}$, at the point $(t, x)$, then we say that $(t, x)$ is a point of crossing.

Definition 4 has a simple geometric meaning: if $(t, \xi) \in S$ is a point of crossing for (2.3) then any Filippov solution $x$ of (2.3) that happen to reach $\xi$ at time $t$, leaves $S$ immediately.

Proposition $4{ }^{12}$ If $x$ is a Filippov solution of (2.3) and the vector field $f$ crosses the hyperplane $x_{i}=0$ at $\left(t_{*}, x\left(t_{*}\right)\right) \in \mathbb{R} \times S$, then there exists $\delta>0$ such that

$$
x_{i}(t) \neq 0, \quad \text { for all } t \in\left[t_{*}-\delta, t_{*}\right) \cup\left(t_{*}, t_{*}+\delta\right] .
$$

If the right-hand-side in (2.3) doesn't depend on $t$ then $\left(t_{*}, x_{*}\right)$ is a point of crossing if and only if $\left(t, x_{*}\right)$ is a point of crossing for any $t$. We say that $x_{*}$ is a point of crossing in such a case.

[^7]Theorem $2{ }^{13}$ Let $f: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be continuous on $(\mathbb{R} \backslash J) \times\left(\mathbb{R}^{n} \backslash S\right)$, where $J$ is a countable set. Let every elements of $\mathbb{R} \times((S \backslash\{0\}) \cap U)$ be either a point of crossing or a point of continuity for (2.3), where $U \subset \mathbb{R}^{n}$ is some neighborhood of the origin. Assume that there exists a function $V: \mathbb{R}^{n} \rightarrow R$ such that

1) $V \in C^{0}\left(\mathbb{R}^{n}, \mathbb{R}\right) \cap C^{1}\left(\mathbb{R}^{n} \backslash S, \mathbb{R}\right)$,
2) $V(x)>0$ for all $x \neq 0$,
3) $V(0)=0$,
4) there exist $k>0$ and $\lambda \in[0,1)$, such that for any Filippov solution $x$ of (2.3) with $x(0) \in U$ the function $v(t)=V(x(t))$ verifies the inequality

$$
\begin{equation*}
\dot{v}(t)+k(v(t))^{\alpha} \leq 0 \quad \text { for all } t \geq 0 \text { such that } x(t) \notin S . \tag{2.26}
\end{equation*}
$$

Then the origin is a finite-time stable equilibrium for (2.3). ${ }^{14}$
Proof. Step 1. Let us show that $v(t)$ reaches 0 on $[0, \bar{T}]$, where $\bar{T}=\frac{v(0)^{-\alpha+1}}{(-\alpha+1) k}$. Assume the contrary. Since $V(x)=0$ if and only if $x=0$, we have that $x(t) \neq 0$ for $t \in[0, \bar{T}]$. This means that $x(t) \in S$ if and only if $x(t)$ is a point of crossing. Therefore, by Proposition 4 , the set $\{t \in[0, \bar{T}]: x(t) \in S\}$ is countable (as a set of isolated elements). This means that inequality (2.26) holds for almost all $t \in[0, \bar{T}]$ (it holds in points of continuity of (2.3) by continuity) and we get a contradiction with the conclusion of Corollary 1. Thus, we proved that there exists $\bar{t} \in[0, \bar{T}]$ such that $v(\bar{t})=0$.

Step 2. Here we show that $v(t)=0$ for all $t \geq \bar{t}$. Assume the contrary, i.e. assume that

$$
\begin{equation*}
v(\overline{\bar{t}})>0 \quad \text { for some } \overline{\bar{t}}>\bar{t} \tag{2.27}
\end{equation*}
$$

Let $\tau=\max \{t \in[\bar{t}, \bar{t}]: v(t)=0\}$. Same arguments as in Step 1 apply to conclude that the inequality 2.26 holds for almost all $t \in[\bar{t}, \bar{t}]$. By the Mean Value Theorem for Integrals there exists $c \in(\tau, \overline{\bar{t}})$ such that

$$
v(\overline{\bar{t}})-v(\tau)=\int_{\tau}^{\bar{t}} \dot{v}(t) d t \leq \int_{\tau}^{\bar{t}}\left(-k(v(t))^{\alpha}\right) d t=-k(v(c))^{\alpha}<0
$$

Since, by definition, $v(\tau)=0$, the latter inequality contradicts 2.27). The proof of the theorem is complete.

[^8]By analyzing the proof of Theorem 2, we see that the only solution of (2.3) that develops along $S$ for positive time intervals is the solution that sticks to the origin forever. The crossing assumption rules out all other possibilities. Thus the title of the section 2.5.3.

Example 1 Prove finite-time stability of the origin in the following system

$$
\begin{array}{ll}
\dot{x}=-\operatorname{sign} x+2 \operatorname{sign} y & =: f_{1}(t, x, y)  \tag{2.28}\\
\dot{y}=-2 \operatorname{sign} x-\operatorname{sign} y & =: f_{2}(t, x, y) .
\end{array}
$$

Let us first reformulate Definition 3 for the case of dimension 2.
Definition 5 (crossing in dimension 2) Consider a time-independent $f(t, x, y)$ which is continuous at any $(x, y) \notin S=(\mathbb{R} \times\{0\}) \cup(\{0\} \times \mathbb{R})$. We say that the vector field $f(t, \cdot)$ crosses the line $y=0$ at the point $(x, 0)$, if

$$
\operatorname{sign} \lim _{y \rightarrow 0^{-}} f_{2}(t, x, y)=\operatorname{sign} \lim _{y \rightarrow 0^{+}} f_{2}(t, x, y) \neq 0
$$

The crossing condition for $(0, y)$ is defined by analogy.
Solution of Example 1. The right-hand-side is continuous outside $S=(\mathbb{R} \times\{0\}) \cup(\{0\} \times \mathbb{R})$. Furthermore,

$$
x=0, y \neq 0 \quad \Longrightarrow \quad \begin{aligned}
& \operatorname{sign} \lim _{x \rightarrow 0^{-}} f_{1}(x, y)=\operatorname{sign}(1+2 \operatorname{sign} y)=\operatorname{sign} y, \\
& \operatorname{sign} \lim _{x \rightarrow 0^{+}} f_{1}(x, y)=\operatorname{sign}(-1+2 \operatorname{sign} y)=\operatorname{sign} y,
\end{aligned}
$$

and

$$
x \neq 0, y=0 \quad \Longrightarrow \quad \begin{aligned}
& \operatorname{sign} \lim _{y \rightarrow 0^{-}} f_{2}(x, y)=\operatorname{sign}(-2 \operatorname{sign} x+1)=-\operatorname{sign} x, \\
& \operatorname{sign} \lim _{y \rightarrow 0^{+}} f_{2}(x, y)=\operatorname{sign}(-2 \operatorname{sign} x-1)=-\operatorname{sign} x .
\end{aligned}
$$

Therefore, all points of $S \backslash\{0\}$ are points of crossing for (2.28). Consider

$$
V(x, y)=|x|+|y|
$$

and define $v:[0, \infty) \rightarrow \mathbb{R}$ as $v(t)=V(x(t), y(t))$, where $(x, y)$ is a solution of (2.28). For those $t \geq 0$ where $x(t) \neq 0$ and $y(t) \neq 0$ one has

$$
\begin{aligned}
\dot{v}(t)= & \operatorname{sign}(x(t)) \dot{x}(t)+\operatorname{sign}(y(t)) \dot{y}(t)=\operatorname{sign}(x(t)) \cdot(-\operatorname{sign}(x(t))+2 \operatorname{sign}(y(t)))+ \\
& +\operatorname{sign}(y(t)) \cdot(-2 \operatorname{sign}(x(t))-\operatorname{sign}(y(t)))=-(\operatorname{sign}(x(t)))^{2}-(\operatorname{sign}(y(t)))^{2}=-2 .
\end{aligned}
$$

Therefore, the finite-time stability of the origin in (2.28) follows from Theorem 2, whose assumptions hold with $k=2$ and $\alpha=0$. Fig. 2.5 illustrates the vector field 2.28) in different quadrants and shows a sample trajectory of 2.28 .

An important application of Theorem 2 is to a one-link manipulator, see [32]. Further results on finite-time stability of the one-link manipulator are obtained in 33].

The following corollary from Theorem 2 holds when $f \in C^{0}\left(\mathbb{R} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right)$.


Figure 2.5: Equations of 2.28 for different quadrants and a part of the solution of 2.28 with the initial condition $(x(0), y(0))=(9,0)$.

Theorem 3 Let $f \in C^{0}\left(\mathbb{R} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right)$. Assume that there exists a function $V: \mathbb{R}^{n} \rightarrow \mathbb{R}$ such that

1) $V \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}\right)$,
2) $V(x)>0$ for all $x \neq 0$,
3) $V(0)=0$,
4) there exist $k>0$ and $\lambda \in[0,1)$, such that for any solution $x$ of (2.3) with $x(0) \in U$ the function $v(t)=V(x(t))$ verifies the inequality

$$
\begin{equation*}
\dot{v}(t)+k(v(t))^{\alpha} \leq 0, \quad \text { for all } t \geq 0 \tag{2.29}
\end{equation*}
$$

Then the origin is a terminal attractor (finite-time stable equilibrium) for (2.3). If

$$
\dot{v}(t)-k(v(t))^{\alpha} \leq 0, \quad \text { for all } t \leq 0
$$

then the origin is a terminal repeller for (2.3).

Theorem 3 allows to establish local finite-time stability of the following normal form continuous system:

$$
\begin{align*}
& \dot{x}=a_{1} x^{\alpha_{1}}(1+f(x, y))+a_{2} y^{\alpha_{2}}  \tag{2.30}\\
& \dot{y}=a_{3} x^{\alpha_{3}}+a_{4} y^{\alpha_{4}}(1+g(x, y)),
\end{align*}
$$

where, by definition, $x^{\alpha_{i}}=\operatorname{sign}(x)|x|^{\alpha_{i}}$ and $y^{\alpha_{i}}=\operatorname{sign}(y)|y|^{\alpha_{i}}$.
Proposition 5 Assume that $\alpha_{1}, \alpha_{4} \in(0,1), \alpha_{2}, \alpha_{3} \geq 0$, and let $f, g$ be continuous functions. The following statements take place in the domain $\left\{(x, y) \in \mathbb{R}^{2}: f(x, y)>-1, g(x, y)>-1\right\}$, if it contains the origin.

1) If $a_{1}, a_{4}<0$ and $a_{2} a_{3}<0$, then the origin is a terminal attractor of (2.30).
2) If $a_{1}, a_{4}>0$ and $a_{2} a_{3}<0$, then the origin is a terminal repeller of (2.30).
3) If $a_{2} a_{3}>0$ and $a_{1} a_{4}<0$, then any solution of (2.30) with initial condition in

$$
S=\left\{(x, y) \in \mathbb{R}^{2}: \operatorname{sign}\left(a_{4}\right)\left(\left(1+\alpha_{3}\right) a_{2}|x|^{1+\alpha_{3}}-\left(1+\alpha_{2}\right) a_{3}|y|^{1-\alpha_{2}}\right)>0\right\}
$$

approaches $\mathbb{R}^{2} \backslash S$ in finite time.
Proof. The proof is carried out by showing that the function

$$
V(x, y)=p|x(t)|^{1+\alpha_{3}}+q|y(t)|^{1+\alpha_{2}}
$$

satisfies the conditions of Theorem 3 for suitable $p, q \in \mathbb{R}, k>0$ and $\alpha \in[0,1)$.
Computing $\dot{V}$ one gets
$\dot{V}(x, y)=p\left(1+\alpha_{3}\right) x^{\alpha_{3}}\left[a_{1} x^{\alpha_{1}}(1+f(x, y))+a_{2} y^{\alpha_{2}}\right]+q\left(1+\alpha_{2}\right) y^{\alpha_{2}}\left[a_{3} x^{\alpha_{3}}+a_{4} y^{\alpha_{4}}(1+g(x, y))\right]$, which leads to

$$
\dot{V}(x, y)=p\left(1+\alpha_{3}\right) a_{1}|x|^{\alpha_{1}+\alpha_{3}}(1+f(x, y))+q\left(1+\alpha_{2}\right) a_{4}|y|^{\alpha_{2}+\alpha_{4}}(1+g(x, y))
$$

if $p, q \in \mathbb{R}$ are selected such that

$$
\begin{equation*}
p\left(1+\alpha_{3}\right) a_{2}=-q\left(1+\alpha_{2}\right) a_{3} \tag{2.31}
\end{equation*}
$$

Since

$$
V(x, y)^{\alpha}=\left(p|x|^{1+\alpha_{3}}+q|y|^{1+\alpha_{2}}\right)^{\alpha} \leq|p|^{\alpha}|x|^{\alpha\left(1+\alpha_{3}\right)}+|q|^{\alpha}|y|^{\alpha\left(1+\alpha_{2}\right)}
$$

we get

$$
\begin{align*}
\dot{V}+k V^{\alpha} \leq & -k\left(-\frac{1}{k} p a_{1}\left(1+\alpha_{3}\right)|x|^{\alpha_{3}+\alpha_{1}}(1+f(x, y))-|p|^{\alpha}|x|^{\alpha\left(1+\alpha_{3}\right)}\right)- \\
& -k\left(-\frac{1}{k} q a_{4}\left(1+\alpha_{2}\right)|y|^{\alpha_{2}+\alpha_{4}}(1+g(x, y))-|q|^{\alpha}|y|^{\alpha\left(1+\alpha_{2}\right)}\right) \tag{2.32}
\end{align*}
$$

and

$$
\begin{align*}
\dot{V}-k V^{\alpha} \geq & k\left(\frac{1}{k} p a_{1}\left(1+\alpha_{3}\right)|x|^{\alpha_{3}+\alpha_{1}}(1+f(x, y))+|p|^{\alpha}|x|^{\alpha\left(1+\alpha_{3}\right)}\right)+ \\
& +k\left(\frac{1}{k} q a_{4}\left(1+\alpha_{2}\right)|y|^{\alpha_{2}+\alpha_{4}}(1+g(x, y))+|q|^{\alpha}|y|^{\alpha\left(1+\alpha_{2}\right)}\right) . \tag{2.33}
\end{align*}
$$

From now on we fix $\alpha \in[0,1)$ such that

$$
\alpha_{3}+\alpha_{1}<\alpha\left(1+\alpha_{3}\right) \quad \text { and } \quad \alpha_{2}+\alpha_{4}<\alpha\left(1+\alpha_{2}\right),
$$

which is possible because $\alpha_{1}, \alpha_{4} \in[0,1)$ by assumption.

Case 1: $a_{1}, a_{4}<0$ and $a_{2} a_{3}<0$. Fix any $p, q>0$ that satisfy (2.31). Choose $k>0$ small enough to verify

$$
-\frac{1}{k} p a_{1}\left(1+\alpha_{3}\right)(1+f(0,0))>|p|^{\alpha} \quad \text { and } \quad-\frac{1}{k} q a_{4}\left(1+\alpha_{2}\right)(1+g(0,0))>|q|^{\alpha} .
$$

Then 2.32 implies $\dot{V}+k V^{\alpha} \leq 0$.
Case 2: $a_{1}, a_{4}>0$ and $a_{2} a_{3}<0$. Following the lines of Step 1 we again fix any $p, q>0$ that satisfy (2.31). If we now choose $k>0$ small enough to verify

$$
\frac{1}{k} p a_{1}\left(1+\alpha_{3}\right)(1+f(0,0))>|p|^{\alpha} \quad \text { and } \quad \frac{1}{k} q a_{4}\left(1+\alpha_{2}\right)(1+g(0,0))>|q|^{\alpha} .
$$

then 2.33 will imply $\dot{V}-k V^{\alpha} \geq 0$.
Case 3: $a_{2} a_{3}>0$ and $a_{1} a_{4}<0$. Let $a_{1}<0$ and $a_{4}>0$. Select any $p>0$ and $q<0$ that satisfy (2.31). Then selecting $k>0$ as in Step 1 , we conclude $\dot{V}+k V^{\alpha} \leq 0$. However, in contrast to Step 1, the function $V$ is no longer positive definite. By following the lines of the proof of Theorem 3 we see that the convergence of $v(t)=V(x(t), y(t)$ to zero will take place only if $V(x(t), y(t))>0$, which is equivalent to saying $(x(t), y(t)) \in S$.

The case $a_{1}<0$ and $a_{4}>0$ can be considered by analogy. The coefficient $\operatorname{sign}\left(a_{4}\right)$ appears (see the formula for $S$ ) because one now considers $p<0$ and $q>0$ to execute the proof of Step 1.

The proof of the proposition is complete.
Another powerful application of Theorem 3 is the design of a so-called super twisting control $u(x, y)$ which drives the system

$$
\begin{align*}
& \dot{x}=y \\
& \dot{y}=u(x, y) \tag{2.34}
\end{align*}
$$

to the origin in finite time. Indeed, the following lemma is due to Bhat-Bernstein [29], which found further applications in designing control for bipedal robot locomotion [34].

Proposition 6 (see [29, Proposition 1]) The origin of (2.34) is globally finite-time stable under the control law

$$
u(x, y)=-y^{1 / 2}-\phi(x, y)^{1 / 3}, \quad \phi(x, y)=x+(3 / 2) y^{3 / 2}
$$

Here, by definition, $h^{\beta}=\operatorname{sign}(h)|h|^{\beta}$.
Proof. ([29]) Consider the Lyapunov function

$$
V(x, y)=\frac{3}{5}|\phi(x, y)|^{5 / 3}+\frac{1}{2} y \phi(x, y)+\frac{4}{5}|y|^{5 / 2}
$$

and put $v(t)=V(x(t), y(t))$, where $(x(t), y(t))$ is any solution of 2.34. Then

$$
\dot{v}(t)=V_{x}^{\prime}(x(t), y(t)) \dot{x}(t)+V_{y}^{\prime}(x(t), y(t)) \dot{y}(t)=V_{x}^{\prime}(x(t), y(t)) y(t)+V_{y}^{\prime}(x(t), y(t)) u(x(t), y(t)) .
$$

The latter expression will be denoted by $\dot{V}(x(t), y(t))$. According to [29],
$\dot{V}(x, y)=-2 y^{2}-\frac{1}{2}|\phi(x, y)|^{4 / 3}-|y|^{1 / 2}|\phi(x, y)|-\frac{1}{2} \phi(x, y) y^{1 / 2}-\frac{5}{2} \operatorname{sign}(y \phi(x, y))|y|^{3 / 2}|\phi(x, y)|^{1 / 3}$.

(a)

(b)

(c)

Figure 2.6: (a) The level curve 2.35; (b) graphs of the functions $w(s)=-2 s^{2}-\frac{1}{2}-\frac{1}{2}|s|^{1 / 2}+\frac{5}{2}|s|^{3 / 2}$ (solid curve) and $w(s)=-2-\frac{1}{2}|s|^{4 / 3}-\frac{1}{2}|s|+\frac{5}{2}|s|^{1 / 3}$ (dashed curve); (c) the set $\mathcal{V}$.

Step 1. Let us prove that $V(x, y)>0$ and $\dot{V}(x, y)<0$ on the level curve (see Fig. 2.6a)

$$
\begin{equation*}
\max \{|\phi(x, y)|,|y|\}=1 \tag{2.35}
\end{equation*}
$$

When $|\phi(x, y)|=1$ and $|y| \leq 1$, we have

$$
\begin{aligned}
& V(x, y)=\frac{3}{5}+\frac{1}{2} y \phi(x, y)+\frac{2 r}{5}|y|^{5 / 2} \geq \frac{3}{5}-\frac{1}{2}=\frac{1}{10} \\
& \dot{V}(x, y) \leq-2 y^{2}-\frac{1}{2}-|y|^{1 / 2}+\frac{1}{2}|y|^{1 / 2}+\frac{5}{2}|y|^{3 / 2}<0 \text { (see Fig. 2.6 } b \text { ). }
\end{aligned}
$$

When $|\phi(x, y)| \leq 1$ and $|y|=1$, we have
$V(x, y) \geq \frac{3}{5} \cdot 0-\frac{1}{2}+\frac{4}{5}=\frac{3}{10}$,
$\dot{V}(x, y) \leq-2-\frac{1}{2}|\phi(x, y)|^{4 / 3}-|\phi(x, y)|+\frac{1}{2} \phi(x, y) y^{1 / 2}+\frac{5}{2}|\phi(x, y)|^{1 / 3}<0$ (see Fig. 2.6 $b$ ).
Step 2. Since the functions $V(x, y)$ and $\dot{V}(x, y)$ verify the homogeneity properties

$$
\begin{align*}
V\left(k^{3 / 2} x, k y\right) & =k^{5 / 2} V(x, y) \\
\dot{V}\left(k^{3 / 2} x, k y\right) & =k^{2} \dot{V}(x, y), \quad k>0, \quad(x, y) \in \mathbb{R}^{2} \tag{2.36}
\end{align*}
$$

the estimates $V(x, y)>0$ and $\dot{V}(x, y)<0$ extend from 2.35 to the entire $\mathbb{R}^{2}$.
Step 3. Since $\mathcal{V}=\{(x, y): V(x, y)=1\}$ is a compact set (see Fig. 2.35c), the minimum $c=\min \{-\dot{V}(x, y):(x, y) \in \mathcal{V}\}>0$ exists. Considering $(x, y) \in \mathcal{V}$ and using 2.36, we have

$$
\dot{V}\left(k^{3 / 2} x, k y\right)=k^{2} \dot{V}(x, y) \geq-k^{2} c=-\left(V\left(k^{3 / 2} x, k y\right)^{2 / 5}\right)^{2} c=-c V\left(k^{3 / 2} x, k y\right)^{4 / 5}
$$

Therefore, $\dot{V}(x, y) \geq-c V(x, y)^{4 / 5}$ for all $(x, y) \in \mathbb{R}^{2}$ and global finite-time stability of the origin follows by applying Theorem 3 .

### 2.5.4 Piecewise continuous systems with nontrivial sliding solutions

Next theorem is a version of Theorem 2 for

$$
\begin{equation*}
V(x)=\left|x_{1}\right|+\ldots+\left|x_{n}\right| \tag{2.37}
\end{equation*}
$$

and $\alpha=0$ in the case where not all of the elements of $S \backslash\{0\}$ are points of crossing for (2.3). Basically, the theorem simply says that all possible derivatives $\dot{x}(t)$ have to be taken into account when examining the validity of 2.26 ). And all these possibilities are given by the convexification $K[f](t, x(t))$, see section 2.3 for definition. Theorem 4 is a simplified combination of [47, Theorem 8], [31, Theorem 2], [48, Theorem 3.1].

Theorem $4{ }^{[5}$ Let $f: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be continuous on $\mathbb{R} \times\left(\mathbb{R}^{n} \backslash S\right)$ and $0 \in U \subset R^{n}$. Assume that there exists $k>0$ such that

$$
\dot{V}=\sum_{i=1}^{n} \operatorname{sign}\left(x_{i}\right) \xi_{i} \leq-k<0 \quad \text { for all } \xi \in K[f](t, x), t \geq 0, x \in U \backslash\{0\}
$$

excluding those $(t, x) \in[0, \infty) \times S$, which are points of hyperplane crossing for (2.3). Then the origin is a finite-time stable equilibrium for (2.3). Furthermore, if $r>0$ is such that $W=\left\{x \in \mathbb{R}^{n}:\left|x_{1}\right|+\ldots+\left|x_{n}\right|<r\right\} \subset U$, then any Filippov solution of (2.3) with the initial condition in $W$ approaches the origin in finite time ${ }^{16}$

In this statement, $\dot{V}$ is not a derivative, but just a notation for the sum to shorten notations in examples. This sum is what we indeed get for the derivative of the Lyapunov function (2.37) in the proof, thus our choice for the notation. For systems (2.2) with continuous right-hand-sides, the symbol $\dot{V}$ can be defined as a function $\dot{V}(t, x)$ because all $K[f](t, x)$ are singletons (sets of single element) for continuous $f$. In this case $\dot{V}$ is called the derivative of $V$ with respect to the system (2.3), see [35, p. 557]. In particular, the conditions (2.26) and (2.29) of Theorems 2 and 3 can be shortened as

$$
\dot{V}(x)+k(V(x))^{\alpha} \leq 0,
$$

[^9]by saying that " $\dot{V}$ is the derivative of $V$ with respect to the system (2.3)". Shall we define $\dot{V}$ as a function of $(t, x)$ for discontinuous $f$, the values of $\dot{V}(t, x)$ would be the sets
$$
\dot{V}(t, x)=\bigcup_{\xi \in K[f](t, x)} \operatorname{sign}\left(x_{i}\right) \xi_{i} .
$$

The proof of the theorem is based on the following lemma, which is a corollary of a result by Clarke (also known as the Chain Rule for regular Lipschitz functions), see [36, Theorem 2.3.9-(iii)].

Lemma 2 Assume that $b: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at $t_{*}$ and $b\left(t_{*}\right)=0$. If the function $a(t)=|b(t)|$ is differentiable at $t_{*}$, then $a^{\prime}\left(t_{*}\right)=b^{\prime}\left(t_{*}\right)=0$.

Proof. If $a^{\prime}\left(t_{*}\right)$ exits, then

$$
\begin{equation*}
a^{\prime}\left(t_{*}\right)=\lim _{\Delta t \rightarrow 0^{-}} \frac{a\left(t_{*}+\Delta t\right)}{\Delta t}=\lim _{\Delta t \rightarrow 0^{+}} \frac{a\left(t_{*}+\Delta t\right)}{\Delta t} \tag{2.38}
\end{equation*}
$$

where we used that $a\left(t_{*}\right)=0$. Computing the one-sided limits one gets

$$
\lim _{\Delta t \rightarrow 0^{-}} \frac{a\left(t_{*}+\Delta t\right)}{\Delta t}=\lim _{\Delta t \rightarrow 0^{-}} \frac{\left|b\left(t_{*}+\Delta t\right)\right|}{\Delta t}=\lim _{\Delta t \rightarrow 0^{-}}\left|\frac{b\left(t_{*}+\Delta t\right)}{\Delta t}\right| \operatorname{sign} \Delta t=\left|b^{\prime}\left(t_{*}\right)\right| \cdot(-1)
$$

and, analogously,

$$
\lim _{\Delta t \rightarrow 0^{+}} \frac{a\left(t_{*}+\Delta t\right)}{\Delta t}=\left|b^{\prime}\left(t_{*}\right)\right| \cdot(+1)
$$

Substituting these values of the one-sided limits to (2.38) gives $\left|b^{\prime}\left(t_{*}\right)\right| \cdot(-1)=\left|b^{\prime}\left(t_{*}\right)\right| \cdot(+1)$, which is only possible when $b^{\prime}\left(t_{*}\right)=0$. The proof of the lemma is complete.

Proof of Theorem 4. Let $x$ be a Filippov solution of (2.3) with the initial condition $x(0) \in U$. Consider

$$
v(t)=V(x(t))=v_{1}(t)+\ldots+v_{n}(t), \quad \text { where } v_{i}(t)=\left|x_{i}(t)\right|, i=1, \ldots, n
$$

Since a composition of a Lipschitz function and an absolutely continuous function is an absolutely continuous function ${ }^{177}$, we have that there exists a zero measure set $J_{i}$ such that $\dot{v}_{i}(t)$ exists for any $t \in \mathbb{R} \backslash J_{i}$. By applying Lemma 2 we can now conclude that

$$
\begin{equation*}
\text { if } t \in \mathbb{R} \backslash J_{i}, \quad \text { then either } \quad\left[x_{i}(t)=0 \text { and } \dot{v}_{i}(t)=0\right] \quad \text { or } \quad x_{i}(t) \neq 0 . \tag{2.39}
\end{equation*}
$$

[^10]Let $I_{i}$ be such a zero measure set that $\dot{x}_{i}(t)$ exists for all $t \in[0, \infty) \backslash I_{i}$. Since for $x_{i}(t) \neq 0$, such that $\dot{x}_{i}(t)$ exists, one has $\dot{v}_{i}(t)=\operatorname{sign}\left(x_{i}(t)\right) \dot{x}_{i}(t)$, then (2.39) implies:
if $t \in[0, \infty) \backslash\left(I_{i} \cup J_{i}\right)$, then either $\left[x_{i}(t)=0\right.$ and $\left.\dot{v}_{i}(t)=0\right]$ or $\left[x_{i}(t) \neq 0\right.$ and $\left.\dot{v}_{i}(t)=\operatorname{sign}\left(x_{i}(t)\right) \dot{x}_{i}(t)\right]$, which compact formulation is

$$
\text { if } t \in[0, \infty) \backslash\left(I_{i} \cup J_{i}\right) \text {, then } \dot{v}_{i}(t)=\operatorname{sign}\left(x_{i}(t)\right) \dot{x}_{i}(t) .
$$

Summing this up from 1 to $n$ and denoting $I=I_{1} \cup \ldots \cup I_{n}, J=J_{1} \cup \ldots \cup J_{n}$, we get

$$
\dot{v}(t)=\sum_{i=1}^{n} \operatorname{sign}\left(x_{i}(t)\right) \dot{x}_{i}(t), \quad \text { for any } t \in[0, \infty) \backslash(I \cup J) .
$$

Since $\dot{x}_{i} \in K[f](t, x(t))$ for almost all $t \in[0, \infty)$, then there exists $H \in[0, \infty)$ of measure zero such that

$$
\dot{v}(t)=\sum_{i=1}^{n} \operatorname{sign}\left(x_{i}(t)\right) \xi_{i}, \quad \text { for some } \xi \in K[f](t, x(t)) \text { and any } t \in[0, \infty) \backslash(I \cup J \cup H),
$$

Using the assumption of the theorem we conclude

$$
\dot{v}(t) \leq-k \quad \text { for any } t \in[0, \infty) \backslash(I \cup J \cup H)
$$

excluding a subset $C$ of $[0, \infty)$ for which $(t, x(t))$ is a point of hyperplane crossing for (2.3) and excluding those $t \in[0, \infty)$ for which $x(t)=0$. Proposition 4 says that $C$ is countable and, therefore, the measure of $C$ is zero (along with $I, J$ and $H$ ). Thus, we finally get

$$
\begin{equation*}
\dot{v}(t) \leq-k, \quad \text { for almost all } t \in[0, \infty) \text { such that } x(t) \neq 0 \tag{2.40}
\end{equation*}
$$

The rest of the proof is similar to Steps 1 and 2 of the proof of Theorem 2. Let $\bar{T}=\frac{v(0)}{k}$ and assume that $x(t)$ doesn't reach 0 on $[0, \bar{T}]$. Therefore, the inequality 2.40 holds for almost all $t \in[0, \bar{T}]$. Integrating 2.40 from 0 to $\frac{v(0)}{k}$ one gets

$$
v(\bar{T})-v(0) \leq-k \cdot \bar{T}
$$

which implies that $v(\bar{T})=x(\bar{T})=0$. This contradiction proves that $x(t)$ reaches 0 at some $\bar{t} \in[0, \bar{T}]$.
If $v(\overline{\bar{t}})>0$ at some $\overline{\bar{t}}>\bar{t}$, then inequality 2.40 holds on $(\tau, \overline{\bar{t}})$, where $\tau=\max \{t \in[\bar{t}, \bar{t}]:$ $v(t)=0\}$. This again implies $v(\overline{\bar{t}})-v(\tau) \leq-k(\bar{t}-\tau)<0$ which contradicts $v(\overline{\bar{t}})=0$.
The proof of the theorem is complete.
A good use of Theorem 4 is for finite-time stability of a two-degree-of-freedom manipulator, see $31{ }^{18}$

[^11]Theorem 4 is a generalization of the ideas of Proposition 3. As a consequence, Theorem 4 is capable to establish finite-time stability of system $(2.11)$, the Proposition 3 was for. Next example implements this statement.

Example 2 Let $\bar{x}=0 \in \mathbb{R}^{n}$. Under the conditions of Proposition 3, the origin is a finite-time stable equilibrium of (2.11).

Solution. From the formula for the convexification of the right-hand-side of (2.11) (see Sec. 2.4.1.2), we have

$$
\begin{equation*}
x_{i} \neq 0 \quad \Longrightarrow \quad \xi_{i}=\phi(t, x)-b_{i} \operatorname{sign}\left(x_{i}\right) \quad \text { for any } \xi \in K[f](t, x) \tag{2.41}
\end{equation*}
$$

To satisfy the assumption of theorem 4 it is, therefore, sufficient to show that

$$
\begin{equation*}
\sum_{i: x_{i} \neq 0} \operatorname{sign}\left(x_{i}\right)\left(\phi(t, x)-b_{i} \operatorname{sign}\left(x_{i}\right)\right) \leq-k \quad \text { for all } t \geq 0 \text { and all } x \in U \backslash\{0\} \tag{2.42}
\end{equation*}
$$

for some $k>0$. Let $x \in U \backslash\{0\}$. Then there exists $i$ such that $x_{i} \neq 0$, for which we have

$$
\operatorname{sign}\left(x_{i}\right)\left(\phi(t, x)-b_{i} \operatorname{sign}\left(x_{i}\right)\right)=\operatorname{sign}\left(x_{i}\right) \phi(t, x)-b_{i} \leq M_{i}-b_{i}=-\left(b_{i}-M_{i}\right) .
$$

Therefore, 2.42 holds with $k=\min _{i=1, . ., n}\left(b_{i}-M_{i}\right)$ and finite-time stability of the origin follows from Theorem 4 .

Example 3 Prove finite-time stability of the origin in the following system

$$
\begin{array}{ll}
\dot{x}=-\operatorname{sign} x & =: f_{1}(t, x, y), \\
\dot{y}=-2 \operatorname{sign} y-\operatorname{sign} x & =: f_{2}(t, x, y)
\end{array}
$$

Solution. The right-hand-side is discontinuous on $S=\left\{(x, y) \in \mathbb{R}^{2}: x y=0\right\}$. One can also check that there are points on $S$ which are not points of hyperplane crossing. Therefore, Theorem 2 is not applicable and Theorem 4 is the only option to attempt. Consider $x y \neq 0$ and $\binom{a}{b} \in K[f](t, x, y)$. We have to show that

[^12]\[

$$
\begin{equation*}
\dot{V}=\operatorname{sign}(x) a+\operatorname{sign}(y) b \leq-k \tag{2.43}
\end{equation*}
$$

\]

for some $k>0$ that doesn't depend on $x, y, a$ and $b$. According to properties (2.9 a) and (2.9 b) of the convexification, it suffices to prove (2.43) for

$$
\begin{aligned}
& a \in-\operatorname{Sign}(x) \\
& b \in-2 \operatorname{Sign}(y)-\operatorname{Sign}(x)
\end{aligned}
$$

Here Sign is the convexification of the sign-function, see (2.12).
$\boldsymbol{x} \neq \mathbf{0}$ and $\boldsymbol{y}=\mathbf{0}: \dot{V}=-\operatorname{sign}(x) \operatorname{sign}(x)=-1$.
$\boldsymbol{x} \neq \mathbf{0}$ and $\boldsymbol{y} \neq \mathbf{0}: \dot{V}=-\operatorname{sign}(x) \operatorname{sign}(x)+\operatorname{sign}(y)(-2 \operatorname{sign} y-\operatorname{sign} x) \leq-2$.
$\boldsymbol{x}=\mathbf{0}$ and $\boldsymbol{y} \neq \mathbf{0}: \dot{V}=\operatorname{sign}(y)(-2 \operatorname{sign}(y)-\tilde{x})$, where $\tilde{x} \in \operatorname{Sign}(0)$. Therefore, $\dot{V} \leq-1$.
In summary, the condition of Theorem 4 holds with $k=1$ for any $(x, y) \neq 0$ and finite-time stability of the origin follows by applying that theorem.

The next example somewhat combines the examples 1 and 3 in dimension 3. Some points of $S$ will appear to be the points of crossing and they need to be identified and excluded for the condition of Theorem 4 to hold.

Example 4 Prove finite-time stability of the origin in the following system

$$
\begin{array}{ll}
\dot{x}=-\operatorname{sign} x+3 \operatorname{sign} y & =: f_{1}(t, x, y, z), \\
\dot{y}=-3 \operatorname{sign} x-\operatorname{sign} y+\operatorname{sign} z & =: f_{2}(t, x, y, z),  \tag{2.44}\\
\dot{z}=-2 \operatorname{sign} z-\operatorname{sign} y & =: f_{3}(t, x, y, z) .
\end{array}
$$

Before proceeding to the solution, it is convenient to formulate the definition 3 of hyperplane crossing in the dimension 3.

Definition 6 (hyperplane crossing in dimension 3) Consider a time-independent $f(t, x, y, z)$ which is continuous outside $S=\left\{(x, y, z) \in \mathbb{R}^{3}: x y z=0\right\}$. The vector field $f(t, \cdot)$ crosses the hyperplane $x=0$ at the point $(0, y, z) \in S$ with $y z \neq 0$, if

$$
\operatorname{sign} \lim _{x \rightarrow 0^{-}} f_{1}(t, x, y, z)=\operatorname{sign} \lim _{x \rightarrow 0^{+}} f_{1}(t, x, y, z) \neq 0
$$

The vector field $f(t, \cdot)$ crosses the hyperplane $x=0$ at the point $(0,0, z) \in S$ with $z \neq 0$, if
$\underset{\substack{x \rightarrow 0^{-} \\ y \rightarrow 0^{-}}}{ } f_{1}(t, x, y, z)=\operatorname{sign} \lim _{\substack{x \rightarrow 0^{-} \\ y \rightarrow 0^{-}}} f_{1}(x, y, z)=\operatorname{sign} \lim _{\substack{x \rightarrow 0^{-} \\ y \rightarrow 0^{+}}} f_{1}(x, y, z)=\operatorname{sign} \lim _{\substack{x \rightarrow 0^{+} \\ y \rightarrow 0^{-}}} f_{1}(x, y, z) \neq 0$.
The hyperplane crossing conditions for $(x, 0, z),(x, y, 0),(0, y, 0),(x, 0,0) \in S$ are defined by analogy.

Solution of example 4. Let $(x, y, z) \in S \backslash\{0\}$ and $(a, b, c) \in \mathbb{R}^{3}$ satisfy

$$
\begin{aligned}
& a \in-\operatorname{Sign} x+3 \operatorname{Sign} y \\
& b \in-3 \operatorname{Sign} x-\operatorname{Sign} y+\operatorname{Sign} z, \\
& c \in-2 \operatorname{Sign} z-\operatorname{Sign} y
\end{aligned}
$$

where Sign is the convexification of the sign-function, see 2.12). We will show that there exists $k>0$ such that

$$
\begin{equation*}
\dot{V}=\operatorname{sign}(x) a+\operatorname{sign}(y) b+\operatorname{sign}(z) c \leq-k \tag{2.45}
\end{equation*}
$$

regardless of the particular choice of $(x, y, z)$ and $(a, b, c)$. We now go through the different cases that can occur when $(x, y, z) \neq 0$ :
$\boldsymbol{x} \neq 0, \boldsymbol{y} \neq 0, \boldsymbol{z} \neq 0$ : Canceling same terms we canceled out in the solution of Example 1, $\dot{V}=-2+\operatorname{sign}(y) \operatorname{sign}(z)+\operatorname{sign}(z) c=-2+\operatorname{sign}(y) \operatorname{sign}(z)-2-\operatorname{sign}(z) \operatorname{sign}(y) \leq-4$.
$\boldsymbol{x} \neq \mathbf{0}, \boldsymbol{y} \neq \mathbf{0}, \boldsymbol{z}=\mathbf{0}: \dot{V}=-2+\operatorname{sign}(y) \tilde{z}$, where $\tilde{z} \in \operatorname{Sign}(0)$. Therefore, $\dot{V} \leq-1$. $\boldsymbol{x}=\mathbf{0}, \boldsymbol{y}=\mathbf{0}, \boldsymbol{z} \neq \mathbf{0}: \dot{V}=\operatorname{sign}(z) c=-2-\operatorname{sign}(z) \tilde{y}$, where $\tilde{y} \in \operatorname{Sign}(0)$. Thus, $\dot{V} \leq-1$.
$\boldsymbol{x}=\mathbf{0}, \boldsymbol{y} \neq 0, \boldsymbol{z} \in(-\infty, \infty):$ selecting $-1 \in \operatorname{Sign}(0)$ leads to $\dot{V}>0$ and this point makes theorem 4 inapplicable, if $(0, y, z)$ is not a point of hyperplane crossing. Fortunately, $f$ does cross $x=0$ at $(0, y, z)$ because the following limits don't vanish and coincide:

$$
\underset{\substack{x \rightarrow 0^{-} \\ s \rightarrow z}}{\operatorname{sign} \lim _{1}(t, x, y, s)=\operatorname{sign}(1+3 \operatorname{sign}(y)),} \underset{\substack{x \rightarrow 0^{+} \\ s \rightarrow z}}{ } f_{1}(t, x, y, s)=\operatorname{sign}(-1+3 \operatorname{sign}(y)) .
$$

$x \neq 0, y=0, z \in \mathbb{R}: \dot{V}$ can exceed 0 in this case and the only chance to apply Theorem 4 is if we can show that $(x, 0, z)$ is a point of hyperplane crossing.
$\boldsymbol{x} \neq \mathbf{0}, \boldsymbol{y}=\mathbf{0}, \boldsymbol{z} \neq \mathbf{0}: f$ crosses $y=0$ at this point as the signs of the one-sided limits coincide and don't vanish:

$$
\operatorname{sign} \lim _{y \rightarrow 0^{ \pm}} f_{2}(t, x, y, z)=\operatorname{sign}(-3 \operatorname{sign} x-( \pm 1)+\operatorname{sign} z)=-\operatorname{sign} x
$$

$\boldsymbol{x} \neq \mathbf{0}, \boldsymbol{y}=\mathbf{0}, \boldsymbol{z}=\mathbf{0}: f$ does cross $y=0$ at this point too, however, per definition 6, four one-sided limits have to coincide and not vanish in this case, that we conclude to be true:

$$
\begin{aligned}
& \operatorname{sign} \lim _{y \rightarrow 0^{ \pm}} f_{2}(t, x, y, z)=\operatorname{sign}(-3 \operatorname{sign} x-( \pm 1)+ \pm 1)=-\operatorname{sign} x . \\
& z \rightarrow 0^{ \pm}
\end{aligned}
$$

Inequality 2.45), therefore, holds with $k=1$, for all $(x, y, z) \in \mathbb{R}^{3} \backslash\{0\}$ except on the hyperplanes $x=0$ and $y=0$, whose points turned out to be points of hyperplane crossing. The finite-time stability of the origin follows by applying Theorem 4.

There are two scenarios as for how the solutions in Example 4 behave: 1) the solution $z(t)$ first sticks to 0 and the solution $(x(t), y(t), z(t))$ develops in the hyperplane $z=0$ similar to Fig. 2.5 since then, 2) the solution $(x(t), y(t))$, making a maneuver of Fig. 2.5, first sticks to 0 and then $(x(t), y(t), z(t))$ develops in the intersection of the hyperplanes $x=0$ and $y=0$ until it finally reaches the origin.

Example 5 Find the region $W \subset \mathbb{R}^{3}$ such that each Filippov solution of

$$
\begin{array}{ll}
\dot{x}=-\operatorname{sign} x+3 \operatorname{sign} y & =: f_{1}(t, x, y, z), \\
\dot{y}=-3 \operatorname{sign} x-\operatorname{sign} y+\operatorname{sign} z & =: f_{2}(t, x, y, z),  \tag{2.46}\\
\dot{z}=-2 \operatorname{sign} z-\operatorname{sign} y+x+0.5 & =: f_{3}(t, x, y, z) .
\end{array}
$$

with the initial condition in $W$ approaches the origin in finite time.
Solution. Following the lines of the solution of Example 4, one gets

$$
\dot{V} \leq-1+|x|+0.5 \leq-0.5+|x| .
$$

If $r \in(0,0.5)$, then $\dot{V} \leq-0.5+r$ for any $(x, y) \in W_{r}=\{(x, y):|x|+|y|+|z|<r\}$ excluding the origin and the points of hyperplane crossing. Therefore any Filippov solution of (2.46) with the initial condition in $W_{r}$, where $0<r<0.5$, approaches the origin in finite time.

### 2.6 Pontryagin Maximum Principle and fastest stabilization

Here I just followed the book [37], pp. 9-27, focusing on Theorem 2 and Examples 1 and 2. 19

### 2.6.1 An example: Draw trajectories for given control signals

Here is a solution of a part of $\# 2$ from Homework 1:
http://www.utdallas.edu/~ makarenkov/Pontryagin-example.pdf

[^13]
### 2.7 Differential equations of sliding motion

Consider $(t, x) \in \mathbb{R} \times \mathbb{R}^{n}$. If the vector field $f$ doesn't cross the hyperplane $x_{i}=0$ at the point $(t, x)$, then the solution that reaches $x_{i}=0$ at the point $(t, x)$ has a chance to slide along $x_{i}=0$.

Definition 7 (hyperplane sliding in dimension $n$ ) Let $f: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be continuous outside $(\mathbb{R} \backslash J) \times S$, where $J$ is a countable set. Consider $(t, x) \in \mathbb{R} \times S$. Let $I_{0} \subset\{1,2, \ldots, n\}$ be the set of indexes of the vanishing components of $x$, i.e.

$$
x_{i}=0 \text { for all } i \in I_{0}, \quad x_{i} \neq 0 \text { for all } i \notin I_{0} .
$$

For $i \in I_{0}$, we say that the vector field $f$ slides along the hyperplane $x_{i}=0$ at the point $(t, x)$, if the following sliding condition holds

$$
\begin{array}{lll}
\lim _{b_{0}\left(\tau-t \rightarrow 0^{+}\right.} f_{i}(\tau, \xi)>0 & \text { and } & \lim _{0}\left(\tau-t \rightarrow 0^{+}\right. \\
b_{j} \xi_{j} \rightarrow 0^{+}, j \in I_{0} \backslash\{i\} & & f_{i}(\tau, \xi)<0 \\
\xi_{i} \rightarrow 0^{-} \\
\xi_{j}=x_{j}, j \notin I_{0} & & b_{0}\left(\tau \xi_{j} \rightarrow 0^{+}, j \in I_{0} \backslash\{i\}\right. \\
\xi_{i} \rightarrow 0^{+} \\
\xi_{j}=x_{j}, j \notin I_{0}
\end{array}
$$

regardless of the choice of $\left|b_{j}\right|=1, j \in\{0\} \cup I_{0}$. In this case we also say that $(t, x)$ is a point of hyperplane crossing.

Note, the particular signs in Definition 7 are important (positive for the limit from the left and negative for the limit from the right). The reversed signs don't give sliding, but lead to so-called escaping instead.

Proposition $7{ }^{20}$ If $x$ is a Filippov solution of (2.3) and the vector field $f$ slides along the hyperplane $x_{i}=0$ at $\left(t_{*}, x\left(t_{*}\right)\right) \in \mathbb{R} \times S$, then there exists $\delta>0$ such that

$$
x_{i}(t)=0, \quad \text { for all } t \in\left[t_{*}-\delta, t_{*}\right) \cup\left(t_{*}, t_{*}+\delta\right] .
$$

2.7.1 ( $n-1$ )-dimensional differential equations of sliding along a single discontinuity hyperplane
Proposition $8{ }^{21}$ If $x_{i}$ is the only component of $(t, x)$ that vanishes, then $K[f](t, x)=$ $\overline{\mathrm{co}}\left\{f^{L}, f^{R}\right\}$, where

$$
\begin{align*}
f^{L} & =\lim _{s \rightarrow 0^{-}} f\left(t, x_{1}, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_{n}\right) \\
f^{R} & =\lim _{s \rightarrow 0^{+}} f\left(t, x_{1}, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_{n}\right) \tag{2.47}
\end{align*}
$$

[^14]If, in addition, $f$ slides along $x_{i}=0$ at the point $(t, x)$, then there exists a unique $\lambda \in(0,1)$ such that

$$
K[f](t, x) \cap\left\{x \in \mathbb{R}^{n}: x_{i}=0\right\}=\left\{\lambda f^{L}+(1-\lambda) f^{R}\right\}
$$

Note, the vectors $f^{L}, f^{R}$ as well as the number $\lambda \in(0,1)$ depend on the point $(t, x)$, so that we write $f^{L}(t, x), f^{R}(t, x)$ and $\lambda(t, x)$ in what follows.


Figure 2.7: Illustration of the sliding vector $K[f](t, x) \cap\left\{x \in \mathbb{R}^{n}: x_{i}=0\right\}$.
Propositions 7 and 8 allow to introduce the following definition.
Corollary 2 Let $x$ be a Filippov solution of (2.3). Assume that the vector field $f$ slides along the hyperplane $x_{i}=0$ at $\left(t_{*}, x\left(t_{*}\right)\right) \in \mathbb{R} \times S$. If $x_{i}\left(t_{*}\right)$ is the only component of $\left(t_{*}, x\left(t_{*}\right)\right)$ that vanishes, then there exists $\delta>0$ such that

$$
\begin{aligned}
\dot{x}_{1}(t)= & \lambda(t, x(t)) \cdot f_{1}^{L}(t, x(t))+(1-\lambda(t, x(t))) \cdot f_{1}^{R}(t, x(t)), \\
& \cdots \\
\dot{x}_{i-1}(t)= & \lambda(t, x(t)) \cdot f_{i-1}^{L}(t, x(t))+(1-\lambda(t, x(t))) \cdot f_{i-1}^{R}(t, x(t)), \\
\dot{x}_{i+1}(t)= & \lambda(t, x(t)) \cdot f_{i+1}^{L}(t, x(t))+(1-\lambda(t, x(t))) \cdot f_{i+1}^{R}(t, x(t)), \\
& \cdots \\
\dot{x}_{n}(t)= & \lambda(t, x(t)) \cdot f_{n}^{L}(t, x(t))+(1-\lambda(t, x(t))) \cdot f_{n}^{R}(t, x(t)),
\end{aligned}
$$

for any $t \in\left[t_{*}-\delta, t_{*}+\delta\right]$, where $f^{L}$ and $f^{R}$ are given by (2.47) and $\lambda(t, x(t)) \in(0,1)$ is found from

$$
0=\lambda(t, x(t)) \cdot f_{i}^{L}(t, x(t))+(1-\lambda(t, x(t))) \cdot f_{i}^{R}(t, x(t)) .
$$

Definition 8 The equations of Corollary 2 are called the equations of sliding motion for a solution $x(t)$ that slides along a single discontinuity hyperplane $x_{i}=0$ (i.e. satisfies $x_{j}(t) \neq 0$ for $i \neq j$.

Simple 2-dimensional example and brief justification of the Fillipov solution concept
http://www.utdallas.edu/ ~ makarenkov/sliding-2d-example.pdf
Example 6 Show that the vector field of example 4 slides along $z=0$ in all points $(x, y, z)$ with $x y \neq 0, z=0$. Find the equation of sliding motion in those points.

For convenience, let us formulate the equations of sliding defined in Corollary 2 in the settings of Example 6. We formulate this equations for the case where the $x=0$ is the sliding hyperplane of interest to ease the comparison of this definition with an analogous of 3d-hyperplane crossing (see Definition 6).

Definition 9 (sliding along a single hyperplane in dimension 3) Consider a time-independent $f(t, x, y, z)$ which is continuous outside $S=\left\{(x, y, z) \in \mathbb{R}^{3}: x y z=0\right\}$. The vector field $f$ slides along $x=0$ at the point $(0, y, z)$ with $y z \neq 0$, if the following sliding condition holds

$$
\begin{equation*}
\lim _{x \rightarrow 0^{-}} f_{1}(t, x, y, z)>0 \quad \text { and } \quad \lim _{x \rightarrow 0^{+}} f_{1}(t, x, y, z)<0 . \tag{2.48}
\end{equation*}
$$

If this sliding condition holds, then the Filippov solution $(x(t), y(t), z(t))$ of (2.3) that passes through the point $(0, y, z)$ at time instance $t$ is differentiable at $t$ and the derivatives satisfy the sliding equations

$$
\begin{aligned}
\dot{y} & =\lambda(y, z) f_{2}^{L}(0, y, z)+(1-\lambda(y, z)) f_{2}^{R}(0, y, z), \\
\dot{z} & =\lambda(y, z) f_{3}^{L}(0, y, z)+(1-\lambda(y, z)) f_{3}^{R}(0, y, z)
\end{aligned}
$$

where

$$
f^{L}(0, y, z)=\lim _{x \rightarrow 0^{-}} f(t, x, y, z) \quad \text { and } \quad f^{R}(0, y, z)=\lim _{x \rightarrow 0^{+}} f(t, x, y, z)
$$

and $\lambda(y, z)$ is the unique constant from the interval $(0,1)$ that satisfies

$$
0=\lambda(y, z) f_{1}^{L}(0, y, z)+(1-\lambda(y, z)) f_{1}^{R}(0, y, z)
$$

Conditions for sliding in $y=0$ and $z=0$ and the respective sliding equations are defined by analogy.

Solution of Example 6. We follow Definition 9 (swapping the roles of $z$ in $x$ ) in each of the four quadrants where $x y \neq 0$.
$\boldsymbol{x}>\mathbf{0}, \boldsymbol{y}>\mathbf{0}: \lim _{z \rightarrow 0^{-}} f_{3}(t, x, y, z)=1, \lim _{z \rightarrow 0^{+}} f_{3}(t, x, y, z)=-3 \Longrightarrow f$ slides along $z=0$. To find the equation of sliding motion we first compute

$$
f^{L}(x, y, 0)=\left(\begin{array}{c}
2 \\
-5 \\
1
\end{array}\right), \quad f^{R}(x, y, 0)=\left(\begin{array}{c}
2 \\
-3 \\
-3
\end{array}\right) .
$$

Then we find $\lambda$ from

$$
0=\lambda f_{3}^{L}(x, y, 0)+(1-\lambda) f_{3}^{R}(x, y, 0) \quad \Longleftrightarrow \quad 0=\lambda \cdot 1+(1-\lambda) \cdot(-3) \quad \Longleftrightarrow \quad \lambda=\frac{3}{4} .
$$

Finally, this value of $\lambda$ is used to write up the equations of sliding as

$$
\begin{aligned}
\dot{x} & =\lambda f_{1}^{L}(x, y, 0)+(1-\lambda) f_{1}^{R}(x, y, 0)=\frac{3}{4} \cdot 2+\left(1-\frac{3}{4}\right) \cdot 2=2, \\
\dot{y} & =\lambda f_{2}^{L}(x, y, 0)+(1-\lambda) f_{2}^{R}(x, y, 0)=\frac{3}{4} \cdot(-5)+\left(1-\frac{3}{4}\right) \cdot(-3)=-4.5 .
\end{aligned}
$$



Figure 2.8: The vector field in the two cubes corresponding $x>0, y>0, z<0$ and $x>0, y>0, z<0$ respectively. Illustration of how the numbers $(2,-4.5)$ in the equation of the sliding motion occur from the vectors $(2,-3,-3)$ and $(2,-5,1)$.

Figure 2.8 illustrates the computations just performed.
$\boldsymbol{x}>\mathbf{0}, \boldsymbol{y}<\mathbf{0}: f^{L}(x, y, 0)=\left(\begin{array}{c}-4 \\ -3 \\ 3\end{array}\right), f^{R}(x, y, 0)=\left(\begin{array}{c}-4 \\ -1 \\ -1\end{array}\right)$, in particular $\lim _{z \rightarrow 0^{-}} f_{3}(t, x, y, z)=3, \lim _{z \rightarrow 0^{+}} f_{3}(t, x, y, z)=-1$, i.e. $f$ slides along $z=0$. Computing $\lambda$, $0=\lambda f_{3}^{L}(x, y, 0)+(1-\lambda) f_{3}^{R}(x, y, 0) \quad \Longleftrightarrow \quad 0=\lambda \cdot 3+(1-\lambda) \cdot(-1) \quad \Longleftrightarrow \quad \lambda=\frac{1}{4} \quad \Longrightarrow$ $\dot{x}=\frac{1}{4} \cdot(-4)+\left(1-\frac{1}{4}\right) \cdot(-4)=-4, \quad \dot{y}=\frac{1}{4} \cdot(-3)+\left(1-\frac{1}{4}\right) \cdot(-1)=-1.5$.
$\boldsymbol{x}<0, \boldsymbol{y}>0$ : do by analogy.
$\boldsymbol{x}<0, \boldsymbol{y}<0$ : do by analogy.

An important observation that one can make by analyzing the solution of Example 6 is that the equation

$$
\begin{aligned}
& \dot{x}=2, \\
& \dot{y}=-4.5,
\end{aligned}
$$

that we obtained for the Filippor solution $(x(t), y(t), z(t))$ of (2.44) when it slides in $\{(x, y, z): x>0, y>0, z=0\}$ is different from

$$
\begin{aligned}
& \dot{x}=2, \\
& \dot{y}=-4,
\end{aligned}
$$

which one gets by simply plugging $z=0$ to (2.44).

Example 7 Find the region of $\left\{(x, y, z) \in \mathbb{R}^{3}: x y \neq 0, z=0\right\}$ where the vector field of

$$
\begin{array}{ll}
\dot{x}=-\operatorname{sign} x+3 \operatorname{sign} y & =: f_{1}(t, x, y, z) \\
\dot{y}=-3 \operatorname{sign} x-\operatorname{sign} y+\operatorname{sign} z & =: f_{2}(t, x, y, z)  \tag{2.49}\\
\dot{z}=-2 \operatorname{sign} z-\operatorname{sign} y+2 y+2 x+1 & =: f_{3}(t, x, y, z)
\end{array}
$$

slides along $z=0$. Find the equation of sliding motion in the region that you obtain.
Solution. $\boldsymbol{x}>\mathbf{0}, \boldsymbol{y}>0$ : The one-sided limits take the form

$$
\lim _{z \rightarrow 0^{-}} f_{3}(t, x, y, z)=1+2 y+2 x+1 \quad \text { and } \quad \lim _{z \rightarrow 0^{+}} f_{3}(t, x, y, z)=-3+2 y+2 x+1
$$

The sliding condition (see Definition 9) requires that
$1+2 y+2 x+1>0 \quad$ and $\quad-3+2 y+2 x+1<0 \quad \Longrightarrow \quad y>-x-1 \quad$ and $y<-x+1$,
see Fig. 2.9. To find the equation of the sliding motion we compute $\lambda(x, y)$ as

$$
\lambda(x, y) \cdot(2+2 y+2 x)+(1-\lambda(x, y)) \cdot(-2+2 y+2 x)=0 \quad \Longrightarrow \quad \lambda(x, y)=\frac{1-y-x}{2}
$$

The equations of the sliding motion are, therefore,

$$
\begin{aligned}
\dot{x} & =\lambda(x, y) \cdot 2+(1-\lambda(x, y)) \cdot 2=2 \\
\dot{y} & =\frac{1-y-x}{2} \cdot(-5)+\left(1-\frac{1-y-x}{2}\right) \cdot(-3)=-4+x+y
\end{aligned}
$$



Figure 2.9: The sliding regions (dark gray) of 2.49) in the hyperplane $z=0$ by quadrants.
$\boldsymbol{x}>\mathbf{0}, \boldsymbol{y}<\mathbf{0}: \lim _{z \rightarrow 0^{-}} f_{3}(t, x, y, z)=3+2 y+2 x+1$ and $\lim _{z \rightarrow 0^{+}} f_{3}(t, x, y, z)=-1+2 y+2 x+1$. The sliding condition is, therefore, $y>-x-2$ and $y<-x$. Compute the equation of sliding motion by analogy with the case $x>0, y>0$.
$\boldsymbol{x}<\mathbf{0}, \boldsymbol{y}>\mathbf{0}: \lim _{z \rightarrow 0^{-}} f_{3}(t, x, y, z)=1+2 y+2 x+1$ and $\lim _{z \rightarrow 0^{+}} f_{3}(t, x, y, z)=-3+2 y+2 x+1$. The sliding condition is, therefore, $y>-x-1$ and $y<-x+1$. Compute the equation of sliding motion by analogy with the case $x>0, y>0$.
$\boldsymbol{x}<\mathbf{0}, \boldsymbol{y}<\mathbf{0}: \lim _{z \rightarrow 0^{-}} f_{3}(t, x, y, z)=3+2 y+2 x+1$ and $\lim _{z \rightarrow 0^{+}} f_{3}(t, x, y, z)=-1+2 y+2 x+1$. The sliding condition is, therefore, $y>-x-2$ and $y<-x$. Compute the equation of sliding motion by analogy with the case $x>0, y>0$.
In summary, the vector field $f(t, x, y, z)$ given by (2.49) slides in $z=0$ when

$$
\begin{aligned}
(x, y) \in & \{(x, y): x>0, y>0, y<-x+1\} \cup\{(x, y): x>0, y<0,-x-2<y<-x\} \cup \\
& \cup\{(x, y): x<0, y>0,-x-1<y<-x+1\} \cup\{(x, y): x<0, y<0, y>-x-2\},
\end{aligned}
$$

see Fig. 2.9.

### 2.7.2 Sliding along an arbitrary $n$ - 1-dimensional plane of an $n$-dimensional space

http://www.utdallas.edu/~makarenkov/sliding-plane.pdf
2.7.3 Lack of ( $n-k$ )-dimensional differential equations for sliding along an intersection of $k$ discontinuity hyperplanes

Sliding along an intersection of switching hyperplanes is usual for switched control systems, see [47, [31], 44], 45], [46, [1, Ch.2, § 10]. In this section I give a simple 3-dimensional example that features sliding along the hyperplanes $y_{2}=0$ and $y_{3}=0$, i.e. we know that the dynamics develops along the $y_{1}$-axis. However, I will show that, for the Filippov solution $y(t)$, the first component $y_{1}(t)$ cannot be described by a single differential equation.

Example 8 Find the equation of sliding motion along the $y_{1}$-axis in the system

$$
\begin{array}{ll}
\dot{y}_{1}=\operatorname{sign}\left(y_{2}\right)\left(\operatorname{sign}\left(y_{3}\right)+1\right) & =: f_{1}\left(y_{1}, y_{2}, y_{3}\right), \\
\dot{y}_{2}=-2 \operatorname{sign}\left(y_{2}\right)+\operatorname{sign}\left(y_{3}\right) & =: f_{2}\left(y_{1}, y_{2}, y_{3}\right),  \tag{2.50}\\
\dot{y}_{3}=-\operatorname{sign}\left(y_{3}\right) & =: f_{3}\left(y_{1}, y_{2}, y_{3}\right) .
\end{array}
$$

The solution uses the following property of the convex hull of a finite number of vectors $\xi_{1}, \ldots, \xi_{m} \in \mathbb{R}^{n}$ (also known as a simplex)

$$
\begin{equation*}
\overline{\operatorname{co}}\left(\xi_{1}, \ldots, \xi_{m}\right)=\bigcup_{\lambda_{1}+\ldots+\lambda_{m}=1}\left(\lambda_{1} \xi_{1}+\ldots+\lambda_{m} \xi_{m}\right), \tag{2.51}
\end{equation*}
$$

see [14, §14.5].
Solution. Any Filippov solution $y$ of 2.50 converges to the $y_{1}$-axis in finite time because for $V(y)=\left|y_{2}\right|+\left|y_{3}\right|$ we have

$$
\frac{d}{d t} \dot{V}(y(t)) \leq-1
$$

for any $t \geq 0$ such that $y_{2}(t) y_{3}(t) \neq 0$. According to Filippov definition,

$$
\begin{equation*}
\dot{y}_{1}(t) \in K_{1}[f]\left(\left(y_{1}(t), 0,0\right)^{T}\right), \quad \text { for almost all } t \geq 0 \tag{2.52}
\end{equation*}
$$

Here, for a set $A \in \mathbb{R}^{3}$, we define the first component $A_{1}$ as $A_{1}=\bigcup_{a \in A}\left\{a_{1}\right\}$. Accordingly, $K_{1}[f](y)$ is the first component of the set $K[f](y)$. We cannot use Proposition 8 to conclude that $K_{1}[f](y)$ with $y=\left(y_{1}(t), 0,0\right)^{T}$ is a singleton because Proposition 8 requires that only one component of $y$ vanishes. In what follows, we compute $K_{1}[f]\left(\left(y_{1}(t), 0,0\right)^{T}\right)$ exactly.


Figure 2.10: (a) The points $A(0,1,1), B(0,-3,1), C(2,-1,-1), D(-2,3,-1)$; (b) projection of the triangle $\Delta_{A B C}$ to the $y_{2}=0$ plane; (c) projection of the triangle $\Delta_{A B D}$ to the $y_{2}=0$ plane.

When $y_{2} \cdot y_{3} \neq 0$ approach zero, the vector $f(y)$ takes one of the four different values depicted at Fig 2.10 . Therefore,

$$
f\left(B_{\delta}\left(\left(y_{1}(t), 0,0\right)^{T}\right) \backslash J\right)=\{A, B, C, D\}, \quad \delta>0, \mu(J)=0,
$$

where $A=(0,1,1), B=(0,-3,1), C=(2,-1,-1), D=(-2,3,-1)$. From the property (2.51) we have

$$
\overline{\mathrm{Co}}(A, B, C) \subset \overline{\mathrm{co}}(A, B, C, D) \quad \text { and } \quad \overline{\mathrm{co}}(A, B, D) \subset \overline{\mathrm{co}}(A, B, C, D)
$$

The triangle $\overline{\mathrm{co}}(A, B, C)$ intersects the $y_{1}$-axis at the point $y_{1}=1$ (see Fig. 2.10(b),(c)). The triangle $\overline{\mathrm{co}}(A, B, D)$ intersects the $y_{1}$-axis at the point $y_{1}=-1$ (see Fig. 2.10(b), (d)). When one changes $\gamma$ from 0 to 1 the set

$$
\Lambda_{\gamma}=\bigcup_{\lambda_{1}+\lambda_{2}+\lambda_{3}=1}\left(\lambda_{1} A+\lambda_{2} B+\gamma \lambda_{3} C+(1-\gamma) \lambda_{3} D\right)
$$

changes from $\overline{\overline{c o}}(A, B, C)$ to $\overline{\mathrm{co}}(A, B, D)$ continuously and overlaps the hyperplanes $(A, B, C)$ and $(A, B, D)$ along the line $(A, B)$ only. Since $\bigcup_{\gamma \in[0,1]} \Lambda_{\gamma}=\overline{\operatorname{co}}(A, B, C, D)$, we get

$$
\overline{\operatorname{co}}(A, B, C, D)=[-1,1] \quad \Longrightarrow \quad K_{1}[f]\left(\left(y_{1}(t), 0,0\right)^{T}\right)=[-1,1], \quad t \geq 0
$$

and (2.52) takes the form

$$
\begin{equation*}
\dot{y}_{1}(t) \in[-1,1] \text {, } \tag{2.53}
\end{equation*}
$$

This is the answer. In contrast to Sec. 2.7.1, the Filippov definition now leads to a differential inclusion for $y_{1}$, rather than to a differential equation.

Note: Differential inclusion (2.53) is still different from $\dot{y}_{1} \in f_{1}\left(\mathbb{R}^{3}\right)=[-2,2]$, which can be one's immediate guess.

The lack of differential equation for the equation of motion can be illustrated through a system of coupled dry friction oscillators, see footnote ${ }^{22}$.
${ }^{22}$ A system of form 2.50 can be obtained from the following mechanical setup


Assuming that the coefficient of the Coulomb dry friction between the bodies $m_{1}$ and $m_{2}$ equals $d_{1}$ and that the coefficient of the Coulomb dry friction between the body $m_{2}$ and the ground is $d_{2}$, the equations of the displacements $x_{1}$ and $x_{2}$ of the bodies read as (see 49])

$$
\begin{aligned}
& m_{1} \ddot{x}_{1}+c_{1}\left(\dot{x}_{1}-\dot{x}_{2}\right)+d_{1} \operatorname{sign}\left(\dot{x}_{1}-\dot{x}_{2}\right)=0 \\
& m_{2} \ddot{x}_{2}-c_{1}\left(\dot{x}_{1}-\dot{x}_{2}\right)-d_{1} \operatorname{sign}\left(\dot{x}_{1}-\dot{x}_{2}\right)+d_{2} \operatorname{sign}\left(\dot{x}_{2}\right)+k_{2} x_{2}+c_{2} \dot{x}_{2}=0 .
\end{aligned}
$$

The change of the variables

$$
y_{1}=x_{2}, \quad y_{2}=\dot{x}_{2}, \quad y_{3}=\dot{x}_{1}-\dot{x}_{2}
$$

brings the system to the form

$$
\begin{align*}
& \dot{y}_{1}=y_{2}+u, \\
& \dot{y}_{2}=d_{1} \operatorname{sign} y_{3}-d_{2} \operatorname{sign} y_{2}-c_{1} y_{3}-k_{2} y_{1}-c_{2} y_{2}  \tag{*}\\
& \dot{y}_{3}=-2 d_{1} \operatorname{sign} y_{3}+d_{2} \operatorname{sign} y_{2}-c_{1} y_{3}+c_{1} y_{3}+k_{2} y_{1}+c_{2} y_{2}
\end{align*}
$$

with $u=0$. When $d_{1}<d_{2}, d_{2}<2 d_{1}$ and $\|y\|$ is small, the hyperplanes $y_{2}=0$ and $y_{3}=0$ are hyperplanes of sliding. If we regard $u$ as a control (invasion into the velocity vector field is used in control in the context of super-twisting controllers, see [50]), then defining

$$
u=\operatorname{sign}\left(y_{2}\right)\left(\operatorname{sign}\left(y_{3}\right)+1\right)
$$

one gets a system that exhibits the properties of system 2.50 .
Exercise 15 Follow the approach that we developed in Example 8 and find the analogue of (2.53), i.e. the differential inclusion of sliding motion, for $y_{1}$ in $\operatorname{system}\left({ }^{*}\right)$ near $y=0$. It will now depend on phase variables.

### 2.8 Examples of finite-time stable limit cycles

### 2.8.1 Terminal limit cycles in polar coordinates

If $(r, \theta)$ are polar coordinates of a point $(x, y)$, then the solution of the system

$$
\begin{aligned}
& \dot{r}=r \operatorname{sign}(R-r)|R-r|^{0.5}, \\
& \dot{\theta}=\omega,
\end{aligned}
$$

where $R, \omega>0$, is a cycle of radius $R$ that attracts all other solutions (excluding a neighborhood of the origin) in finite time, see [51, §2.4].

### 2.8.2 Finite-time stability of stick-slip limit cycles in dry friction oscillators

The equation of the coordinate $x$ of a mass $m$ placed on the moving (frictional) belt and attached to an immovable wall through a spring is governed by the dry friction oscillator

$$
\begin{array}{ll}
\dot{x}=y & =: f_{1}(x, y) \\
\dot{y}=-x-c y-F(y-V) & =: f_{2}(x, y) \tag{2.54}
\end{array}
$$

see Fig. 2.11. Here $F$ is the friction law that describes the dry friction between the mass and


Figure 2.11: Dry friction oscillator on a moving belt.
the ground. This law is continuously differentiable everywhere besides $y=0$ and it verifies

$$
\lim _{y \rightarrow 0^{-}} F(y)=-1 \quad \text { and } \quad \lim _{y \rightarrow 0^{+}} F(y)=1
$$

at $y=0$. As a consequence, the right-hand-side $f$ is continuously differentiable everywhere besides $y=V$.
In accordance with Definition 9 (that we now use in dimension 2), define
$f^{L}(x, V)=\lim _{y \rightarrow V^{-}} f(x, y)=\binom{1}{-x-c+1}, \quad f^{R}(x, V)=\lim _{y \rightarrow V^{+}} f(x, y)=\binom{1}{-x-c-1}$,
see Fig. 2.12, and find that the points $(x, V)$ with

$$
f_{2}^{L}(x, V)=-x-c+1>0 \quad \text { and } \quad f_{2}^{R}(x, V)=-x-c-1<0
$$



Figure 2.12: Vector fields $f^{L}$ and $f^{R}$ along the line $y=V$.
are points of sliding. This leads to the following interval and equation of sliding motion (see Definition 9)

$$
\dot{x}(t)=V, \quad \text { for all } t \geq 0 \text { such that } x(t) \in(-1-c, 1-c) .
$$

Proposition 9 Let $(x(t), y(t))$ be a Filippov solution of (2.54) with the initial condition $(x(0), y(0))=(1-c, V)$. If

$$
x(\bar{t}) \in(-1-c, 1-c) \quad \text { and } \quad y(\bar{t})=V
$$

for some $\bar{t}>0$, then $(x(t), y(t))$ is a finite-time stable limit cycle of (2.54).
The proof is a combination of 3 facts: 1 ) any solution of (2.54) with the initial condition in $(-1-c, 1-c) \times\{V\}$ reaches the point $(1-c, V) ; 2)$ without loss of generality we can assume that $y(t)<V, t \in(0, \bar{t})$, and $(x(t), y(t))$ is the unique solution of (2.54) on the interval $(0, \bar{t})$; 3) the solution $(x(t), y(t))$ reaches $y=V$ at $t=\bar{t}$ transversally and so the solutions of (2.54) that originate near $(x(0), y(0))$ do reach $y=V$ too.

### 2.8.2.1 The case of Coulomb dry friction

In this case $F(y)=\operatorname{sign}(y)$ and 2.54 takes the form

$$
\begin{array}{lr}
\dot{x}=y, & \text { if } y>V, \\
\dot{y}=-x-c y-1, & \\
\dot{x}=y, & \text { if } y<V . \\
\dot{y}=-x-c y+1, &
\end{array}
$$

In this case the equilibrium $(1,0)$ of (2.56) is stable and, therefore, the solution $(x(t), y(t))$ of (2.56) with the initial condition $(x(0), y(0))=(1-c, V)$ intersects the $y>0$ part of the line $-x-c y+1=0$ at those $t>0$ for which $y(t) \in(0, V)$, see Fig. 2.13(b). Indeed, if we assume that $(x(t), y(t))$ intersects the $y>0$ part of the line $-x-c y+1=0$ at those $t>0$ for which $y(t)>V$, then this contradicts the stability of $(1,0)$ (in this case we will construct a second trajectory that spirals towards $(1,0)$ and conclude the existence of a limit cycle for (2.56), which cannot happen). Therefore, system (2.55)-(2.56) doesn't have close orbits passing through the sliding region $(-1-c, 1-c) \times\{V\}$.


Figure 2.13: (a) The trajectories of 2.55 - 2.56 ) with $c=0$; (b) the solutions ( $x, y$ ) of 2.56 (smaller loop) and (2.57) (larger loop) with the initial condition $(x(0), y(0))=(1-c, V)$.

### 2.8.2.2 The case of Coulomb dry friction and Stribeck effect

In this section we consider the following dry friction law

$$
F(y)=\operatorname{sign}(y)\left(\frac{1-\alpha}{1+\gamma|y|}+\alpha+\beta y^{2}\right)
$$

which is a possible way to account for so-called Stribeck effect, see [52, §4.1-4.2]. This friction characteristics is utilized in [54, 55, 53, 56]. We have

$$
\begin{array}{ll}
\dot{x}=y & =: f_{1}^{L}(x, y), \\
\dot{y}=-x-c y+\frac{1-\alpha}{1-\gamma(y-V)}+\alpha+\beta(y-V)^{2} & =: f_{2}^{L}(x, y), \tag{2.57}
\end{array} \quad \text { if } y<V .
$$

The equilibrium of this system is $\left(x_{0}, 0\right)=\left(\frac{1-\alpha}{1+\gamma V}+\alpha+\beta V^{2}, 0\right)$, which is unstable if

$$
-c+\frac{\gamma(1-\alpha)}{(1+\gamma V)^{2}}-2 \beta V>0
$$

Denote by $L$ the $y>0$ part of the line through $\left(x_{0}, 0\right)$ and $(1-c, 1)$. If the parameters $c \geq 0$, $\alpha \geq 0, \beta \geq 0, \gamma \geq 0$ are small enough, then the solution $(x(t), y(t))$ of equation (2.57) with the initial condition $(x(0), y(0))=(1-c, 1)$ will intersect the $y>0$ part of line $L$ at some time moment $\tau>0$ again. Assume that $\tau>0$ is the first time moment when $(x(t), y(t))$ intersects the $y>0$ part of $L$. There are two cases to consider:
$\boldsymbol{y}(\boldsymbol{\tau}) \leq 1$ : Since the equilibrium $\left(x_{0}, 0\right)$ is unstable, then there is an orbit $(\hat{x}(t), \hat{y}(t))$ with the initial condition on $L$ and close to $\left(x_{0}, 0\right)$ such that for the next intersection $(\hat{x}(\hat{t}), \hat{y}(\hat{t}))$ of this orbit with $L$ we have $\hat{y}(\hat{t})>\hat{y}(0)$. This implies that 2.57) possesses a closed orbit $(\tilde{x}, \tilde{y})$ with the initial condition on $L$. Assume, in addition, that for some $r>0$, we have

$$
\begin{array}{r}
\operatorname{div} f^{L}(x, y)=\frac{\partial f_{1}^{L}}{\partial x}(x, y)+\frac{\partial f_{2}^{L}}{\partial y}(x, y)=-c+\frac{\gamma(1-\alpha)}{(1-\gamma(y-V))^{2}}+2 \beta(y-V)>0  \tag{2.57,1}\\
y \in[-V-r, V+r]
\end{array}
$$

Then, if $|c|+|\gamma|+|\beta|$ is small enough, then (2.57) cannot have close orbits in the stripe $y \in[-V-r, V+r]$ by the Criterion of Bendixson [57, Ch. X, $\S 7]$. In particular, the closed orbit $(\tilde{x}, \tilde{y})$ is impossible. Therefore, the case $y(\tau)<1$ cannot take place.
$\boldsymbol{y}(\boldsymbol{\tau})>1:$ If $|c|+|\gamma|+|\beta|$ is small enough, then the solution $(x, y)$ is close to circle of radius $V$ and centered at (1,0). Therefore, the assumptions of Proposition 9 hold and (2.54) has a finite-time stable limit cycle by applying this proposition.

Same kind of result is obtained for (2.54) in [52, Example 9.1] over measure differential inclusions. Numeric simulations are performed in [55].

### 2.8.3 Electromagnetic relay model

This model is investigated in [59] and [58, Ch. II, §4.10] and it provides a natural example of a finite-time stable limit cycle in $\mathbb{R}^{3}$, see Fig. 2.14 ${ }^{33}$.


Figure 2.14: Finite-time stable limit cycle in $\mathbb{R}^{3}$.

[^15]Exercise 16 Design a 3-dimensional discontinuous system that exhibits the finite-time stable limit cycle of Fig. 2.14.

## 3 Asymptotic stability of switched equilibria

Consider the following version of switched system (2.3)

$$
\dot{x}=f(x), \quad f(x)= \begin{cases}f^{L}(x), & \text { if } x \in D^{L}  \tag{3.1}\\ f^{R}(x), & \text { if } x \in D^{R}\end{cases}
$$

where $f^{L}, f^{R}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ are smooth vector fields and $D^{L}, D^{R} \subset \mathbb{R}^{n}$ are open regions such that $D^{L} \cup S \cup D^{R}=\mathbb{R}^{n}$, where $S$ is a smooth surface (that separates $f^{L}$ and $f^{R}$ ). I will stick to the case where $S$ is a hyperplane, when it doesn't diminish the class of applications that I have in mind. If $S$ is a hyperplane, then the letter $L$ will be used in place of $S$.

Definition 10 A point $\bar{x} \in S$ is a switched equilibrium of system (3.1) or, equivalently, $\bar{x}$ is a switched equilibrium for the vector fields $f^{R}$ and $f^{L}$, if $f^{R}(\bar{x}) \neq 0$ and $f^{L}(\bar{x}) \neq 0$, and there exists $\bar{\lambda} \in(0,1)$ such that

$$
\begin{equation*}
\bar{\lambda} f^{R}(\bar{x})+(1-\bar{\lambda}) f^{L}(\bar{x})=0 \tag{3.2}
\end{equation*}
$$

With this definition, we don't require that the sets $D^{L}$ and $D^{R}$ are defined for the notion of switched equilibrium to make sense.

### 3.1 Structural instability of regular equilibria lying on the switching hyperplane

Let $S=L$ be a hyperplane (the statement of this section remains true for surfaces). Assume that $f^{L}\left(x_{0}\right)=f^{R}\left(x_{0}\right)=0$ for some $x_{0} \in L$ and $\operatorname{det}\left(f^{L}\right)^{\prime}\left(x_{0}\right) \cdot \operatorname{det}\left(f^{R}\right)^{\prime}\left(x_{0}\right) \neq 0$. Let $f_{\varepsilon}^{L}$ and $f_{\varepsilon}^{R}$ be the vector fields obtained from $f_{\varepsilon}^{L}$ and $f_{\varepsilon}^{R}$ by $\varepsilon$-small smooth perturbations. The Implicit Function Theorem (see [13, Theorem 13.7], [60, § 8.5.4, Theorem 1]) implies that $f_{\varepsilon}^{L}\left(x_{\varepsilon}^{L}\right)=f_{\varepsilon}^{R}\left(x_{\varepsilon}^{R}\right)=0$ for some $x_{\varepsilon}^{L}, x_{\varepsilon}^{R} \in \mathbb{R}^{n}$ that converge to $x_{0}$ as $\varepsilon \rightarrow 0$. The formula for the derivative of the implicit function (see [60, §8.5.4, Theorem 1, p. 490])) allows to see that if none of the vectors $\left.\left(\left(f^{L}\right)^{\prime}\left(x_{0}\right)\right)^{-1}(d / d \varepsilon) f_{\varepsilon}^{L}\left(x_{0}\right)\right|_{\varepsilon=0}$ and $\left.\left(\left(f^{R}\right)^{\prime}\left(x_{0}\right)\right)^{-1}(d / d \varepsilon) f_{\varepsilon}^{R}\left(x_{0}\right)\right|_{\varepsilon=0}$ are parallel $L$, then $x_{\varepsilon}^{L} \notin L$ and $x_{\varepsilon}^{R} \notin L$ for all $|\varepsilon|>0$ sufficiently small. Therefore, any generic perturbation brings $f$ to a vector field that doesn't longer have an equilibrium on the switching hyperplane $L$. Such a phenomenon is often accompanied by new (nonsmooth) bifurcations whose classification was actively pursued lately, see [65, 66, 67, 68]. Next section introduces a non-standard equilibrium for switched systems that doesn't run away from $L$ under perturbations.

If system (3.1) respects a certain structure and perturbations don't destroy this structure (the main example is when both the vector fields in (3.1) are linear as well as the perturbations allowed), then the equilibrium $x_{0}$ may remain on the line $L$ even in the presence of perturbations. We refer the reader to [61, 62, 64] for the theory of stability of (3.1) in the case where both $f^{L}$ and $f^{R}$ are matrices. In particular, one may enjoy [63, Example 3], where the matrices $f^{L}$ and $f^{R}$ are unstable, but the origin of (3.1) appears to be asymptotically stable.


Figure 3.1: Response of a common equilibrium of two vector fields $f^{L}$ and $f^{R}$ to perturbations.

### 3.2 Structural stability of switched equilibria lying on the switching hyperplane

Let $\bar{x} \in S$ be a switched equilibrium and let $\bar{\lambda}$ be the respective value of $\lambda$. Assume that $S$ is parameterized as $S=S\left(s_{1}, \ldots, s_{n-1}\right)$. Consider $g(\lambda, s)=\lambda f^{R}(S(s))+(1-\lambda) f^{L}(S(s))$. We have $g: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ and $g(\bar{\lambda}, \bar{s})=0$, where $\bar{s}$ is such that $S(\bar{s})=\bar{x}$. The derivative $g^{\prime}(\bar{\lambda}, \bar{s})$ is an $n \times n$-matrix. Assume that

$$
\begin{equation*}
\operatorname{det}\left\|g^{\prime}(\bar{\lambda}, \bar{s})\right\| \neq 0 \tag{3.3}
\end{equation*}
$$

Introduce smooth functions $f_{\varepsilon}^{R}, f_{\varepsilon}^{L}$, such that $f_{\varepsilon}^{R} \rightarrow f^{R}$ and $f_{\varepsilon}^{L} \rightarrow f^{L}$ as $\varepsilon \rightarrow 0$. If $g(\lambda, s, \varepsilon)=$ $\lambda f_{\varepsilon}^{R}(S(s))+(1-\lambda) f_{\varepsilon}^{L}(S(s))$, then the Implicit Function Theorem ensures the existence of $(\lambda(\varepsilon), s(\varepsilon))$, such that $g(\lambda(\varepsilon), s(\varepsilon), \varepsilon)=0$ and $(\lambda(\varepsilon), s(\varepsilon)) \rightarrow(\bar{\lambda}, \bar{s})$ as $\varepsilon \rightarrow 0$. By the other words, condition (3.3) ensures that the perturbed system

$$
f(x)= \begin{cases}f_{\varepsilon}^{L}(x), & \text { if } x \in D^{L} \\ f_{\varepsilon}^{R}(x), & \text { if } x \in D^{R}\end{cases}
$$

has a switched equilibrium $\bar{x}(\varepsilon)=S(s(\varepsilon))$ near $\bar{x}$.
Observe, that 3.3 never holds, if $f^{R}(\bar{x})=f^{L}(\bar{x})=0$ because $g_{\lambda}^{\prime}(\lambda, \bar{s})$ is a zero vector in this case.

### 3.3 Reduction to the sliding switching surface

### 3.3.1 2-dimensional systems

Let $L \subset \mathbb{R}^{2}$ be a line that splits $\mathbb{R}^{2}$ into two open regions $D^{L}$ and $D^{R}$.
Definition 11 If $\bar{x} \in L$ is such a point that $f^{R}(\bar{x})=k f^{L}(\bar{x})$, where $k<0$, then $\bar{x}$ is a switched equilibrium of (3.1).

The definition implies that the constant function $x(t)=\bar{x}, t \geq 0$, is a Fillipov solution of (3.1), thus $\bar{x}$ is indeed an "equilibrium" of (3.1). The notion of switched equilibrium is
common in engineering literature. The same equilibrium is termed an invisible equilibrium in mathematics literature.

The Definition 7 of sliding can be formulated at this instance as follows.
Definition 12 A point $x \in S=\partial D^{L}=\partial D^{R}$ is a point of sliding for system (3.1), if $f^{L}(x)$ points strictly towards $D^{R}$ and $f^{R}(x)$ points strictly towards $D^{L}$.

The word "strictly" in this definition needs clarification and that is what we do in Definition 13 and Definition 14 below. As for the earlier Definition 7, the "strict" situation was guaranteed by strict inequalities entering the definition.

Assume that $L$ is parameterized by a parameter $s \in \mathbb{R}$. From now on we, therefore, regard $L$ as a smooth function $s \mapsto L(s)$, whose graph is the original line $L$ (the double sense of the letter $L$ is not going to create a confusion). Furthermore, we assume that the parameterization $L(s)$ is non-singular, i.e. $\left\|L^{\prime}(s)\right\| \neq 0, s \in \mathbb{R}$. The function $L^{\prime}(s)$ is, therefore, a constant vector for now. If $x=L(s)$ is close to $\bar{x}=L(0)$, then $\overline{\operatorname{co}}\left\{f^{L}(x), f^{R}(x)\right\}$ contains a (unique) vector $\vec{F}(s)$ collinear $L^{\prime}(s)$. Let $F(s)=\left\langle\vec{F}(s), L^{\prime}(s)\right\rangle$.

Let us fix $s \in \mathbb{R}$ and consider $x=L(s)$. Next lemma provides a formula to express $F(s)$ in terms of $f^{L}(x), f^{R}(x)$ and $L^{\prime}(s)$. We omit the variables $s$ and $x$ to shorten notations (and to present the lemma as a purely geometrical exercise). For $l \in \mathbb{R}^{2}$, the vector $l^{\perp}$ is defined by

$$
\binom{l_{1}}{l_{2}}^{\perp}=\binom{l_{2}}{-l_{1}}
$$



Figure 3.2: Illustration of the notations of this section.

Lemma 3 Let $f^{L}, f^{R}, l \in \mathbb{R}^{2}$ be such that $\left\langle f^{L}, l^{\perp}\right\rangle \cdot\left\langle f^{R}, l^{\perp}\right\rangle<0$ and $\|l\|=1$. Then, for the constant $F=\left\langle\overline{\operatorname{co}}\left\{f^{L}, f^{R}\right\} \cap \bigcup_{\lambda \in \mathbb{R}} \lambda l\right\rangle$, the following formula holds

$$
F=\frac{\left\langle f^{R}, l^{\perp}\right\rangle \cdot\left\langle f^{L}, l\right\rangle-\left\langle f^{L}, l^{\perp}\right\rangle \cdot\left\langle f^{R}, l\right\rangle}{\left\langle f^{R}, l^{\perp}\right\rangle-\left\langle f^{L}, l^{\perp}\right\rangle} .
$$

Proof. We prove the formula for the case where all the vectors are pointing as at Fig. 3.2. The validity of the formula for all other configurations of the vectors can be established by analogy.

The triangles $\triangle O A B$ and $\triangle O C D$ are similar, therefore

$$
\frac{O B}{O D}=\frac{\left\langle f^{R}, l^{\perp}\right\rangle}{-\left\langle f^{L}, l^{\perp}\right\rangle}
$$

In addition,

$$
\left\langle f^{R}, l\right\rangle+O B+O D=\left\langle f^{L}, l\right\rangle .
$$

Since $F=\left\langle f^{R}, l\right\rangle+O B$, the conclusion follows by solving the system of two equations with two unknowns $O B$ and $O D$.

Lemma 3 implies that the equation of the sliding motion along $L$ can be described by the following equation of sliding motion

$$
\begin{equation*}
\dot{s}=F(s), \tag{3.4}
\end{equation*}
$$

where
$F(s)=\frac{\left\langle f^{R}(L(s)), L^{\prime}(s)^{\perp}\right\rangle \cdot\left\langle f^{L}\left(L(s), L^{\prime}(s)\right\rangle-\left\langle f^{L}(L(s)), L^{\prime}(s)^{\perp}\right\rangle \cdot\left\langle f^{R}(L(s)), L^{\prime}(s)\right\rangle\right.}{\left\|L^{\prime}(s)\right\| \cdot\left(\left\langle f^{R}(L(s)), L^{\prime}(s)^{\perp}\right\rangle-\left\langle f^{L}(L(s)), L^{\prime}(s)^{\perp}\right\rangle\right)}$.
Theorem 5 [12, p. 217-218, Lemma 3 (p. 223)] Consider $f^{L}, f^{R} \in C^{1}\left(\mathbb{R}^{2}, \mathbb{R}^{2}\right)$. Let $L(\cdot) \in$ $C^{2}\left(\mathbb{R}, \mathbb{R}^{2}\right)$ be a non-singular parameterization of the switching line $L$ that separates the open regions $D^{L}$ and $D^{R}$. Assume that $\bar{x}=L(0)$ is a switched equilibrium of (3.1).

1) If $F^{\prime}(0) \neq 0$, then $\bar{x}$ is structurally stable.
2) If $\bar{x}$ is a point of sliding and $F^{\prime}(0)<0$, then $\bar{x}$ is asymptotically stable.

Proof. 1) The statement claims that for any $C^{1}$-smooth $\varepsilon$-small perturbations of $f^{L}$ and $f^{R}$, the respective function $F_{\varepsilon}$ will have an equilibrium $\bar{x}_{\varepsilon}$ and $\bar{x}_{\varepsilon} \rightarrow \bar{x}$ as $\varepsilon \rightarrow 0$. This statement is a consequence of the Implicit Function Theorem (see [13, Theorem 13.7], [60, § 8.5.4, Theorem 1]).
2) Consider $\varepsilon>0$ such that all points of $L([-\varepsilon, \varepsilon])$ are points of sliding. Let $B_{\delta}(\bar{x})$ be such a neighborhood of $\bar{x}$ that any Filippov solution $x$ of (3.1) with the initial condition
in $B_{\delta}(\bar{x})$ reaches $L([-\varepsilon, \varepsilon])$ at some time moment $\bar{t} \geq 0$ (which is different for different solutions $x$ ). Fix some Filippov solution $x$ with the initial condition in $B_{\delta}(\bar{x})$. This can be proved by applying the Comparison Lemma (see Lemma 1$]^{24}$. Thus, $x(t)=L(s(t))$ starting from $t=\bar{t}$ and (at least ) until $x(t)$ escapes from $L([-\varepsilon, \varepsilon])$, where $s(t)$ is a solution of (3.4) with the initial condition $s(0) \in[-\varepsilon, \varepsilon]$. The assumption $F^{\prime}(0)<0$ ensures that 0 is an asymptotically stable equilibrium of (3.4) by the linearization theorem (see e.g. 69, Theorem 1.44]). Therefore, we can consider $\varepsilon>0$ so small, that $s(t) \rightarrow 0$ (monotonically) as $t \rightarrow \infty$. This implies that $x(t)$ never escapes from $L([-\varepsilon, \varepsilon])$ and $x(t) \rightarrow \bar{x}$ as $t \rightarrow \infty$ uniformly in $x(0) \in B_{\delta}(\bar{x})$.

## S17 revision of Example 9

https://www.utdallas.edu/~makarenkov/automatic-pilot-example.pdf
Example 9 ([70, Ch. VIII, § 6, pp. 501-515]) Prove asymptotic stability of the origin in the following model of a two-position automatic pilot with velocity correction, where $\beta>0$,

$$
\begin{array}{ll}
\dot{x}=y & =: f_{1}^{R}(x, y), \quad \text { if } x+\beta y>0, \\
\dot{y}=-y-1 & =: f_{2}^{R}(x, y), \quad \text { f } \tag{3.6}
\end{array}
$$



Figure 3.3: The notations of the solution of Example 9 .
It is convenient to reformulate Definition 12 for the case of systems (3.1) defined by

$$
\dot{x}= \begin{cases}f^{L}(x), & \text { if }\left\langle x-\bar{x}, l^{\perp}\right\rangle>0  \tag{3.7}\\ f^{R}(x), & \text { if }\left\langle x-\bar{x}, l^{\perp}\right\rangle<0 .\end{cases}
$$

[^16]Exercise 17 1) Detail the arguments that conclude the existence of $\bar{t}$ from the Comparison Lemma in the proof of Theorem 5-2), 2) explain rigorously why for any $\varepsilon>0$ the value $\delta>0$, that we use in the proof of Theorem 52), exists.

Definition 13 (sliding along a hyperplane in $\mathbb{R}^{n}$ ) A point $x \in\left\{x \in \mathbb{R}^{n}:\left\langle x-\bar{x}, l^{\perp}\right\rangle=0\right\}$ is a point of sliding for the switched system (3.7), if

$$
\left\langle f^{L}(x), l^{\perp}\right\rangle<0 \quad \text { and } \quad\left\langle f^{R}(x), l^{\perp}\right\rangle>0
$$

Solution. The normalized parameterization of the line $x+\beta y=0$ can be defined as

$$
L(s)=\frac{1}{\sqrt{\beta^{2}+1}}\binom{\beta s}{-s}
$$

for which

$$
\bar{x}=L(0)=0, l=L^{\prime}(s)=\frac{1}{\sqrt{\beta^{2}+1}}\binom{\beta}{-1}, l^{\perp}=L^{\prime}(s)^{\perp}=\frac{1}{\sqrt{\beta^{2}+1}}\binom{-1}{-\beta}
$$

Accordingly, the open set $D^{L}$ is the part of $\mathbb{R}^{2}$ under $L$ and $D^{R}$ is the part of $\mathbb{R}^{2}$ above $L$. Thus, system (3.6) takes form (3.7). Testing $\bar{x}=0$ for sliding leads to

$$
\begin{aligned}
& \left\langle f^{L}(0), l^{\perp}\right\rangle=\left\langle\binom{ 0}{1}, \frac{1}{\sqrt{\beta^{2}+1}}\binom{-1}{-\beta}\right\rangle<0 \\
& \left\langle f^{R}(0), l^{\perp}\right\rangle=\left\langle\binom{ 0}{-1}, \frac{1}{\sqrt{\beta^{2}+1}}\binom{-1}{-\beta}\right\rangle>0
\end{aligned}
$$

Therefore, the origin is a point of sliding for (3.6) by Definition 13.
Since

$$
f^{L}(L(s))=\frac{1}{\sqrt{\beta^{2}+1}}\binom{-s}{s+\sqrt{\beta^{2}+1}}, f^{R}(L(s))=\frac{1}{\sqrt{\beta^{2}+1}}\binom{-s}{s-\sqrt{\beta^{2}+1}}
$$

computing the scalar products gives

$$
\begin{aligned}
& \left\langle f^{L}(L(s)), L^{\prime}(s)\right\rangle=\frac{1}{\beta^{2}+1}\left(-s \cdot \beta+\left(s+\sqrt{\beta^{2}+1}\right) \cdot(-1)\right) \\
& \left\langle f^{R}(L(s)), L^{\prime}(s)^{\perp}\right\rangle=\frac{1}{\beta^{2}+1}\left(-s \cdot(-1)+\left(s-\sqrt{\beta^{2}+1}\right) \cdot(-\beta)\right) \\
& \left\langle f^{R}(L(s)), L^{\prime}(s)\right\rangle=\frac{1}{\beta^{2}+1}\left(-s \cdot \beta+\left(s-\sqrt{\beta^{2}+1}\right) \cdot(-1)\right) \\
& \left\langle f^{L}(L(s)), L^{\prime}(s)^{\perp}\right\rangle=\frac{1}{\beta^{2}+1}\left(-s \cdot(-1)+\left(s+\sqrt{\beta^{2}+1}\right) \cdot(-\beta)\right)
\end{aligned}
$$

and

$$
F(s)=-\frac{s}{\beta} .
$$

The origin is asymptotically stable for any $\beta>0$ by Theorem 5 .

Note: It follows from the solution that the equation of sliding motion (3.4) for (3.6) is given by

$$
\begin{equation*}
\dot{s}=-\frac{1}{\beta} s \tag{3.8}
\end{equation*}
$$

which agrees with [70, p. 507].

Remark 1 Formula (3.5) and Theorem 5 remain valid when $L(\cdot) \in C^{1}\left(\mathbb{R}, \mathbb{R}^{2}\right)$. Items 1) and 2) of Theorem 5 can be replaced by

1) If $F$ is strictly monotone at 0 , then $\bar{x}$ persists.
2) If the exists $r>0$ such that the points of $L([-r, 0) \cup(0, r])$ are points of sliding and $s=0$ is a finite-time stable equilibrium of the equation of sliding motion (3.4), then $\bar{x}$ is finite-time stable switched equilibrium of (3.1).
The following clarification of Definition 12 is required to test points of $S=\partial D^{L}=\partial D^{R}$ for sliding when $L$ is a curve (Fig. 3.4 illustrates the construction of the function $F$ in the case where $L$ is a curve).


Figure 3.4: Illustration of notations for the case where $L$ is an arbitrary smooth curve.
Definition 14 (sliding along a curve in $\left.\mathbb{R}^{2}\right)$ Assume that a non-singular curve $L \in C^{1}\left(\mathbb{R}, \mathbb{R}^{2}\right)$ parameterizes the set $S=\partial D^{L}=\partial D^{R}$. If $L^{\prime}(s)^{\perp}$ points towards $D^{L}$ then $L(s)$ is a point of sliding for (3.1) provided that

$$
\left\langle f^{L}(L(s)), L^{\prime}(s)^{\perp}\right\rangle<0 \quad \text { and } \quad\left\langle f^{R}(L(s)), L^{\prime}(s)^{\perp}\right\rangle>0
$$

If $L^{\prime}(s)^{\perp}$ points towards $D^{R}$ then $L(s)$ is a point of sliding for (3.1) provided that

$$
\left\langle f^{L}(L(s)), L^{\prime}(s)^{\perp}\right\rangle>0 \quad \text { and } \quad\left\langle f^{R}(L(s)), L^{\prime}(s)^{\perp}\right\rangle<0
$$

The Kaveh's solution of $\# 2 \mathrm{~b}$ of Homework 2 suggested two questions whose answers are unknown to me ${ }^{25}$

### 3.3.1.1 Return to the $(x, y)$-coordinates

One-dimensional formulas (3.4)-(3.5) are convenient for analyzing the stability of the switched equilibrium $\bar{x}$. Sometimes it is of interest to plot the trajectory in the original coordinates. The formulas (3.4)-(3.5) can be then turned into

$$
\dot{x}_{i}=G\left(x_{i}\right), \quad \text { where } G\left(x_{i}\right)=\left.\left[F(s) \frac{L^{\prime}(s)}{\left\|L^{\prime}(s)\right\|}\right]_{i}\right|_{s=L^{-1}\left(x_{i}\right)}, \quad i=1,2
$$

${ }^{25}$ Consider the exercises.
Exercise 18 (5 pts) Let $(x, y)$ be a Filippov solution of

$$
\binom{\dot{x}}{\dot{y}}= \begin{cases}\left(\begin{array}{l}
y \\
1 \\
y \\
-1
\end{array}\right), & \text { if } x+b \operatorname{sign}(y) y^{2}<0 \\
& \text { if } x+b \operatorname{sign}(y) y^{2}>0\end{cases}
$$

Assume that $0<b<\frac{1}{2}$. As Kaveh observed, the solution $t \mapsto(x(t), y(t))$ crosses the curve $x+b \operatorname{sign}(y) y^{2}=0$ as long as $(x(t), y(t)) \neq 0$. Let $\left\{t_{i}\right\}_{i=1}^{\infty}$ be the increasing sequence of the respective times of crossing. Examine whether or not the series $\sum_{i=1}^{\infty}\left(t_{i+1}-t_{i}\right)$ converges. The convergence corresponds to finite-time stability of the origin. That would be an acceptable solution even if you just observe the convergence/divergence of the series on the computer (I can then do the proof myself).

Exercise 19 (10 pts) Consider the switched system

$$
\binom{\dot{x}}{\dot{y}}=f(x, y), \quad \text { where } \quad f(x, y)= \begin{cases}f^{L}(x, y), & \text { if } x+b \operatorname{sign}(y) y^{2}<0 \\ f^{R}(x, y), & \text { if } x+b \operatorname{sign}(y) y^{2}>0\end{cases}
$$

The vector $\binom{2 b \operatorname{sign}(y)}{-1}$ is tangent to the switching curve $x+b \operatorname{sign}(y) y^{2}=0$. Therefore, the vector

$$
F(x, y) \in K[f](x, y) \bigcap\left(\bigcup_{k \in \mathbb{R}} k\binom{2 b \operatorname{sign}(y)}{-1}\right)
$$

is tangent to $x+b \operatorname{sign}(y) y^{2}=0$ in all those points of this curve, which are points of sliding. Kaveh noticed that if $(\star \star)$ is $(\star)$, then the following Fact holds: the solution of the initial value problem

$$
\begin{aligned}
& \binom{\dot{x}}{\dot{y}}=F(x, y), \\
& x(0)+b \operatorname{sign}(y(0)) y(0)^{2}=0
\end{aligned}
$$

doesn't leave the curve $x+b \operatorname{sign}(y) y^{2}=0$ as long as all points of $x+b \operatorname{sign}(y) y^{2}=0$ are points of sliding. Does this fact hold for any switched system ( $\star \star$ ) with smooth $f^{L}$ and $f^{R}$ ? Construct a counter-example, if not.

In particular, one can check that equation (3.8) of Example 9 leads to

$$
\dot{x}=-\frac{1}{\beta} x
$$

for the $x$-coordinate of the sliding solution.
Note, there is a more direct approach of obtaining the solution of (3.1) in the original coordinates, that doesn't involve $F(s)$. We don't consider that approach here because our focus is on stability, where construction of $F(s)$ or similar reduction in original coordinates is always required.

### 3.3.2 3-dimensional systems

### 3.3.2.1 Switched equilibrium inside the sliding region

Lemma 4 Let $l_{1}, l_{2}, l^{\perp} \in \mathbb{R}^{3}$ be orthogonal unit vectors and let $f^{L}, f^{R} \in \mathbb{R}^{3}$ be such that $\left\langle f^{L}, l^{\perp}\right\rangle \cdot\left\langle f^{R}, l^{\perp}\right\rangle<0$. Then, for the two-dimensional vector $F=$ $\left\langle\overline{\operatorname{co}}\left\{f^{L}, f^{R}\right\} \cap \bigcup_{\lambda_{1}, \lambda_{2} \in \mathbb{R}} \lambda_{1} l_{1}+\lambda_{2} l_{2}\right\rangle$, the following formula holds

$$
F_{i}=\frac{\left\langle f^{R}, l^{\perp}\right\rangle \cdot\left\langle f^{L}, l_{i}\right\rangle-\left\langle f^{L}, l^{\perp}\right\rangle \cdot\left\langle f^{R}, l_{i}\right\rangle}{\left\langle f^{R}, l^{\perp}\right\rangle-\left\langle f^{L}, l^{\perp}\right\rangle}
$$

in the coordinates $\left(l_{1}, l_{2}\right)$.
Proof. As in the proof of Lemma 3, we fix particular directions of all vectors and establish Lemma 4 for those specific directions. One can verify that the formula of Lemma 4 remains valid for all other directions of the vectors.

Step 1. Consider the notations of Fig. 3.5. Similar to the proof of Lemma 3, we have

$$
\frac{O B}{O D}=\frac{\left\langle f^{R}, l^{\perp}\right\rangle}{-\left\langle f^{L}, l^{\perp}\right\rangle}
$$

This is the only information that gain from the 3d drawing of Fig. 3.5.(Left). All the further computations refer to the planar drawing of Fig. 3.5(Right).

Step 2. Here we compute $O N$ and $B N$.

$$
\begin{aligned}
D Q & =P K+P L=-\left\langle f^{R}, l_{1}\right\rangle+\left\langle f^{L}, l_{1}\right\rangle, \\
D M & =D Q-O N, \\
\frac{D M}{O N} & =\frac{O D}{O B}
\end{aligned} \quad \Longrightarrow O N=\frac{-\left\langle f^{R}, l_{1}\right\rangle+\left\langle f^{L}, l_{1}\right\rangle}{1+\frac{-\left\langle f^{L}, l^{\perp}\right\rangle}{\left\langle f^{R}, l^{\perp}\right\rangle}}
$$



Figure 3.5: Illustration of notations of Lemma 4 :
Left and Right: $P A=\left\|f^{R}\right\|, A B=\left\|\left\langle f^{R}, l^{\perp}\right\rangle\right\|, P C=\left\|f^{L}\right\|, C D=\left\|\left\langle f^{L}, l^{\perp}\right\rangle\right\|, P O=\|F\|, \overrightarrow{P O}=\vec{F}$,
Right: $P K=\left\|\left\langle f^{R}, l_{1}\right\rangle\right\|, K B=\left\|\left\langle f^{R}, l_{2}\right\rangle\right\|, P L=\left\|\left\langle f^{L}, l_{1}\right\rangle\right\|, L D=\left\|\left\langle f^{L}, l_{2}\right\rangle\right\|$.

$$
\begin{aligned}
& B Q=L D-K B=\left\langle f^{L}, l_{2}\right\rangle-\left\langle f^{R}, l_{2}\right\rangle, \\
& O M=B Q-B N, \\
& \frac{O M}{B N}=\frac{O D}{O B}
\end{aligned} \quad \Longrightarrow \quad B N=\frac{-\left\langle f^{R}, l_{2}\right\rangle+\left\langle f^{L}, l_{2}\right\rangle}{1+\frac{-\left\langle f^{L}, l^{\perp}\right\rangle}{\left\langle f^{R}, l^{\perp}\right\rangle}}
$$

Step 3. We finally compute the projections of the vector $F$ on both $l_{1}$ and $l_{2}$ as

$$
F_{1}=-(P K-O N) \quad \text { and } \quad F_{2}=K B+B N
$$

that leads to the required formula.
Let $\bar{x}$ be a switched equilibrium for the vector fields $f^{L}$ and $f^{R}$. Fix two arbitrary linearly-independent vectors $l_{1}, l_{2} \in \mathbb{R}^{3}$, such that

$$
\begin{equation*}
\left\langle l^{\perp}, l_{1}\right\rangle=0, \quad\left\langle l^{\perp}, l_{2}\right\rangle=0, \quad\left\langle l_{1}, l_{2}\right\rangle=0 \tag{3.9}
\end{equation*}
$$

and parameterize the switching hyperplane as

$$
\begin{equation*}
L\left(s_{1}, s_{2}\right)=(\bar{x}, \bar{y}, \bar{z})^{T}+s_{1} l_{1}+s_{2} l_{2} . \tag{3.10}
\end{equation*}
$$

Lemma 4 suggests that if $F \in C^{1}\left(\mathbb{R}^{2}, \mathbb{R}^{2}\right)$ is defined as

$$
\begin{aligned}
& F_{1}\left(s_{1}, s_{2}\right)=\frac{\left\langle f^{R}\left(L\left(s_{1}, s_{2}\right)\right), l^{\perp}\right\rangle \cdot\left\langle f^{L}\left(L\left(s_{1}, s_{2}\right)\right), l_{1}\right\rangle-\left\langle f^{L}\left(L\left(s_{1}, s_{2}\right)\right), l^{\perp}\right\rangle \cdot\left\langle f^{R}\left(L\left(s_{1}, s_{2}\right)\right), l_{1}\right\rangle}{\left\|l_{1}\right\| \cdot\left(\left\langle f^{R}\left(L\left(s_{1}, s_{2}\right)\right), l^{\perp}\right\rangle-\left\langle f^{L}\left(L\left(s_{1}, s_{2}\right)\right), l^{\perp}\right\rangle\right)}, \\
& F_{2}\left(s_{1}, s_{2}\right)=\frac{\left\langle f^{R}\left(L\left(s_{1}, s_{2}\right)\right), l^{\perp}\right\rangle \cdot\left\langle f^{L}\left(L\left(s_{1}, s_{2}\right)\right), l_{2}\right\rangle-\left\langle f^{L}\left(L\left(s_{1}, s_{2}\right)\right), l^{\perp}\right\rangle \cdot\left\langle f^{R}\left(L\left(s_{1}, s_{2}\right)\right), l_{2}\right\rangle}{\left\|l_{2}\right\| \cdot\left(\left\langle f^{R}\left(L\left(s_{1}, s_{2}\right)\right), l^{\perp}\right\rangle-\left\langle f^{L}\left(L\left(s_{1}, s_{2}\right)\right), l^{\perp}\right\rangle\right)},
\end{aligned}
$$



Figure 3.6: Left: a switched equilibrium $\bar{x}$ in $\mathbb{R}^{2}$ in the case where both $f^{L}(\bar{x})$ and $f^{R}(\bar{x})$ are parallel to the switching surface $S$ (which is a line in this example); Right: the response of this switched equilibrium to a small perturbation.
then

$$
\begin{align*}
& \dot{s}_{1}=F_{1}\left(s_{1}, s_{2}\right), \\
& \dot{s}_{2}=F_{2}\left(s_{1}, s_{2}\right) \tag{3.11}
\end{align*}
$$

are the equations of sliding motion along the switching hyperplane $\left\langle(x-\bar{x}, y-\bar{y}, z-\bar{z})^{T}, l^{\perp}\right\rangle=0$. Then, the origin is a regular equilibrium of (3.11) and the following analogue of Theorem 5 can be proposed.

Theorem 6 Consider $f^{L}, f^{R} \in C^{1}\left(\mathbb{R}^{3}, \mathbb{R}^{3}\right)$. Let $\bar{x}$ be a switched equilibrium for $f^{L}$ and $f^{R}$. Define $D^{L}$ and $D^{R}$ as

$$
D^{L}=\left\langle(x-\bar{x}, y-\bar{y}, z-\bar{z})^{T}, l^{\perp}\right\rangle<0, \quad D^{R}=\left\langle(x-\bar{x}, y-\bar{y}, z-\bar{z})^{T}, l^{\perp}\right\rangle>0,
$$

where $l^{\perp} \in \mathbb{R}^{3}$ is an arbitrary vector. Let $l_{1}, l_{2} \in \mathbb{R}^{3}$ be any two vectors that, when combined with $l^{\perp}$, form an orthogonal basis of $\mathbb{R}^{3}$, i.e. (3.9) holds. Let $L\left(s_{1}, s_{2}\right)$ be the parameterization $(3.10)$ of the hyperplane $\left\{(x, y, z) \in \mathbb{R}^{3}:\left\langle(x-\bar{x}, y-\bar{y}, z-\bar{z})^{T}, l^{\perp}\right\rangle\right\}$ induced by the vectors
$l_{1}, l_{2}$.

1) If $\operatorname{det}\left\|F^{\prime}(0)\right\| \neq 0$, then $\bar{x}$ is a structurally stable switched equilibrium of (3.1).
2) If $\bar{x}$ is a point of sliding and the real parts of the eigenvalues of $F^{\prime}(0)$ are negative, then $\bar{x}$ is an asymptotically stable switched equilibrium of (3.1).

### 3.3.2.2 Switched equilibrium on the boundary of the sliding region

If $\bar{x}$ is a switched equilibrium of a 2-dimensional switched system (3.1) then condition (3.3) ensures that this switched equilibrium persists under perturbations (in the sense of $\S(3.2)$. However, the situation where both $f^{L}(\bar{x})$ and $f^{R}(\bar{x})$ are tangent to the switching surface $S$ is structurally unstable in $\mathbb{R}^{2}$. The vector fields (generically) don't feature same property at the perturbed switched equilibrium, see Fig. 3.6 However, the situation where both $f^{L}(\bar{x})$ and $f^{R}(\bar{x})$ are tangent to the switching surface $S$ is structurally stable in the space $\mathbb{R}^{3}$. This leads to a new type of stable equilibria in 3-dimensional switched systems, not found in dimension 2.

To have a brief idea as for why the above-mentioned tangent situation persists, assume that the switching hyperplane is given by $S=\left\{(x, y, z) \in \mathbb{R}^{3}: z=0\right\}$ and both $f^{L}$ and $f^{R}$ are


Figure 3.7: Directions of the vector fields $f^{L}$, $f^{R}$ (left figure) and $f_{\varepsilon}^{L}, f_{\varepsilon}^{R}$ (right figure) against the curves $h^{L}, h^{R}$ (left figure) and $h_{\varepsilon}^{L}, h_{\varepsilon}^{R}$ (right figure). The solid trajectories with arrows are governed by $f^{R}$ and they correspond to $z>0$. The dashed trajectories with arrows are governed by $f^{L}$ and they correspond to $z<0$. The gray regions are regions of sliding on the both figures.
tangent to $S$ at the origin. This means that $f_{3}^{L}(0)=f_{3}^{R}(0)=0$. If now $\frac{\partial f^{L}}{\partial y}(0) \neq 0$, then by the Implicit Function Theorem, there is a curve $h^{L}(x)$ such that $f_{3}^{L}\left(x, h^{L}(x), 0\right)=0$ for all $|x|$ sufficiently small. Analogously, if a suitable derivative is different from zero, then there will be a curve $h^{R}(x)$ such that $f_{3}^{R}\left(x, h^{R}(x), 0\right)=0$ for all $|x|$ sufficiently small. The two curves will generically (i.e. if a suitable derivative is different from zero) intersect transversally. This implies that small $\varepsilon$-perturbations of the vector fields $f^{L}$ and $f^{R}$ will not destroy the transversal intersection of the curves $h^{L}$ and $h^{R}$. The intersection of the perturbed curves $h_{\varepsilon}^{L}$ and $h_{\varepsilon}^{R}$ will give a new point $\left(\bar{x}_{\varepsilon}, \bar{y}_{\varepsilon}, \bar{z}_{\varepsilon}\right)$ near the origin, where the perturbed vector fields $f_{\varepsilon}^{L}$ and $f_{\varepsilon}^{R}$ will still be tangent to $S$, see Fig. 3.7.

In this example, the curves $h^{L}$ and $h^{R}$ split the $(x, y)$-plane into four regions, see Fig. 3.7 for the directions of the vector fields $f^{R}$ and $f^{L}$ against the two curves. One of these regions is a region of sliding (gray region at Fig. 3.7). It is possible to impose the conditions that ensure that each Filippov solution that reaches the sliding region, sticks to the sliding region forever while approaching the origin asymptotically. One of the four regions will be a region of escaping. The remaining two regions are regions of crossing. Teixeira proposed conditions that ensure that any Filippov solution that crosses the regions of crossing multiple number of times will finally lend to the region of sliding. This allowed Teixeira [81] to obtain conditions for asymptotic stability of the point of the intersection of $h^{L}$ and $h^{R}$. This point is termed $U$-singularity or Teixeira singularity. We refer the reader to [81] and [12, § 22].

A prototypic example (normal form) of Teixeira singularity is given by the system

$$
\begin{equation*}
(\dot{x}, \dot{y}, \dot{z})^{T}=F(x, y, z)+\operatorname{sign}(z) \cdot G(x, y, z) \tag{3.12}
\end{equation*}
$$

where

$$
F(x, y, z)=\frac{1}{2}\left(a_{1}+b_{1}, a_{2}+b_{2}, x+y\right), \quad G(x, y, z)=\frac{1}{2}\left(a_{1}-b_{1}, a_{2}-b_{2}, x-y\right) .
$$

The curves that divide the $(x, y)$-plane in four regions are here the lines $x=0$ and $y=0$. The result in [81, p. 29] states that 0 is an asymptotically stable point of (3.12), if

$$
b_{1}<a_{2}, \quad 2 a_{1} b_{2}<a_{2} b_{1}<a_{1} b_{2}<0 .
$$

I don't devote much time to the analysis of Teixeira singularity because, despite of its mathematical beauty, I know only one paper on an application of this singularity, see [82] (I would be happy to learn more applications).

### 3.4 Design of switching surfaces using Lyapunov theory

Let $\bar{x}$ be a switched equilibrium for the vector fields $f^{L}, f^{R}$, i.e.

$$
\begin{equation*}
\bar{\lambda} f^{L}(\bar{x})+(1-\bar{\lambda}) f^{R}(\bar{x})=0 \tag{3.13}
\end{equation*}
$$

for some $\bar{\lambda} \in(0,1)$. In this section we will design the sets $D^{L}, D^{R}$ and the respective switching surface $S$ that make $\bar{x}$ an asymptotically stable switched equilibrium of (3.1).

Because of (3.13), the point $\bar{x}$ is an equilibrium for

$$
\begin{equation*}
\dot{x}=\bar{\lambda} f^{L}(x)+(1-\bar{\lambda}) f^{R}(x) . \tag{3.14}
\end{equation*}
$$

The whole analysis will be based on the assumption that $\bar{x}$ is an asymptotically stable equilibrium of (3.14). Asymptotic stability of $\bar{x}$ is equivalent (see Theorem 34.1 and Theorem 51.1 in [83]) to the existence of a (Lyapunov) function $V \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}\right)$ such that

$$
\begin{align*}
& V(x)>0, x \neq \bar{x}, V(\bar{x})=0 \\
& V^{\prime}(x)\left(\bar{\lambda} f^{L}(x)+(1-\bar{\lambda}) f^{R}(x)\right)<0 \quad \text { for all } x \neq \bar{x} \tag{3.15}
\end{align*}
$$

Introduce two open sets

$$
\begin{aligned}
& \Omega^{L}=\left\{x: V^{\prime}(x) f^{L}(x)<0\right\}, \\
& \Omega^{R}=\left\{x: V^{\prime}(x) f^{R}(x)<0\right\} .
\end{aligned}
$$

Lemma 5 Let $\bar{x}$ be a switched equilibrium for the vector fields $f^{L}$ and $f^{R}$, i.e. (3.13) holds. Assume that the equilibrium $\bar{x}$ of system (3.14) is asymptotically stable and consider the respective (Lyapunov) function $V \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}\right)$ that satisfies (3.15). Then, the sets $\Omega^{L}$ and $\Omega^{R}$ satisfy the properties:

1) $\Omega^{L} \cup \Omega^{R} \cup\{\bar{x}\}=\mathbb{R}^{n}, \overline{\Omega^{L}} \cup \overline{\Omega^{R}}=\mathbb{R}^{n}$,
2) $\partial \Omega^{L} \backslash\{\bar{x}\} \subset \Omega^{R}, \partial \Omega^{R} \backslash\{\bar{x}\} \subset \Omega^{L}$,


Figure 3.8: Relative locations of sets $\Omega^{L}$ and $\Omega^{R}$.
3) $\bar{x} \in \partial \Omega^{L}, \bar{x} \in \partial \Omega^{R}$.

Figure 3.8 illustrates the conclusion of Lemma 5. In particular, I cannot see that $\Omega^{L} \cap \Omega^{R}$ always includes a hyperplane passing through $\bar{x}$. The role of this note will be clear after one gets through $\S$ 3.4.1.4 and $\S$ 3.4.1.6.

Proof. 1) The first part follows from (3.15) directly. The first part implies that either $\bar{x} \in \overline{\Omega^{L}}$ or $\bar{x} \in \overline{\Omega^{R}}$, thus the second part holds.
2) Consider $x \in \partial \Omega^{L}$. Then $x \notin \Omega^{L}$ because $\Omega^{L}$ is open. Then $x \in \overline{\Omega^{R}}$ by 1$)$. The property $\partial \Omega^{R} \subset \overline{\Omega^{L}}$ can be proved by analogy.
3) It is sufficient to show that $V^{\prime}(\bar{x})=0$. To observe this, consider the vector $\xi^{j} \in \mathbb{R}^{n}$ defined as $\xi_{i}^{j}=0, i \neq j$, and $\xi_{j}^{j}=1$. Since $V(x)>0, x \neq \bar{x}$, we have

$$
\begin{array}{ll}
0<V\left(\bar{x}+\alpha \xi^{j}\right)-V(\bar{x})=V^{\prime}\left(\bar{x}+\alpha_{*} \xi^{j}\right) \xi^{j} \cdot \alpha=\frac{\partial V}{\partial x_{j}}\left(\bar{x}+\alpha_{*} \xi^{j}\right) \alpha, \\
0<V\left(\bar{x}-\alpha \xi^{j}\right)-V(\bar{x})=-V^{\prime}\left(\bar{x}-\alpha_{* *} \xi^{j}\right) \xi^{j} \cdot \alpha=-\frac{\partial V}{\partial x_{j}}\left(\bar{x}-\alpha_{* *} \xi^{j}\right) \alpha, & \text { for all } \alpha>0
\end{array}
$$

This can happen only if $\frac{\partial V}{\partial x_{j}}(\bar{x})=0$. Since $j \in \overline{1, n}$ is chosen arbitrary, one gets $V^{\prime}(\bar{x})=0$.
The proof of the lemma is complete.

### 3.4.1 State feedback switching rule

### 3.4.1.1 Lyapunov stability theorem for smooth Lyapunov functions

Theorem 7 (similar to [48, Theorem 3.1], [73, Theorem 2.3]) Let $\bar{x} \in S$, where $S$ is a smooth surface that separates $D^{L}$ and $D^{R}$. Assume that $V \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}\right)$ is such that $V(x)>0, x \neq \bar{x}$, and $V(\bar{x})=0$. Let $W_{r}$ be an open neighborhood of $\bar{x}$ defined by $W_{r}=\left\{x \in \mathbb{R}^{n}: V(x)<r\right\}$, where $r$ is some positive constant. Consider a piecewise continuous function $w: \mathbb{R}^{n} \rightarrow \mathbb{R}$ such that for any $\rho>0$ there exists $\varepsilon>0$ for which $w(x) \geq \varepsilon$ as long as $x \in \overline{W_{r}} \backslash B_{\rho}(\bar{x})$. If

$$
V^{\prime}(x) \xi \leq-w(x) \quad \text { for any } \xi \in K[f](x), \text { and any } x \in \overline{W_{r}} \backslash\{\bar{x}\}
$$

where $w \in C^{0}\left(\mathbb{R}^{n}, \mathbb{R}\right)$ is strictly positive for all $x \neq \bar{x}$, then $\bar{x}$ is an asymptotically stable point of (3.1), which attracts all Filippov solutions of (3.1) that originate in $\overline{W_{r}}$.

Proof. Let $x$ be a Filippov solution of (3.1) that originates in $W_{r}$. We pick any $0<\rho<r$ such that $x(0) \notin W_{\rho}$ and prove that $x(t) \in W_{\rho}$ beginning from some $t=t_{\rho}$.
Step 1. Let us prove that $x(t) \in W_{r}$ for all $t>0$. We prove by contradiction, i.e. assume that $x(0) \in \partial W_{r}$, but $x(\tau) \notin W_{r}$ for some $\tau>0$. Without loss of generality we can assume that $x([0, \tau]) \subset W$, where $W$ is an open neighborhood of $\overline{W_{r}}$, such that $w(x)$ is strictly positive in $W \backslash\{0\}$. For the function

$$
v(t)=V(x(t))
$$

we have

$$
\begin{equation*}
v(0)=r \quad \text { and } \quad v(\tau) \geq r . \tag{3.16}
\end{equation*}
$$

Step 1.1 We claim that $v(t)>r / 2$ for all $t \in[0, \tau]$. Indeed, if the latter is wrong, then defining

$$
s=\max \{t \in[0, \tau]: v(t) \leq r / 2\}
$$

one gets

$$
\begin{equation*}
v(s)=r / 2, \quad v(\tau)=r, \quad v(t) \in[r / 2, r], \quad \text { for any } t \in[s, \tau] . \tag{3.17}
\end{equation*}
$$

In particular, $x(t) \neq \bar{x}$ for all $t \in[s, \tau]$ and, therefore,

$$
v^{\prime}(t)=V^{\prime}(x(t)) \xi<0, \quad \text { for some } \xi \in K[f](x(t)) \text { and almost any } t \in[s, \tau]
$$

This contradicts (3.17) and proves that $v(t)>r / 2$ for all $t \in[0, \tau]$.
Step 1.2 Step 1.1 implies that

$$
x(t) \neq 0, \text { for any } t \in[0, \tau], \quad \text { and, as a consequence, } \quad v^{\prime}(t)<0, \text { for any } t \in[0, \tau] .
$$

This contradicts (3.16) and completes the proof of the fact that $x(t) \in W_{r}$ for all $t>0$.
Step 2. Let us show that $x(t)$ reaches $\overline{W_{\rho}}$ at some time moment. Assume that $x(t)$ never reaches $\overline{W_{\rho}}$. Then

$$
v^{\prime}(t)=V^{\prime}(x(t)) \xi<-w(x(t)), \quad \text { for some } \xi \in K[f](x(t)) \text { and almost any } t>0 .
$$

The definition of function $w$ implies that

$$
w_{\min }=\min \left\{w(x), x \in \overline{W_{r}} \backslash W_{\rho}\right\}>0 .
$$

Therefore,

$$
v(t)=v(0)+\int_{0}^{t} v^{\prime}(t) d t<v(0)-w_{\min } t
$$

and $v(t)$ becomes negative, if $x(t)$ never reaches $\overline{W_{\rho}}$. Since $\rho \in(0, r)$ was chosen arbitrary, our conclusion implies that $x(t) \rightarrow \bar{x}$ as $t \rightarrow \infty$.


Figure 3.9: Implicitly defined surface $H(x)=0$ along with its tangent hyperplane constructed at $x=\bar{x}$.

### 3.4.1.2 The gradient of a surface

Lemma 6 Let $H \in C^{2}\left(\mathbb{R}^{n}, \mathbb{R}\right)$ and $\bar{x} \in \mathbb{R}^{n}$. If $H^{\prime}(\bar{x}) \neq(0, \ldots, 0)$, then

1) the hyperplane $\left\{x \in \mathbb{R}^{n}:\left\langle x-\bar{x}, H^{\prime}(\bar{x})\right\rangle=0\right\}$ is tangent to the surface $H(x)=0$.
2) the vector $H^{\prime}(\bar{x})$ points to the positive side of $H(x)=0$, i.e. $H\left(\alpha H^{\prime}(\bar{x})\right)>0$ for all $\alpha>0$ sufficiently small.

Fig. 3.9 illustrates the notations of Lemma 6 .
Proof. 1) By the assumption of the lemma there exists $i \in \overline{1, . ., n}$ such that $\frac{\partial H}{\partial x_{i}}(\bar{x}) \neq 0$. To shorten notations we consider the case where $i=1$ (the cases $i=2, \ldots, n$ can be considered by analogy). By the Implicit Function Theorem, there exists $h\left(x_{2}, \ldots, x_{n}\right)$ such that $H\left(h\left(x_{2}, \ldots, x_{n}\right), x_{2}, \ldots, x_{n}\right)=0$, for all $\left(x_{2}, \ldots, x_{n}\right)$ close to $\left(\bar{x}_{2}, \ldots, \bar{x}_{n}\right)$ and $h\left(\bar{x}_{2}, \ldots, \bar{x}_{n}\right)=\bar{x}_{1}$. By taking the derivative of this equality with respect to $x_{i}$ at $\left(\bar{x}_{2}, \ldots, \bar{x}_{n}\right)$, one gets

$$
\begin{equation*}
H^{\prime}(\bar{x}) l_{i}=0, \quad i=2, \ldots, n \tag{3.18}
\end{equation*}
$$

where

$$
l_{i}=\left(\frac{\partial h}{\partial x_{i}}\left(\bar{x}_{2}, \ldots, \bar{x}_{n}\right), 0, \ldots, 0,1,0, \ldots, 0\right)^{T} \text { and " } 1 \text { " is the value of the } i-\text { th component. }
$$

The linearly independent vectors $l_{2}, \ldots, l_{n}$ form a basis of the tangent hyperplane to $H(x)=0$ at $\bar{x}$. Therefore, by (3.18), $H^{\prime}(\bar{x})$ is normal to this hyperplane.
2) Given $\alpha>0$, we have $H\left(\alpha H^{\prime}(\bar{x})\right)-H(\bar{x})=H^{\prime}\left(\alpha_{*} H^{\prime}(\bar{x})\right) H^{\prime}(\bar{x})$, which converges to $H^{\prime}(\bar{x}) H^{\prime}(\bar{x})>0$ as $\alpha \rightarrow 0$. The latter can happen only if $H\left(\alpha H^{\prime}(\bar{x})\right)>0$ for all $\alpha>0$ sufficiently small.

### 3.4.1.3 Lyapunov functions for linear systems

Consider

$$
V(x)=x^{T} P x=\langle x, P x\rangle,
$$

where $P$ is a symmetric matrix.
Fact 1: The matrix $P$ has real eigenvalues. The function $V$ is positive for all $x \neq 0$ if and only if all eigenvalues of $P$ are positive.
Proof. See [76, Theorem 5.3.6] for the fact that the eigenvalues of $P$ are real. [76, Theorem 7.2.1] then says that $P$ has an orthonormal basis $U=\left(u_{1}, \ldots, u_{n}\right)$ of eigenvectors corresponding to the eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$. The diagonal matrix with the eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$ will be denoted by $\Lambda$. By the definition of the orthonormal basis, $U^{T} U=I$. Since $U$ diagonalizes $P$ to $\Lambda$ (see [76, Theorem 5.2.1]) we have $\Lambda=U^{T} P U$. Therefore,

$$
\langle x, P x\rangle=\left\langle x, U \Lambda U^{T} x\right\rangle=\left\langle U^{T} x, \Lambda U^{T} x\right\rangle=\sum_{i=1}^{n} \lambda_{i}\left\langle U^{T} x, U^{T} x\right\rangle=\sum_{i=1}^{n} \lambda_{i}\left\langle x, U U^{T} x\right\rangle=\sum_{i=1}^{n} \lambda_{i}\|x\|,
$$

from which the statement follows.
Consider a linear system of differential equations

$$
\begin{equation*}
\dot{x}=A x \tag{3.19}
\end{equation*}
$$

where $A$ is an $n \times n$-matrix.
Fact 2: For any $x, \xi \in \mathbb{R}^{n}$ we have $V^{\prime}(x) \xi=\xi^{T} P x+x^{T} P \xi$. In particular, $V^{\prime}(x) A x=-x^{T} Q x$, where

$$
\begin{equation*}
-Q=A^{T} P+P A \tag{3.20}
\end{equation*}
$$

is a symmetric matrix and, therefore, $V^{\prime}(x) A x<0$ for all $x \neq 0$ if and only if all the eigenvalues of $Q$ are positive.
Proof. If $\alpha \in \mathbb{R}$, then $\frac{d}{d \alpha} V(x+\alpha \xi)=V^{\prime}(x+\alpha \xi) \xi$ and so $\left.\frac{d}{d \alpha} V(x+\alpha \xi)\right|_{\alpha=0}=V^{\prime}(x) \xi$. On the other hand,

$$
\left.\frac{d}{d \alpha}\langle x+\alpha \xi, P(x+\alpha \xi)\rangle\right|_{\alpha=0}=\left.\langle\xi, P(x+\alpha \xi)\rangle\right|_{\alpha=0}+\left.\langle x+\alpha \xi, P \xi\rangle\right|_{\alpha=0}=\xi^{T} P x+x^{T} P \xi
$$

The matrix Q is symmetric because $\left(A^{T} P+P A\right)^{T}=P^{T} A+A^{T} P=P A+A^{T} P$. And the proof of the fact about eigenvalues follows the lines of the proof of Fact 1.

Fact 3: If the real parts of all eigenvalues of $A$ are negative, then, given any symmetric matrix $Q$ with positive eigenvalues, the equation $Q=A^{T} P+P A$ can always be solved for $P$. This solution $P$ is unique and has positive eigenvalues.

Proof. See [77, Sec 5.4, Theorem 42, p. 199] or [78, Theorem 3.6, p. 127].

Example 10 (an analogue of [78, Example 3.13, p. 128]) Find a Lyapunov function $V$ for linear system (3.19) with $A=\left(\begin{array}{ll}0 & 1 \\ -1 & -c\end{array}\right)$.
Solution. Solving 3.20 with $Q=\left(\begin{array}{ll}-1 & 0 \\ 0 & -1\end{array}\right)$ in $P$ one finds $P=\left(\begin{array}{ll}\frac{2+c^{2}}{2 c} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{c}\end{array}\right)$. This matrix $P$ allows to define a Lyapunov function as $V(x)=x^{T} P x$.

Fact 4: If $H(x)=V^{\prime}(x) \xi$, where $\xi \in \mathbb{R}^{n}$, then

$$
H^{\prime}(x)=V^{\prime \prime}(x) \xi=2(P \xi)^{T}
$$

Proof. Use the formula for $V^{\prime}(x) \xi$ of Fact 2, replace $x$ by $x+\alpha \zeta$, then differentiate in $\alpha$ and plug $\alpha=0$. This leads to $H^{\prime}(x) \zeta=2\langle\xi, P \zeta\rangle$ for any $\zeta \in \mathbb{R}^{n}$, that coincides with the statement.

### 3.4.1.4 Linear state feedback switching rule

For this switching rule we need to assume that (3.15) holds in a stronger sense. Specifically, we need that $\bar{x}$ is an asymptotically stable equilibrium for each of the two systems

$$
\begin{equation*}
\dot{x}=f^{L}(x)-f^{L}(\bar{x}) \quad \text { and } \quad \dot{x}=f^{R}(x)-f^{R}(\bar{x}) \tag{3.21}
\end{equation*}
$$

and, moreover, we assume that there exists a common (Lyapunov) function $V \in C^{2}\left(\mathbb{R}^{n}, \mathbb{R}\right)$, which verifies

$$
\begin{align*}
& V(x)=(x-\bar{x})^{T} P(x-\bar{x}) \text {, where } P \text { is symmetric with positive eigenvalues, } \\
& V^{\prime}(x)\left(f^{L}(x)-f^{L}(\bar{x})\right) \leq-\alpha\|x-\bar{x}\|^{2},  \tag{3.22}\\
& V^{\prime}(x)\left(f^{R}(x)-f^{R}(\bar{x})\right) \leq-\alpha\|x-\bar{x}\|^{2}, \quad \text { where } \alpha>0 \text { is a fixed constant. }
\end{align*}
$$

Given a vector $l^{\perp} \in \mathbb{R}^{n}$, define the open regions $D^{L}$ and $D^{R}$ as

$$
\begin{equation*}
D^{L}=\left\{x \in \mathbb{R}^{n}:\left\langle x-\bar{x}, l^{\perp}\right\rangle<0\right\}, \quad D^{R}=\left\{x \in \mathbb{R}^{n}:\left\langle x-\bar{x}, l^{\perp}\right\rangle>0\right\} \tag{3.23}
\end{equation*}
$$

so that system (3.1) takes the form

$$
\dot{x}=\left\{\begin{array}{ll}
f^{L}(x), & \text { if }\left\langle x-\bar{x}, l^{\perp}\right\rangle<0, \\
f^{R}(x), & \text { if }\left\langle x-\bar{x}, l^{\perp}\right\rangle>0 .
\end{array} \quad\right. \text { (system (3.1) in settings of Theorem 8) }
$$

Theorem 8 (ideas of [80], [79], 61, §3.4.1] refined and rephrased for nonlinear systems) Let $\bar{x}$ be a switched equilibrium for the vector fields $f^{L}$ and $f^{R}$, i.e. (3.13) holds. Assume that the common equilibrium $\bar{x}$ of systems (3.21) is asymptotically stable for each of the two systems. Assume that systems (3.21) admit a common quadratic Lyapunov function $V \in C^{2}\left(\mathbb{R}^{n}, \mathbb{R}\right)$,
i.e. (3.22) holds. Then $\bar{x}$ is an asymptotically stable switched equilibrium of switched system (3.1) with $D^{L}, D^{R}$ given by (3.23) and with $l^{\perp}$ defined by

$$
\left(l^{\perp}\right)^{T}=V^{\prime \prime}(\bar{x}) f^{L}(\bar{x})
$$

Specifically, the point $\bar{x}$ asymptotically attracts all Filippov solutions of (3.1) that originate in $W=\bigcup_{r>0}\left\{x \in \mathbb{R}^{n}: V(x)<r\right\}$.

Introduce two open sets

$$
\begin{aligned}
& \Omega_{\alpha}^{L}=\left\{x \in \mathbb{R}^{n}:-\alpha\|x-\bar{x}\|^{2}+V^{\prime}(x) f^{L}(\bar{x})<0\right\}, \\
& \Omega_{\alpha}^{R}=\left\{x \in \mathbb{R}^{n}:-\alpha\|x-\bar{x}\|^{2}+V^{\prime}(x) f^{R}(\bar{x})<0\right\} .
\end{aligned}
$$

Lemma 7 Consider $f^{L}, f^{R} \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$. Let $\bar{x}$ be a switched equilibrium for the vector fields $f^{L}$ and $f^{R}$, i.e. (3.13) holds. Assume that systems (3.21) admit a common quadratic Lyapunov function $V \in C^{2}\left(\mathbb{R}^{n}, \mathbb{R}\right)$ that satisfies (3.22) with some $\alpha>0$. Then $\Omega_{\alpha}^{L}$ and $\Omega_{\alpha}^{R}$ verify the following properties:

1) $\Omega^{L} \supset \Omega_{\alpha}^{L}, \Omega^{R} \supset \Omega_{\alpha}^{R}$,
2) $\bar{x} \in \partial \Omega_{\alpha}^{L}, \quad \bar{x} \in \partial \Omega_{\alpha}^{R}$,
3) both $\partial \Omega_{\alpha}^{L}$ and $\partial \Omega_{\alpha}^{R}$ are ellipsoids,
4) the hyperplane $\left\{x:\left\langle x-\bar{x}, l^{\perp}\right\rangle=0\right\}$ is tangent to both $\Omega_{\alpha}^{L}$ and $\Omega_{\alpha}^{R}$ at $\bar{x}$,
5) $\Omega_{\alpha}^{L} \subset\left\{x:\left\langle x-\bar{x}, l^{\perp}\right\rangle<0\right\}, \Omega_{\alpha}^{R} \subset\left\{x:\left\langle x-\bar{x}, l^{\perp}\right\rangle>0\right\}$.

The notations and statements of Lemma 7 are illustrated at Fig. 3.10,
Proof. 1) Let $x \in \Omega_{\alpha}^{L}$. Then

$$
\begin{aligned}
& V^{\prime}(x) f^{L}(x)=V^{\prime}(x)\left(f^{L}(x)-f^{L}(\bar{x})\right)+V^{\prime}(x) f^{L}(\bar{x}) \leq \\
& \leq-\alpha\|x-\bar{x}\|^{2}+V^{\prime}(x) f^{L}(\bar{x})<0
\end{aligned}
$$

Therefore, $x \in \Omega^{L}$. The proof for $\Omega_{\alpha}^{R}$ and $\Omega_{\alpha}^{R}$ is analogous.
2) In the proof of item 3) of Lemma 5 we established that $V^{\prime}(\bar{x})=0$. This implies 2).
3) We execute the proof for $\bar{x}=0$. The proof in the general case doesn't differ. The change of the coordinates $y=x-\Delta$ transforms the equation

$$
-\alpha\|x-\bar{x}\|^{2}+V^{\prime}(x) f^{L}(\bar{x})=0
$$

into

$$
-\alpha\|y\|^{2}-2 \alpha\langle\Delta, y\rangle+2\left\langle P f^{L}(0), y\right\rangle-\alpha\|\Delta\|^{2}+2\left\langle\Delta, P f^{L}(0)\right\rangle=0
$$



Figure 3.10: The sets $\Omega^{L}, \Omega_{\alpha}^{L}$ (left figure) and $\Omega^{R}, \Omega_{\alpha}^{R}$ (right figure) relative to the hyperplane $\left\{x:\left\langle x-\bar{x}, l^{\perp}\right\rangle=0\right\}$. At this figure:
Left and Right: The straight line $=$ the hyperplane $\left\{x:\left\langle x-\bar{x}, l^{\perp}\right\rangle=0\right\}$,
Left: The inner dotted line $=\partial \Omega^{L}$. The exterior of the inner dotted line $=\Omega^{L}$.
Left: The outer dotted line $=\partial \Omega_{\alpha}^{L}$. The exterior of the outer dotted line $=\Omega_{\alpha}^{L}$.
Right: The inner dotted line $=\partial \Omega^{R}$. The exterior of the inner dotted line $=\Omega^{R}$.
Right: The outer dotted line $=\partial \Omega_{\alpha}^{R}$. The exterior of the outer dotted line $=\Omega_{\alpha}^{R}$.
If $\Delta=\frac{P f^{L}(0)}{\alpha}$, then we further get

$$
-\alpha\|y\|^{2}-\frac{1}{\alpha}\left\|P f^{L}(0)\right\|^{2}+\frac{2}{\alpha}\left\|P f^{L}(0)\right\|^{2}=0
$$

which is the equation of ellipsoid centered at 0 and radius $\frac{1}{\alpha^{2}}\left\|P f^{L}(0)\right\|^{2}$.
The proof for $\partial \Omega_{\alpha}^{R}$ is analogous.
4) This follows from the equality

$$
\left.\frac{d}{d x}\left(-\alpha\|x-\bar{x}\|^{2}+V^{\prime}(x) f^{L}(\bar{x})\right)\right|_{x=\bar{x}}=l^{\perp}
$$

and the property (3.13) of switched equilibrium.
5) Let $H(x)=-\alpha\|x-\bar{x}\|^{2}+V^{\prime}(x) f^{L}(\bar{x})$. The interior of the ellipsoid $\partial \Omega_{\alpha}^{L}$ corresponds to $H(x)>0$. Therefore, the exterior of the ellipsoid $\partial \Omega_{\alpha}^{L}$ (which, by definition, coincides with the set $\left.\Omega_{\alpha}^{L}\right)$ is where $H(x)<0$. Analogously for $\Omega_{\alpha}^{R}$.

The proof of the lemma is complete.
Remark: The sets $\Omega_{\alpha}^{L}$ and $\Omega_{\alpha}^{R}$ are introduced because $\partial \Omega^{L}$ and $\partial \Omega^{R}$ are not necessary ellipsoids. I don't see that $\partial \Omega^{L}$ and $\partial \Omega^{R}$ are ellipsoids even in $\mathbb{R}^{2}$ because the "quadratic" parts in the definition of $V^{L}$ and $V^{R}$ are defined over fully nonlinear functions $V^{\prime}(x)\left(f^{L}(x)-\right.$ $\left.f^{L}(\bar{x})\right)$ and $\left.V^{\prime}(x)\left(f^{R}(x)-f^{R}(\bar{x})\right)\right)$.

Proof of Theorem 8. We will show that the conditions of Theorem 7 hold with

$$
w(x)= \begin{cases}-V^{\prime}(x) f^{L}(x), & x \in D^{L} \\ -\max \left\{V^{\prime}(x) f^{L}(x), V^{\prime}(x) f^{R}(x)\right\}, & x \in S \\ -V^{\prime}(x) f^{R}(x), & x \in D^{R}\end{cases}
$$

If $x \in \overline{D^{L}} \backslash\{\bar{x}\}$, then $x \in \Omega_{\alpha}^{L} \subset \Omega^{L}$ by items 5) and 1) of Lemma 7 , which implies $w(x)<0$. Analogously, $w(x)>0$, if $x \in \overline{D^{R}} \backslash\{\bar{x}\}$. This implies that $\max _{x \in \partial B_{\rho}(\bar{x})} w(x)$ is a positive function that approaches 0 as $\rho \rightarrow 0$. Since $K[f](x)=\left\{f^{L}(x)\right\}$, when $x \in D^{L}$, and $K[f](x)=$ $\left\{f^{R}(x)\right\}$, when $x \in D^{R}$, then condition $V^{\prime}(x) \xi \leq-w(x)$ of Theorem 7 holds for $D^{L} \cup D^{R}$.

Consider $\left\langle x-\bar{x}, l^{\perp}\right\rangle=0$. Then each $\xi \in K[f](x)$ has the form $\xi=\lambda f^{L}(x)+(1-\lambda) f^{R}(x)$, where $\lambda$ is a constant from the interval $[0,1]$. We have

$$
V^{\prime}(x) \xi=\lambda V^{\prime}(x) f^{L}(x)+(1-\lambda) V^{\prime}(x) f^{R}(x) \leq \max \left\{V^{\prime}(x) f^{L}(x), V^{\prime}(x) f^{R}(x)\right\}=-w(x)
$$

that completes the proof of the theorem.

### 3.4.1.5 Example

## S17 revision of Example 11:

http://www.utdallas.edu/~ makarenkov/switching-line-design-example.pdf
Example 11 For the vector fields $f^{L}$ and $f^{R}$ given, propose a state feedback linear switching strategy that stabilizes the switched equilibrium $\bar{x}=0$. By the other words, find the vector $l^{\perp}$, such that $\bar{x}$ is a globally asymptotically stable switched equilibrium of the system

$$
\begin{gathered}
\dot{x}= \begin{cases}f^{L}(x), & \text { if }\left\langle x, l^{\perp}\right\rangle>0, \\
f^{R}(x), & \text { if }\left\langle x, l^{\perp}\right\rangle<0 .\end{cases} \\
f^{L}(x)=A^{L} x+a^{L}, \quad A^{L}=A=\left(\begin{array}{ll}
0 & 1 \\
-1 & -1
\end{array}\right), \quad a^{L}=\binom{0}{L}, \quad \text { where } L<0, \\
f^{R}(x)=A^{R} x+a^{R}, \quad A^{R}=A=\left(\begin{array}{ll}
0 & 1 \\
-1 & -1
\end{array}\right), \quad a^{R}=\binom{0}{R}, \quad \text { where } R>0 .
\end{gathered}
$$

Note, the system of Example 11 can be rewritten as

$$
\dot{x}=A x+\binom{0}{u(x)}, \quad \text { where } u(x)= \begin{cases}L, & \text { if } l_{1}^{\perp} x_{1}+l_{2}^{\perp} x_{2}>0  \tag{3.24}\\ R, & \text { if } l_{1}^{\perp} x_{1}+l_{2}^{\perp} x_{2}<0\end{cases}
$$

i.e. $u(x)$ switches between $L$ and $R$ when the Filippov solution $x$ crosses the line $l_{1}^{\perp} x_{1}+l_{2}^{\perp} x_{2}=$ 0 . The whole problem is about finding the line, i.e. the vector $l^{\perp}$ that renders $\bar{x}=0$ a globally asymptotically stable switched equilibrium of (3.24).

Solution. The point $\bar{x}=0$ is a switched equilibrium because the vectors $a^{L}$ and $a^{R}$ are opposite each other. Systems (3.21) both reduce to

$$
\dot{x}=A x,
$$

whose Lyapunov function can be taken (see $\S$ 3.4.1.3) as

$$
V(x)=x^{T} P x \quad \text { with } \quad P=\left(\begin{array}{cc}
\frac{3}{2} & \frac{1}{2} \\
\frac{1}{2} & 1
\end{array}\right)
$$

Based on Fact 4 of $\S$ 3.4.1.3, we compute $l^{\perp}$ as

$$
\left(l^{\perp}\right)^{T}=V^{\prime \prime}(0) f^{R}(0)=V^{\prime \prime}(0)\binom{0}{R}=2\left(P\binom{0}{R}\right)^{T}=(R, 2 R)
$$

(Here we used the fact that $f^{R}$ of this example corresponds to $f^{L}$ of Theorem 8). Plugging this value of $l^{\perp}$ into (3.24), one gets

$$
u(x)=\left\{\begin{array}{ll}
L, & \text { if } R \cdot x_{1}+2 R \cdot x_{2}>0, \\
R, & \text { if } R \cdot x_{1}+2 R \cdot x_{2}<0
\end{array} \quad \text { or } \quad u(x)= \begin{cases}L, & \text { if } x_{1}+2 x_{2}>0 \\
R, & \text { if } x_{1}+2 x_{2}<0\end{cases}\right.
$$

The asymptotic stability of switched equilibrium $\bar{x}=0$ of (3.24) follows by applying Theorem 8 .

The point $\bar{x}=0$ attracts all Filippov solutions of (3.24) because for the set $W_{r}$ (defined in Theorem 88 one has $\bigcup_{r>0} W_{r}=\mathbb{R}^{2} \backslash\{0\}$, i.e. the level sets of the Lyapunov function $V$ fill in the whole space.

### 3.4.1.6 Nonlinear state feedback switching rule

If $\bar{x}$ is not an asymptotically stable equilibrium for each of the systems (3.21), then I am not aware of sufficient conditions to ensure that $\partial \Omega^{L}$ and $\partial \Omega^{R}$ are separated by a hyperplane (which would play a role of the switching hyperplane, if exists), i.e. to ensure that option 2 of Fig. 3.8 takes place. However, the following approach can always be used to design a (nonlinear) switching surface $S$ (i.e. to design the sets $D^{L}$ and $D^{R}$ ) that renders $\bar{x}$ asymptotically stable.

Proposition 10 (ideas of [80], [79], [61, §3.4.1]) Let $\bar{x}$ be a switched equilibrium for the vector fields $f^{L}$ and $f^{R}$, i.e. (3.13) holds. Assume that the equilibrium $\bar{x}$ of system (3.14) is asymptotically stable and consider the respective (Lyapunov) function $V \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}\right)$ that satisfies (3.15). Let $\Omega^{0} \subset \mathbb{R}^{n}$ be an arbitrary open set such that

1) $\bar{x} \in \partial \Omega^{0}$,


Figure 3.11: Relative locations of sets $\Omega^{0}$ (striped light gray), $\Omega^{R}$ (all light gray, i.e. the union of solid light gray and striped light gray), $\Omega^{L}$ (dark gray).
2) $\overline{\Omega^{0}} \backslash\{\bar{x}\} \subset \Omega^{R}$,
3) $\left(\mathbb{R}^{n} \backslash \Omega^{0}\right) \backslash\{\bar{x}\} \subset \Omega^{L}$.

Then $\bar{x}$ is an asymptotically stable switched equilibrium of switched system (3.1) with $D^{L}$ and $D^{R}$ defined by

$$
D^{L}=\mathbb{R}^{n} \backslash \overline{\Omega^{0}}, \quad D^{R}=\Omega^{0}
$$

Specifically, the point $\bar{x}$ asymptotically attracts all Filippov solutions of (3.1) that originate in $W=\bigcup_{r>0}\left\{x \in \mathbb{R}^{n}: V(x)<r\right\}$.

The proof of the Proposition follows the lines of the proof of Theorem 8. The notations and possible location of the set $\Omega^{0}$ relative to $\Omega^{R}$ and $\Omega^{L}$ are illustrated in Fig. 3.11.

### 3.4.2 Hybrid Feedback Switching Rule

I am still about to put a precise definition of solution and a suitable version of the Lyapunov stability theorem here. And I will of course type the scans. Please refer to the scans meanwhile:
http://www.utdallas.edu/ makarenkov/hybrid_switching_scan.pdf

This material follows the ideas of [80], [79], [61, §3.4.1].

### 3.5 Time-dependent switching. Dwell time

### 3.5.1 Global convergence

Consider a switched system

$$
\begin{equation*}
\dot{x}=f_{u(t)}(x), \tag{3.25}
\end{equation*}
$$

where $f_{u} \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$ and $u:\left[t_{0}, \infty\right) \mapsto \mathbb{R}$ is a piecewise constant right-continuous control signal. Assume that for each fixed $u \in \mathbb{R}$ the system

$$
\dot{x}=f_{u}(x)
$$

admits a unique equilibrium $x_{u} \in \mathbb{R}^{n}$. Let $V_{u}$ be the Lyapunov function of equilibrium $x_{u}$ which verifies the following two conditions

$$
\begin{align*}
& \alpha_{u}\left(\left\|x-x_{u}\right\|\right) \leq V_{u}(x) \leq \beta_{u}\left(\left\|x-x_{u}\right\|\right), \quad x \in \mathbb{R}^{n}  \tag{3.26}\\
& \left(V_{u}\right)^{\prime}(x) f_{u}(x) \leq-k_{u} V_{u}(x), \quad x \in \mathbb{R}^{n}, \tag{3.27}
\end{align*}
$$

where $\alpha, \beta$ are strictly monotonically increasing functions with $\alpha_{u}(0)=\beta_{u}(0)$, and $k_{u} \in \mathbb{R}$. Note, we don't assume that $k_{u}>0$, so we don't assume that $x_{u}$ are stable equilibria.
Let $t_{1}<t_{2}<t_{3} \ldots$ be the points of discontinuity of the function $t \mapsto u(t)$.
Definition 15 The control signal $t \mapsto u(t)$ has an average weighted dwell time $\tau_{a w}$, if for any $i \in \mathbb{N}$, there exists $N_{i} \in \mathbb{N}$ such that

$$
\begin{equation*}
\frac{1}{N_{i}}\left(k_{u\left(t_{i}\right)}\left(t_{i+1}-t_{i}\right)+k_{u\left(t_{i+1}\right)}\left(t_{i+2}-t_{i+1}\right)+\ldots+k_{u\left(t_{i+N_{i}-1}\right)}\left(t_{i+N_{i}}-t_{i+N_{i}-1}\right)\right) \geq \tau_{a w} \tag{3.28}
\end{equation*}
$$

The inequality of Definition 15 can be reformulated as

$$
\int_{t_{i}}^{t_{i+N_{i}}} k_{u}(s) d s \geq \tau_{a w}
$$

For a given $\varepsilon>0$ let the neighborhood $N_{u}^{\varepsilon}$ of the equilibrium $x_{u}$ be defined as

$$
N_{u}^{\varepsilon}=\left\{x: V_{u}(x) \leq \varepsilon\right\} .
$$

Theorem 9 Let $\varepsilon>0$ be a given constant and let $u:\left[t_{0}, \infty\right) \mapsto \mathbb{R}$ be a piecewise constant right-continuous control. Assume that there exists $\mu>0$ such that

$$
\begin{equation*}
\frac{V_{u\left(t_{i+1}\right)}(x)}{V_{u\left(t_{i}\right)}(x)} \leq \mu, \quad x \in \mathbb{R}^{n} \backslash N_{u\left(t_{i}\right)}^{\varepsilon}, \quad i \in \mathbb{N} . \tag{3.29}
\end{equation*}
$$

If the average weighted dwell time $\tau_{\text {aw }}$ of the control signal $t \mapsto u(t)$ satisfies

$$
\tau_{a w}>\ln \mu,
$$

then, for any solution solution $x$ of switched system (3.25), there exists $i \in \mathbb{N}$ such that $x\left(t_{i}\right) \in N_{u\left(t_{i}\right)}^{\varepsilon}$.
The proof of the theorem follows the ideas of [90] and it is based base on the following lemma about the rate of convergence of quadratic Lyapunov functions to zero.
Lemma 8 If $f \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$ and $V \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}\right)$ satisfy

$$
V^{\prime}(x) f(x) \leq-k V(x)
$$

for some constant $k \in \mathbb{R}$ and any $x \in W \subset \mathbb{R}^{n}$, then

$$
\begin{equation*}
V(x(\tau)) \leq e^{-k(\tau-s)} V(x(s)), \tag{3.30}
\end{equation*}
$$

for any solution $t \mapsto x(t)$ of $\dot{x}=f(x)$ and any $\tau>s$ such that $x([s, \tau]) \subset W$.

Proof. The assumption of the theorem implies that

$$
\dot{v}(t) \leq-k v(t), \quad t \in[s, \tau]
$$

for $v(t)=V(x(t))$. By the comparison lemma (lemma 1) we have

$$
\begin{equation*}
v(t) \leq w(t), \quad t \in[s, \tau] \tag{3.31}
\end{equation*}
$$

where $w$ is the solution of

$$
\dot{w}(t)=-k w(t)
$$

with the initial condition $w(\tau)=x(\tau)$. Inequality (3.31) coincides with the required (3.30). The proof is complete.
Proof of Theorem 9. Using inequalities (3.29) and (3.30), one concludes

$$
\begin{equation*}
V_{u\left(t_{i+1}\right)}\left(x\left(t_{i+1}\right)\right) \leq \mu V_{u\left(t_{i}\right)}\left(x\left(t_{i+1}\right)\right) \leq \mu e^{-k_{u\left(t_{i}\right)}\left(t_{i+1}-t_{i}\right)} V_{u\left(t_{i}\right)}\left(x\left(t_{i}\right)\right), \text { if } x\left(t_{i+1}\right) \notin N_{u\left(t_{i+1}\right)}^{\varepsilon} . \tag{3.32}
\end{equation*}
$$

Assume that the statement of the theorem doesn't hold. Therefore

$$
\begin{equation*}
x\left(t_{i+1}\right) \notin N_{u\left(t_{i+1}\right)}^{\varepsilon}, \quad \text { for all } i \in \mathbb{N}, \tag{3.33}
\end{equation*}
$$

and so inequality (3.32) holds for all $i \in \mathbb{N}$. By applying inequality (3.32) successively $N_{i}$ times we get

$$
V_{u\left(t_{i+N_{i}}\right)}\left(x\left(t_{i+N_{i}}\right)\right) \leq \mu_{i}^{N} e^{\left(-k_{u\left(t_{i}\right)}\left(t_{i+1}-t_{i}\right)-k_{u\left(t_{i+1}\right)}\left(t_{i+2}-t_{i+1}\right)-\ldots-k_{u\left(t_{i+N_{i}-1}\right)}\left(t_{i+N_{i}}-t_{i+N_{i}-1}\right)\right)} V_{u\left(t_{i}\right)}\left(x\left(t_{i}\right)\right)
$$ and taking into account (3.28),

$$
V_{u\left(t_{i+N_{i}}\right)}\left(x\left(t_{i+N_{i}}\right)\right) \leq \mu_{i}^{N} e^{-\tau_{a w} N_{i}} V_{u\left(t_{i}\right)}\left(x\left(t_{i}\right)\right)=\left(\mu e^{-\tau_{a w}}\right)^{N_{i}} V_{u\left(t_{i}\right)}\left(x\left(t_{i}\right)\right), \quad i \in \mathbb{N}
$$

Applying same process to $V_{u\left(t_{1+N_{1}}\right)}\left(x\left(t_{1+N_{1}}\right)\right)$ we obtain

$$
V_{u\left(t_{1+N_{1}+\tilde{N}_{1}}\right)}\left(x\left(t_{1+N_{1}+\tilde{N}_{1}}\right)\right) \leq\left(\mu e^{-\tau_{a w}}\right)^{N_{1}+\tilde{N}_{1}} V_{u\left(t_{1}\right)}\left(x\left(t_{1}\right)\right)
$$

Repeating iteratively, we will get a sequence of increasing time instances in the left-hand-side and a sequence of increasing exponents in the right-hand-side. Since $\mu e^{-\tau_{a w}}<1$ we have $V_{u\left(t_{1+k}\right)}\left(x\left(t_{1+k}\right)\right) \rightarrow 0$ as $k \rightarrow \infty$, which implies that $x\left(t_{1+k}\right) \in N_{u\left(t_{1+k}\right)}^{\varepsilon}$ for some $k \in \mathbb{N}$. This contradicts (3.33) and completes the proof.

Example 12 For a right-continuous $u: \mathbb{R} \rightarrow \mathbb{Z}$, consider the system

$$
\begin{align*}
& \dot{x}=\left(\begin{array}{cc}
-1 & -1 \\
1 & -1
\end{array}\right) x+\binom{1}{-1} u(t), \quad \text { if } u(t) \text { is even, }  \tag{3.34}\\
& \dot{x}=-\left(\begin{array}{cc}
-1 & -1 \\
1 & -1
\end{array}\right) x-\binom{1}{-1} u(t), \quad \text { if } u(t) \text { is odd, }
\end{align*}
$$

whose unique equilibrium for each $u(t) \in \mathbb{Z}$ is given by

$$
x_{u(t)}=\binom{u(t)}{0} .
$$

Let $\varepsilon=0.04$ and $V_{u}(x)=\left\|x-x_{u}\right\|^{2}$. Find the average dwell time $\tau_{\text {aw }}$ such that if $t \mapsto u(t)$ satisfies $\left|u\left(t_{i-1}\right)-u\left(t_{i}\right)\right|=1, i \in \mathbb{N}$, and has average weighted dwell time $\tau_{\text {aw }}$, then any solution $x$ of (3.34) verifies $x\left(t_{i}\right) \in N_{u\left(t_{i}\right)}^{\varepsilon}=\left\{x \in \mathbb{R}^{2}:\left\|x-x_{u\left(t_{i}\right)}\right\|^{2} \leq \varepsilon\right\}$ for some $i \in \mathbb{N}$.

Solution. We are simply supposed to find the constant $\mu$ for which the condition (3.29) of Theorem 9 holds. Denoting $x_{i}=x_{u\left(t_{i}\right)}$, we have

$$
\frac{V_{u\left(t_{i+1}\right)}(x)}{V_{u\left(t_{i}\right)}(x)}=\frac{\left\|x-x_{i+1}\right\|^{2}}{\left\|x-x_{i}\right\|^{2}} \leq \frac{\left(\left\|x-x_{i}\right\|+\left\|x_{i}-x_{i+1}\right\|\right)^{2}}{\left\|x-x_{i}\right\|^{2}}
$$

Since the scalar function $f(t)=\frac{(t+c)^{2}}{t^{2}}$ decreases on $t>0$ for each fixed $c>0$,

$$
\begin{aligned}
\frac{\left(\left\|x-x_{i}\right\|+\left\|x_{i}-x_{i+1}\right\|\right)^{2}}{\left\|x-x_{i}\right\|^{2}} & \leq \max _{y:\left\|y-x_{i}\right\|^{2}=\varepsilon} \frac{\left(\left\|y-x_{i}\right\|+\left\|x_{i}-x_{i+1}\right\|\right)^{2}}{\left\|y-x_{i}\right\|^{2}}= \\
& =\frac{\left(\sqrt{\varepsilon}+\left\|x_{i}-x_{i+1}\right\|\right)^{2}}{\varepsilon}, \quad \text { for all } x \in \mathbb{R}^{2} \backslash N_{u(i)}^{\varepsilon}
\end{aligned}
$$

Since the control signal $t \mapsto u(t)$ is chosen such that $\left|u\left(t_{i-1}\right)-u\left(t_{i}\right)\right|=1$, we have

$$
\begin{equation*}
\left\|x_{i}-x_{i+1}\right\|=\left\|\binom{u\left(t_{i}\right)-u\left(t_{i-1}\right)}{0}\right\|=\left|u\left(t_{i}\right)-u\left(t_{i-1}\right)\right|=1 \tag{3.35}
\end{equation*}
$$

Finally,

$$
\begin{equation*}
\frac{V_{u\left(t_{i+1}\right)}(x)}{V_{u\left(t_{i}\right)}(x)} \leq \frac{(0.2+1)^{2}}{0.04}=36 \tag{3.36}
\end{equation*}
$$

Therefore, $\mu$ can be taken as $\mu=36$, whose natural logarithm is $\ln \mu \approx 3.583$.
Answer: $\tau_{\text {aw }} \geq 3.59$.
There is big step to go from knowing the average weighted dwell time $\tau_{a w}$ to knowing the specific control signals $t \mapsto u(t)$.

Example 13 Consider the switched system of Example 12. Find the conditions for $T_{\text {stab }}>0$ and $T_{\text {unstab }}>0$ such that the control signal

$$
u(t)= \begin{cases}m, & \text { if } t \in\left[0, T_{\text {stab }}\right)+m T \text { for some } m \in \mathbb{Z},  \tag{3.37}\\ m+1, & \text { if } t \in\left[T_{\text {stab }}, T\right)+m T \text { for some } m \in \mathbb{Z}\end{cases}
$$

where $T=T_{\text {stab }}+T_{\text {unstab }}$, has the average weighted dwell time $\tau_{a w}=3.59$.

Solution. When $u\left(t_{i}\right)$ is even the Lyapunov function $V_{u\left(t_{i}\right)}$ verifies the inequality

$$
\frac{d}{d t} V_{u\left(t_{i}\right)}(x(t)) \leq-2 V_{u\left(t_{i}\right)}(x(t))
$$

Therefore, When $u\left(t_{i}\right)$ is odd one has

$$
\frac{d}{d t} V_{u\left(t_{i}\right)}(x(t)) \leq 2 V_{u\left(t_{i}\right)}(x(t))
$$

Therefore, $k_{u\left(t_{i}\right)}=2$ or $k_{u\left(t_{i}\right)}=-2$ according to whether $u\left(t_{i}\right)$ is even or odd. To fulfill the requirement of Definition 15 we consider $N=2$. Since, any two successive constant phases of control (3.37) always include an even and an odd value, inequality (3.28) takes the form

$$
\frac{1}{2}\left(k_{u\left(t_{i}\right)}\left(t_{i+1}-t_{i}\right)+k_{u\left(t_{i+1}\right)}\left(t_{i+2}-t_{i+1}\right)\right)=\frac{1}{2}\left(2 T_{\text {stab }}-2 T_{\text {unstab }}\right) \geq 3.59
$$

Answer: $T_{\text {stab }}-T_{\text {unstab }} \geq 3.59$.
Combining Examples 12 and 13 we conclude that if $T_{\text {stab }}-T_{\text {unstab }} \geq \tau_{a w}$, where $\tau_{a w}=3.59$, then any solution $x$ of (3.34) verifies $x\left(t_{i}\right) \in\left\{x \in \mathbb{R}^{2}:\left\|x-x_{u\left(t_{i}\right)}\right\| \leq 0.2\right\}$ for some $i \in \mathbb{N}$.

Example 14 Consider the switched system (3.34) with a right-continuous control $u: \mathbb{R} \rightarrow$ $\{0,1,2,3\}$. Let $\varepsilon=0.04$ and $V_{u}(x)=\left\|x-x_{u}\right\|^{2}$.

1) Find $\tau_{\text {aw }}$ such that if $t \mapsto u(t)$ has an average weighted dwell time $\tau_{a w}$, then any solution $x$ of (3.34) satisfies $x\left(t_{i}\right) \in N_{0}^{\varepsilon} \cup N_{1}^{\varepsilon} \cup N_{2}^{\varepsilon} \cup N_{3}^{\varepsilon}$ for some $i \in \mathbb{N}$.
2) Assume, that for each $i \in \mathbb{N}$ such that $u\left(t_{i}\right)$ is odd, we have the following:
(i) both $u\left(t_{i+1}\right)$ and $u\left(t_{i+2}\right)$ are even,
(ii) $t_{i+1}-t_{i} \leq T_{1}, t_{i+2}-t_{i+1} \geq T_{2}, t_{i+3}-t_{i+2} \geq T_{3}$, where $T_{1}>0, T_{2}>0$, and $T_{3}>0$ don't depend on $i$.

Find a relationship for $T_{1}, T_{2}$, and $T_{3}$ such that the control signal $t \mapsto u(t)$ has an average weighted dwell time $\tau_{a w}$.

Solution. 1) The solution follows the lines of Example 12 with the only difference that $\left|u\left(t_{i}\right)-u\left(t_{i-1}\right)\right|$ in (3.35) can now be as large as $\left|u\left(t_{i}\right)-u\left(t_{i-1}\right)\right| \leq 3$. Thus, (3.36) changes to

$$
\frac{V_{u\left(t_{i+1}\right)}(x)}{V_{u\left(t_{i}\right)}(x)} \leq \frac{(0.2+3)^{2}}{0.04}=256
$$

Since $\ln (256) \approx 5.545$, an average weighted dwell time $\tau_{a w}$ can be taken as $\tau_{a w} \geq 5.55$.
2) We will seek for $T_{1}, T_{2}$, and $T_{3}$ such that Definition 15 holds with $N=3$. We have

$$
\begin{aligned}
& \frac{1}{3}\left(k_{u\left(t_{i}\right)}\left(t_{i+1}-t_{i}\right)+k_{u\left(t_{i+1}\right)}\left(t_{i+2}-t_{i+1}\right)+k_{u\left(t_{i+2}\right)}\left(t_{i+3}-t_{i+2}\right)\right)= \\
& =\frac{1}{3}\left(-2\left(t_{i+1}-t_{i}\right)+2\left(t_{i+2}-t_{i+1}\right)+2\left(t_{i+3}-t_{i+2}\right)\right) \geq \frac{1}{3}\left(-2 T_{1}+2 T_{2}+2 T_{3}\right)
\end{aligned}
$$

The left-hand-side of this inequality will be greater than $\tau_{a w}$, if $\frac{1}{3}\left(-2 T_{1}+2 T_{2}+2 T_{3}\right) \geq \tau_{a w}$, which gives $\frac{1}{3}\left(-2 T_{1}+2 T_{2}+2 T_{3}\right) \geq 5.55$, or $-T_{1}+T_{2}+T_{3} \geq 16.65$.
Answer: $\tau_{a w} \geq 5.55,-T_{1}+T_{2}+T_{3} \geq 16.65$.
When $k_{u\left(t_{i}\right)}=k_{u\left(t_{i-1}\right)}, i \in \mathbb{N}$, the inequality (3.28) reduces to

$$
\frac{k}{N}\left(t_{i+N}-t_{i}\right) \geq \tau_{a w}
$$

Definition 16 The control signal $t \mapsto u(t)$ has an average dwell time $\tau_{a}$, if, for any $i \in \mathbb{N}$, there exists $N_{i} \in \mathbb{N}$ such that

$$
\begin{equation*}
\frac{1}{N_{i}}\left(t_{i+N_{i}}-t_{i}\right) \geq \tau_{a} . \tag{3.38}
\end{equation*}
$$

Corollary 3 (Alpcan-Basar [90]) Let $\varepsilon>0$ be a given constant and let $u:\left[t_{0}, \infty\right) \mapsto \mathbb{R}$ be a piecewise constant right-continuous control. Assume that there exists $\mu>0$ and $k>0$ such that both (3.29) and

$$
k_{u\left(t_{i}\right)} \geq k, \quad i \in \mathbb{N}
$$

hold. If the average dwell time $\tau_{a}$ of the control signal $t \mapsto u(t)$ satisfies

$$
\tau_{a}>\frac{\ln \mu}{k}
$$

then, for any solution solution $x$ of switched system 3.25), there exists $i \in \mathbb{N}$ such that $x\left(t_{i}\right) \in N_{u\left(t_{i}\right)}^{\varepsilon}$.

Example 15 Consider the switched system (3.34) with a right-continuous control $u: \mathbb{R} \rightarrow$ $\{0,2,4\}$. Let $\varepsilon=0.04$ and $V_{u}(x)=\left\|x-x_{u}\right\|^{2}$.

1) Find an average dwell time $\tau_{a}$ for the control signal $t \mapsto u(t)$ such that any solution $x$ of (3.34) reaches $N_{0}^{\varepsilon} \cup N_{2}^{\varepsilon} \cup N_{4}^{\varepsilon}$ at some time.
2) Estimate the maximal number of switchings allowed over any time-interval of 10 units for which the control signal $t \mapsto u(t)$ has an average dwell time $\tau_{a}$.

Solution. 1) Following the solution of Example 12 , we find

$$
\mu=\frac{(0.2+4)^{2}}{0.04}=441, \quad \ln \mu \approx 6.09, \quad \tau_{a} \geq 3.05
$$

2) From formula (3.38)

$$
\frac{1}{N} \cdot 10 \geq 3.05, \quad N \leq \frac{10}{3.05} \approx 3.28
$$

Answer: $\tau_{a}=3.05, N=3$.
Note, Corollary 3 cannot be applied to Examples 13 and 14 because some of the $k_{u\left(t_{i}\right)}$ in those examples are negative.

Definition 17 The control signal $t \mapsto u(t)$ has a dwell time $\tau_{d}$, if

$$
\begin{equation*}
t_{i+1}-t_{i} \geq \tau_{d}, \quad i \in \mathbb{N} \tag{3.39}
\end{equation*}
$$

A dwell time $\tau_{d}$ of control input $t \mapsto u(t)$ is also its average dwell time $\tau_{a}$, but the converse is not. In particular, $\tau_{a}>\frac{\ln \mu}{k}$ doesn't imply $\tau_{d}>\frac{\ln \mu}{k}$. For example, a 10-periodic control

$$
u(t)= \begin{cases}0, & t \in[0,1) \\ 2, & t \in[1,2) \\ 4, & t \in[2,10)\end{cases}
$$

has an average dwell time $\tau_{a}=\frac{1}{3}\left(t_{3}-t_{0}\right) \approx 3.33$, which satisfies the condition $\tau_{a} \geq 3.05$ obtained in Example 15. At the same time, the dwell time $\tau_{d}$ of the same control signal is at most $\tau_{d}=1$, which fails the condition $\tau_{d} \geq 3.05$.
For a right-continuous piecewise constant control $u:\left[t_{0}, \infty\right) \mapsto \mathbb{R}$, let $T_{\text {stab }}(t)$ and $T_{\text {unstab }}(t)$ be the total activation times of stable $\left(k_{u(t)}>0\right)$ and unstable $\left(k_{u(t)}<0\right)$ subsystems of switched system (3.25) on the time interval $\left[t_{0}, t\right]$, i.e.

$$
T_{s}\left(t_{1}, t_{2}\right)=\int_{t_{1}}^{t_{2}} \mathbb{1}_{s: u(s)>0}(\tau) d \tau, \quad T_{u}\left(t_{1}, t_{2}\right)=\int_{t_{1}}^{t_{2}} \mathbb{1}_{s: u(s)<0}(\tau) d \tau
$$

where $\mathbb{1}_{A}(x)=\left\{\begin{array}{ll}1, & \text { if } x \in A, \\ 0, & \text { otherwise }\end{array}\right.$ is the indicator function of set $A \subset \mathbb{R}$.
The next corollary extends the result of Zhai et al [101] by providing an explicit formula for the coefficient $k$.

Corollary 4 Let $\varepsilon>0$ be a given constant and let $u:\left[t_{0}, \infty\right) \mapsto \mathbb{R}$ be a piecewise constant right-continuous control. Assume that there exists $\mu>0$ such that (3.29) holds. Assume that there exist $k_{u}<0$ and $k_{s}>0$ such that, for any $t \geq t_{0}$,

$$
k_{u} \leq k_{u(t)}, \text { if } k_{u(t)}<0, \quad \text { and } \quad k_{s} \leq k_{u(t)} \text {, if } k_{u(t)}>0
$$

Finally, assume that for any $t_{1} \geq t_{0}$ there exists $t_{2} \geq t_{1}$ such that

$$
\begin{equation*}
-\frac{k_{u} T_{u}\left(t_{1}, t\right)}{k_{s} T_{s}\left(t_{1}, t\right)} \leq \lambda<1, \quad t \geq t_{2} \tag{3.40}
\end{equation*}
$$

If the average dwell time $\tau_{a}$ of the control signal $t \mapsto u(t)$ satisfies

$$
\tau_{a}>\frac{\ln \mu}{k}, \quad \text { where } k=-\frac{k_{u} k_{s}(1-\lambda)}{-k_{u}+\lambda k_{s}}
$$

then, for any solution solution $x$ of switched system (3.25), there exists $i \in \mathbb{N}$ such that $x\left(t_{i}\right) \in N_{u\left(t_{i}\right)}^{\varepsilon}$.

Proof. The goal of the proof is to show that $\tau_{a w} \geq \ln \mu$. One can check that if $-\frac{k_{u} T_{u}}{k_{s} T_{s}} \leq \lambda$ then $\frac{-k_{s} T_{s}-k_{u} T_{u}}{T_{s}+T_{u}} \leq-k$ and so

$$
-k_{s} T_{s}-k_{u} T_{u} \leq-k\left(T_{s}+T_{u}\right)
$$

Take an arbitrary $i \in \mathbb{N}$. According to (3.40), there exists $N_{0} \in \mathbb{N}$ such that

$$
-\frac{k_{u} T_{u}\left(t_{i}, t_{i+N}\right)}{k_{s} T_{s}\left(t_{i}, t_{i+N}\right)} \leq \lambda, \quad \text { for any } N \geq N_{0}
$$

Note that the constant $N \in \mathbb{N}$ in the definition of the average dwell time can be taken arbitrary large. Indeed, if $\frac{1}{N}\left(t_{i+N}-t_{i}\right) \geq \tau_{a}$, then we can always obtain $N_{1} \in \mathbb{N}$ such that $\frac{1}{N_{1}}\left(t_{i+N+N_{1}}-t_{i+N}\right) \geq \tau_{a}$, from where $\frac{1}{N+N_{1}}\left(t_{i+N+N_{1}}-t_{i}\right) \geq \tau_{a}$. Therefore, there exists $N \geq N_{0}$ such that

$$
\frac{1}{N}\left(t_{i+N}-t_{i}\right) \geq \tau_{a}
$$

We now use the three inequalities established above to show that $\tau_{a w}$ can be taken as $\tau_{a w}=$ $\ln \mu$. We have

$$
\begin{aligned}
& k_{u\left(t_{i}\right)}\left(t_{i+1}-t_{i}\right)+\ldots+k_{u\left(t_{i+N-1}\right)}\left(t_{i+N}-t_{i+N-1}\right) \geq \\
& \geq k_{u} T_{u}\left(t_{i}, t_{i+N}\right)+k_{s} T_{s}\left(t_{i}, t_{i+N}\right) \geq \\
& \geq k\left(T_{u}\left(t_{i}, t_{i+N}\right)+T_{s}\left(t_{i}, t_{i+N}\right)\right)=k\left(t_{i+N}-t_{i}\right) \geq \\
& \geq k N \tau_{a} \geq k N \frac{\ln \mu}{k}=\ln \mu .
\end{aligned}
$$

Therefore, the conclusion of the corollary follows by applying Theorem 9 ,

### 3.5.2 Local dynamics

http://www.utdallas.edu/~makarenkov/dwell_time_scan.pdf

## 4 Orbital asymptotic stability of limit cycles

### 4.1 The method of Poincaré section for systems with impacts

A differential equation with impact is a smooth differential equation

$$
\begin{equation*}
\dot{x}=f(x), \quad x \in \mathbb{R}^{n}, \tag{4.1}
\end{equation*}
$$

coupled with an impact map $\Delta: L \rightarrow \mathbb{R}^{n}$ which creates jumps in any solution $x$ that happen to reach a region $L$ of a hyperplane of $\mathbb{R}^{n}$. Specifically, if for a solution $x$ of (4.1) reaches $L$
at some $t \in \mathbb{R}$ in the sense that $x\left(t^{-}\right)=\lim _{s \rightarrow t-0} x(s) \in L$, then $\Delta$ applies instantaneously and moves the point $x(t)$ to the position $\Delta\left(x\left(t^{-}\right)\right)$. The solution $x$ continuous its way along the flow of (4.1) then until it reaches $L$ again. This rule can be formulated as

$$
\begin{align*}
& \dot{x}=f(x)  \tag{4.2}\\
& x\left(t^{+}\right)=\Delta\left(x\left(t^{-}\right)\right), \quad \text { if } x\left(t^{-}\right) \in L .
\end{align*}
$$

Let us denote by $t \rightarrow X(t, \xi)$ the solution $x(t)$ of (4.1) with the initial condition $x(0)=\xi$. The map $t \rightarrow X(t, \xi)$ is know as the general solution of 4.1).

Definition 18 Let $L$ be a region of a hyperplane of $\mathbb{R}^{n}$. A Poincaré map $P$ of smooth system (4.1) induced by $L$ maps a point of $\xi \in L$ to the point $P(\xi)$, where $t \rightarrow X(t, \xi)$ intersects $L$ again. By the other words,

$$
P(\xi)=X(T(\xi), \xi)
$$

where $T(\xi)$ is the minimum of $\{t>0: X(t, \xi) \in L\}$.
Two assumptions are required to make this definition correct. First of all, the set $\{t>0$ : $X(t, \xi) \in L\}$ has to be nonempty, i.e. $t \mapsto X(t, \xi)$ does reach $L$ again. Secondly, we need that $\min \{t>0: X(t, \xi) \in L\}>0$, i.e. the solution of (4.1) that originates at $\xi \in L$ must leave $L$ immediately. Furthermore, $P$ is not required to be defined at all points of $L$.


Figure 4.1: Illustration of the notations of this section.

Definition 19 If $L$ is the impact surface from (4.2) then the Poincaré map $P$ of (4.2) induced by $L$ is

$$
P(\xi)=X(T(\Delta(\xi)), \Delta(\xi))
$$

where the function $T(\xi)=\min \{t>0: X(t, \xi) \in L\}$ is known as time to impact, see Fig. 4.1.

Impact system (4.2) doesn't require that $\Delta$ is defined outside $L$, however, the later property can, in principle, hold. If $\Delta$ is defined outside $L$, then the Poincaré map is formally defined outside $L$. However, only points of $L$ are mapped back to $L$ over the action of $P$. If $\xi_{0}$ is a fixed point of $P$ (i.e. $P\left(\xi_{0}\right)=\xi_{0}$ ) then $\xi_{0} \in L$.

Each fixed point of $P$ is the initial condition of a periodic orbit $x$ of 4.2). The orbit $x$ is attractive, if the $P$ is a contraction in a neighborhood of $x(0)$ of $L$.

Theorem 10 If $P$ is defined in a closed convex set $\tilde{C} \subset L$ and $P(\tilde{C}) \subset \tilde{C}$, then $\tilde{C}$ contains a fixed point $\xi_{0}$ of $P$, i.e. $P\left(\xi_{0}\right)=\xi_{0}$. In particular, the orbit $x$ of (4.2) with the initial condition $x(0)=\xi_{0}$ is periodic. If, in addition,

$$
\left\|P\left(\xi_{1}\right)-P\left(\xi_{2}\right)\right\| \leq \rho\left\|\xi_{1}-\xi_{2}\right\|, \quad \text { for all } \xi_{1}, \xi_{2} \in \tilde{C}
$$

for a fixed $\rho \in[0,1)$ and some norm $\|\cdot\|$, then the graph of $x$ attracts all other solution of (4.2) that originate in $\tilde{C}$.

The existence part is the Brouwer theorem (see [92, Theorem 3.1]). The stability part is a version of [24, Lemma 9.2], see also [34, Theorem 1].

When a fixed point $\xi_{0}$ of $P$ is known, the stability of the respective periodic solution can be analyzed via the linearization approach. For the next theorem to hold (specifically, for the derivative $P^{\prime}\left(\xi_{0}\right)$ to be defined), the Poincare map $P$ needs to be defined in a little $n$-dimensional ball $B_{\delta}\left(\xi_{0}\right)$ near $\xi_{0}$. That is equivalent to saying that the jump $\Delta$ has to be artificially defined in the whole $B_{\delta}\left(\xi_{0}\right)$ (i.e. slightly outside $L$ ), which is often the case in applications. However, the way how $\Delta$ is defined on $B_{\delta}\left(\xi_{0}\right) \backslash L$ doesn't influence the eigenvalues that Theorem 11 is talking about (see Remark 2 that follows Theorem 11). We get rid of this assumption in Theorem 12 by reducing the dimension of $P$ explicitly.

Let the cross-section $L$ be given by

$$
L=\left\{x \in C: c^{T} x+d=0\right\},
$$

where $C$ is a bounded set of $\mathbb{R}^{n}$ and $c \in \mathbb{R}^{n}, d \in \mathbb{R}$.
Theorem 11 Consider $f \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$ and $\Delta \in C^{1}\left(L, \mathbb{R}^{n}\right)$. Consider a $T$-periodic orbit $x$ of (4.2) such that $x\left(T^{-}\right) \in L$ and $x(t) \notin L, t \in(0, T)$, see Fig. 4.1. Assume that $\Delta$ is defined in a small ball $B_{\delta}\left(x\left(T^{-}\right)\right)$of $\mathbb{R}^{n}$ centered at $x\left(T^{-}\right)$and

$$
\begin{equation*}
\Delta \in C^{1}\left(B_{\delta}\left(x\left(T^{-}\right)\right), \mathbb{R}^{n}\right) \tag{4.3}
\end{equation*}
$$

Assume that

$$
\begin{equation*}
\left\langle f\left(x\left(T^{-}\right)\right), c\right\rangle \neq 0 \tag{4.4}
\end{equation*}
$$

so that $P$ is defined near $x\left(T^{-}\right)$. If the real parts of all the eigenvalues of the $n \times n$-matrix $P^{\prime}\left(x\left(T^{-}\right)\right)$are inside the unit circle, then $x$ is an orbitally asymptotically stable limit cycle.

A limit cycle $x$ is orbital asymptotic stable, if the graph of $x$ asymptotically attracts all solutions from its neighborhood, see [83]. This theorem will follow from Theorem 12 .

Remark 2 The way how $\Delta$ is defined on $B_{\delta}\left(\xi_{0}\right) \backslash L$ doesn't influence the eigenvalues of $P^{\prime}\left(x\left(T^{-}\right)\right)$. Indeed, the definition of $\Delta$ outside $L$ doesn't influence the values of $P$ on $L$. That means that $n-1$ eigenvalues of $P^{\prime}\left(x\left(T^{-}\right)\right)$that correspond to the $n-1$-dimensional eigenspace of $P^{\prime}\left(x\left(T^{-}\right)\right)$in $L$ do not depend on the definition of $\Delta$ in $B_{\delta}\left(\xi_{0}\right) \backslash L$. The one eigenvalue of $P^{\prime}\left(x\left(T^{-}\right)\right)$that remains is always zero regardless the definition of $\Delta$. Next proposition is a proof of this fact.

Proposition 11 Let $\xi_{0} \in L$ be a fixed point of the Poincaré map $P$ and the assumptions (4.3)-(4.4) of Theorem 11 hold $\left(x(0)=\xi_{0}\right)$. Then, generically, there exists $v \nVdash L$ (i.e. $\langle v, c\rangle \neq 0)$ such that $P^{\prime}\left(\xi_{0}\right) v=0$. Specifically, if

$$
\text { the matrix } \Delta^{\prime}\left(\xi_{0}\right) \text { is invertible, }
$$

then $v \in \mathbb{R}^{n}$ satisfying $P^{\prime}\left(\xi_{0}\right) v=0$ exists. If, in addition,

$$
\left\langle\Delta^{\prime}\left(\xi_{0}\right)^{-1} f\left(\Delta\left(\xi_{0}\right)\right), c\right\rangle \neq 0
$$

then $\langle v, c\rangle \neq 0$.
Proof. We are going to show that, if $\Delta^{\prime}\left(\xi_{0}\right)$ is invertible, then a function $\xi(\delta)$ is defined near 0 , such that $P(\xi(\delta))=\xi_{0}$ for all $|\delta|$ sufficiently small.
Step 1. In this step we show that finding $\xi(\delta)$ that vanishes the function

$$
F(\delta, \xi)=X\left(-T_{0}-\delta, \xi_{0}\right)-\Delta(\xi)
$$

is sufficient for the proof. Here $T_{0}$ is the period of the cycle of (4.2) with the initial condition $\xi_{0}$. Indeed, if $F(\delta, \xi)=0$, then

$$
X\left(T_{0}+\delta, X\left(-T_{0}-\delta, \xi_{0}\right)\right)=X\left(T_{0}+\delta, \Delta(\xi)\right)
$$

The left-hand-side equals $\xi_{0}$ for all $\delta$. Therefore, it remains to show that

$$
\begin{equation*}
X\left(T_{0}+\delta, \Delta(\xi)\right)=X(T(\Delta(\xi)), \Delta(\xi)) \tag{4.5}
\end{equation*}
$$

for all $\xi$ from a neighborhood of $\xi_{0}$. Let $\vec{L}$ be an $n \times(n-1)$-matrix, whose columns are parallel to $L$ and linearly independent. Consider the function

$$
G(T, v)=H(X(T, v)), \quad \text { where } \quad H(x)=c^{T} x+d(\text { the equation of the hyperplane } L) .
$$

We have $G\left(T_{0}, \Delta\left(\xi_{0}\right)\right)=0$ and $G_{T}^{\prime}\left(T_{0}, \Delta\left(\xi_{0}\right)\right)=\left\langle c, f\left(\xi_{0}\right)\right\rangle$ which is different from zero by (4.4). Therefore, by the Implicit Function Theorem, in the neighborhood of $\left(T_{0}, \Delta\left(\xi_{0}\right)\right)$, there
exists only one $T(v)$ such that $X(T(v), v) \in L$. We noticed earlier that $X\left(T_{0}+\delta, \Delta(\xi)\right)=$ $\xi_{0} \in L$. In consequence, $T_{0}+\delta$ must coincide with $T(\Delta(\xi))$ for 4.5) to hold.

We proved that $F(\delta, \xi(\delta))=0$ implies that $P(\xi(\delta))=\xi_{0}$. By taking the derivative of the last relation in $\delta$ at $\delta=0$, we conclude $P^{\prime}\left(\xi_{0}\right) \xi^{\prime}(0)=0$. So, the required $v$ is $v=\xi^{\prime}(0)$.
Step 2. In this step we show that the function $\xi(\delta)$ that vanishes $F(\delta, \xi)$ in a neighborhood of $\left(0, \xi_{0}\right)$ exists. We have $F\left(0, \xi_{0}\right)=0$ and $F_{\xi}^{\prime}\left(0, \xi_{0}\right)=-\Delta^{\prime}\left(\xi_{0}\right)$, which is invertible by our assumption. Thus, the required $\xi(\delta)$ exists by the Implicit Function Theorem.

Step 3. To establish the condition that ensures $\left\langle\xi^{\prime}(0), c\right\rangle \neq 0$, we use the formula for the derivative of the implicit function

$$
\xi^{\prime}(0)=-F_{\xi}^{\prime}\left(0, \xi_{0}\right)^{-1} F_{\delta}^{\prime}\left(0, \xi_{0}\right)=\Delta^{\prime}\left(\xi_{0}\right)\left(-X_{t}^{\prime}\left(-T_{0}, \xi_{0}\right)\right)=-\Delta^{\prime}\left(\xi_{0}\right) f\left(\Delta\left(\xi_{0}\right)\right)
$$

The proof of the Proposition is complete.

### 4.1.1 An illustrative example where the Poincaré map can be obtained in a closed form

http://www.utdallas.edu/ makarenkov/Poincare-map-design.pdf

### 4.1.2 An example where the Poincaré map can be obtained in a closed form: the simplest clock model

The Poincaré map defined in Definition 19 maps vectors of $n$ components into vectors of $n$ components. For some analytic computations it is useful to pass to an $n-1$ dimensional equivalent Poincaré map. Let $L(s): L \rightarrow \mathbb{R}^{n}$ is a non-singular parameterization of the region $L$. Then

$$
\tilde{P}(s)=L^{-1}(X(T(\Delta(L(s))), \Delta(L(s)))
$$

maps vectors of $n-1$ components into vectors of $n-1$ components and $\tilde{P}$ coincides with $P$ on $L$ in the sense that

$$
P(L(s))=L(\tilde{P}(s)) \quad \text { or } \quad P(\xi)=L\left(\tilde{P}\left(L^{-1}(\xi)\right)\right) \quad \text { for all } s \in \mathbb{R}^{n-1} \text { and all } \xi \in L
$$

The advantage of $\tilde{P}$ is that the contraction property can hold for $\tilde{P}$ in a closed convex set of the entire space of this maps (e.g. $C$ can be a closed ball of $\mathbb{R}^{n-1}$ ), not just in a part of the space that we restricted to in Theorem 10.

Theorem 12 Let $s \rightarrow L(s)$ be a non-singular parameterization of $L$ and $L(0)=\xi_{0}$, where $\xi_{0}$ is the initial condition of a periodic solution $x$ of (4.2). Assume that $f \in C^{2}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$. Assume that $x_{0}$ is the only point of intersection of $x$ and $L$ and that the intersection is transversal. The derivative $\tilde{P}^{\prime}(0)$ is then defined. Assume, that the eigenvalues of $\tilde{P}^{\prime}(0)$ are strictly inside the unit circle of $\mathbb{C}$. Then $\tilde{P}$ is a contraction in a sufficiently small neighborhood of 0 . In particular, $x$ is an orbitally asymptotically stable limit cycle of (4.2).

The proof of this theorem can be found in [92, Theorem 9.1].
Example 16 Prove the existence and orbital asymptotic stability of a limit cycle in the impact system

$$
\begin{aligned}
& \ddot{\theta}+b \dot{\theta}+a^{2} \theta=0, \\
& \dot{\theta}\left(t^{+}\right)=\dot{\theta}\left(t^{-}\right)+k, \quad \text { if } \theta\left(t^{-}\right)=0 \text { and } \dot{\theta}\left(t^{-}\right)>0
\end{aligned}
$$

where $a, b, k$ are given positive constant and $a, b$ satisfy $b^{2}<4 a^{2}$.
Solution. The respective first-order system of differential equations with impulsive effect reads as

$$
\begin{align*}
& \dot{\theta}=w, \\
& \dot{w}=-b w-a^{2} \theta,  \tag{4.6}\\
& w\left(t^{+}\right)=w\left(t^{-}\right)+k, \quad \text { if }\left(\theta\left(t^{-}\right), w\left(t^{-}\right)\right)^{T} \in L \tag{4.7}
\end{align*}
$$

where $L=\{0\} \times(0, \infty)$. The general solution of (4.6) is given by

$$
X\left(t,\binom{0}{w_{0}}\right)=\binom{w_{0} \frac{1}{\nu} \exp \left(-\frac{1}{2} b t\right) \cos \left(\nu t-\frac{\pi}{2}\right)}{-w_{0} \exp \left(-\frac{1}{2} b t\right) \sin \left(\nu t-\frac{\pi}{2}\right)-w_{0} \frac{b}{2 \nu} \exp \left(-\frac{1}{2} b t\right) \cos \left(\nu t-\frac{\pi}{2}\right)}
$$

By defining $\Delta$ as $\Delta\left(\left(0, w_{0}\right)^{T}\right)=\left(0, w_{0}+k\right)^{T}$ and by noticing that it takes $\frac{2 \pi}{\nu}$ for any solution of (4.6) to get from $L$ to $L$, one can write the following formula for the Poincaré map

$$
\begin{aligned}
P\left(\left(0, w_{0}\right)^{T}\right) & =X\left(T\left(\Delta\left(\left(0, w_{0}\right)^{T}\right)\right), \Delta\left(\left(0, w_{0}\right)^{T}\right)\right)=X\left(\frac{2 \pi}{\nu},\binom{0}{w_{0}+k}\right)= \\
& =\binom{0}{\left(w_{0}+k\right) \exp \left(-\frac{\pi b}{\nu}\right)}
\end{aligned}
$$

see Fig. 16. By parameterizing $L$ as $L(w)=(0, w)^{T}$, one gets $L^{-1}\left((0, w)^{T}\right)=w$ and

$$
P\left(w_{0}\right)=\left(w_{0}+k\right) \exp \left(-\pi \frac{b}{\nu}\right)
$$

Solving $P\left(w_{0}\right)=w_{0}$, one gets

$$
w_{0}=\frac{k \exp (-\pi b / \nu)}{1-\exp (-\pi b / \nu)} \quad \text { and } \quad P^{\prime}\left(w_{0}\right)=\exp (-\pi b / \nu)<1
$$

Existence and stability of a limit cycle follows by applying Theorem 12 .


Figure 4.2: Illustration of the solution of Example 16.


Figure 4.3: The diagram of 3 -link planar bipedal robot and definitions of the angles $\theta_{1}, \theta_{2}, \theta_{3}$.

### 4.1.3 An example where the fixed points and contraction of the Poincaré map can be established geometrically by reducing the dimension to $\mathbb{R}^{2}$ : switched control of 3-link biped robot

The equation of motion 3-link bipedal walker of Fig. 4.3 reads as

$$
\begin{equation*}
\ddot{\theta}=F(\theta, \dot{\theta})+G(\theta) u, \quad F: \mathbb{R}^{3} \times \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}, G(\theta) \text { is a } 3 \times 2 \text { - matrix, } u \in \mathbb{R}^{2} \tag{4.8}
\end{equation*}
$$

as long as leg $\theta_{1}$ contacts the ground and $\operatorname{leg} \theta_{2}$ doesn't. The two-dimensional control $u$ represents the two torques applied between the torso and the stance leg, and the torso and the swing leg, respectively. The control $u$ that we are to design makes sure that swing leg just slide along the floor (i.e. almost touch the floor, but doesn't touch). The controller has two more inputs: additional motors on the legs allow to push the swing leg just slightly out of the saggital plane during the swing phase and to pull the leg back into the saggital plane
whenever the contact is desired. Specifically, the contact will be initiated when the angle of the stance leg attains a desired value, $\theta_{1}^{d}$. This leads to the following impact event

$$
\begin{equation*}
\left(\theta\left(t^{+}\right), \dot{\theta}\left(t^{+}\right)\right)=\Delta\left(\left(\theta\left(t^{-}\right), \dot{\theta}\left(t^{-}\right)^{T}\right), \quad \text { if } \theta_{1}\left(t^{-}\right)=\theta_{1}^{d}\right. \tag{4.9}
\end{equation*}
$$

The goal of the control is to maintain the angle of the torso at some constant value, say $\theta_{3}^{d}$, while command the swing leg to behave as the mirror image of the stance leg, that is $\theta_{2}=-\theta_{1}$. We, therefore, want that the control $u$ is introduced in 4.8) in such a way that the relations

$$
\begin{align*}
& y_{1}(t)=\theta_{3}(t)-\theta_{3}^{d} \rightarrow 0 \quad \text { as } \quad t \rightarrow \infty \\
& y_{2}(t)=\theta_{2}(t)+\theta_{1}(t) \rightarrow 0 \quad \text { as } t \rightarrow \infty \tag{4.10}
\end{align*}
$$

hold for a range of solutions $\theta$ of (4.8)-(4.9).

### 4.1.3.1 Reduction to 2-dimensional equations of Zero dynamics

Notation 4.10 can be rewritten as

$$
\binom{y(t)}{\theta_{1}(t)}=A \theta(t)+b, \quad \text { where } A=\left(\begin{array}{lll}
0 & 0 & 1  \tag{4.11}\\
1 & 1 & 0 \\
1 & 0 & 0
\end{array}\right), \quad b=\left(\begin{array}{l}
-\theta_{3}^{d} \\
0 \\
0
\end{array}\right)
$$

Since the change of the variables (4.11) brings (4.8) to

$$
\begin{equation*}
\binom{\ddot{y}}{\ddot{\theta}_{1}}=A\left(F\left(A^{-1}\left(\binom{y}{\theta_{1}}-b\right), A^{-1}\binom{\dot{y}}{\dot{\theta}_{1}}\right)+G\left(A^{-1}\left(\binom{y}{\theta_{1}}-b\right)\right) u\right), \tag{4.12}
\end{equation*}
$$

then, by assuming that

$$
\text { the matrix } K(\theta)=\left(\begin{array}{ccc}
0 & 0 & 1 \\
1 & 1 & 0
\end{array}\right) G(\theta) \text { is invertible, }
$$

equation (4.12) can further be rewritten as

$$
\binom{\ddot{y}}{\ddot{\theta_{1}}}=\left(\begin{array}{l}
v  \tag{4.13}\\
\left.F_{1}\left(A^{-1}\left(\binom{y}{\theta_{1}}-b\right), A^{-1}\binom{\dot{y}}{\dot{\theta}_{1}}\right)+G_{1}\left(A^{-1}\left(\binom{y}{\theta_{1}}-b\right)\right) U\left(v, y, \dot{y}, \theta_{1}, \dot{\theta}_{1}\right)\right), ~
\end{array}\right.
$$

where

$$
U\left(v, y, \dot{y}, \theta_{1}, \dot{\theta}_{1}\right)=K\left(A^{-1}\left(\binom{y}{\theta_{1}}-b\right)\right)^{-1}\left(v-\left(\begin{array}{lll}
0 & 0 & 1 \\
1 & 1 & 0
\end{array}\right) F\left(A^{-1}\left(\binom{y}{\theta_{1}}-b\right), A\binom{\dot{y}}{\dot{\theta}_{1}}\right)\right)
$$

If we can find a control $v(y)$ that ensures (4.10) for solutions of 4.13), then the control

$$
u\left(y, \dot{y}, \theta_{1}, \dot{\theta}_{1}\right)=U\left(v(y), y, \dot{y}, \theta_{1}, \dot{\theta}_{1}\right)
$$

ensures (4.10) for solutions of 4.12.
For each of the double integrators $\ddot{y}_{1}=v_{1}$ and $\ddot{y}_{2}=v_{2}$, the respective controls $v_{1}\left(y_{1}, \dot{y}_{1}\right)$ and $v_{2}\left(y_{2}, \dot{y}_{2}\right)$ can be taken as finite-time controllers of Exercise ??, see [34, formula (21)] for the precise formula. Any solution of (4.13) with $v=\left(v_{1}\left(y_{1}, \dot{y}_{1}\right), v_{2}\left(y_{2}, \dot{y}_{2}\right)\right)^{T}$ approaches the hyperplane $\{0\} \times\{0\} \times\{0\} \times\{0\} \times \mathbb{R} \times \mathbb{R}$ in finite-time.

In what follows, we consider only those initial conditions $\Omega$, for which the solution of (4.13) reaches the hyperplane $\{0\} \times\{0\} \times\{0\} \times\{0\} \times \mathbb{R} \times \mathbb{R}$ (called Zero dynamics hyperplane $)$ transversally and before it crosses the hyperplane $\theta_{1}=\theta_{1}^{d}$ (i.e. before an impulse occurs). This set (in coordinates $(\theta, \dot{\theta})$ ) is denoted by $\hat{S}$ in [34, formula (29)]. For each initial condition from $\Omega$ the solution converges to the Zero dynamics hyperplane in finite time. After a solution $\left(y, \theta_{1}\right)$ of 4.13 reaches the Zero dynamics hyperplane, the dynamics develops according to the second-order differential equation with impacts

$$
\begin{align*}
& \ddot{\theta}_{1}=\bar{\zeta}_{a}\left(\theta_{1}\right)+\bar{\zeta}_{b}\left(\theta_{1}\right) \dot{\theta}_{1}^{2}, \\
& \theta_{1}\left(t^{+}\right)=-\theta_{1}\left(t^{-}\right),  \tag{4.14}\\
& \dot{\theta}_{1}\left(t^{+}\right)=m \dot{\theta}_{1}\left(t^{-}\right)+m_{0}, \quad \text { if } \theta_{1}\left(t_{0}\right)=\theta_{1}^{d},
\end{align*}
$$

see [34, formulas (52), (49), (50)]. Equation (4.14) is called equation of zero dynamics.

### 4.1.3.2 Phase plane analysis of the equation of Zero Dynamics

The existence of a limit cycle can be established under the following assumptions (that I expect must be the case in [34 ${ }^{26}$ ):

1) Let $L$ denotes the homoclinic orbit that originates at the origin and let $v_{1}$ be the ordinate of the intersection of $L$ with the line $\theta_{1}=\theta_{1}^{d}$. Assume that the ordinate of the intersection of $L$ with the line $\theta_{1}=-\theta_{1}^{d}$ is smaller than $m v_{1}+m_{0}$.
2) Assume that the constant $m<0$,
3) Let $L(v)$ denotes the ordinate of the intersection of the solution that starts at $\left(\theta_{1}^{d}, v\right)$ with the line $\theta_{1}=-\theta_{1}^{d}$. Assume that $\frac{|v-L(v)|}{v} \rightarrow 0$ as $v \rightarrow \infty$.

By using Fig. 4.4 we conclude that assumption 1) implies that $P\left(v_{1}\right)>v_{1}$. Assumption 2) implies that the difference between $m v+m_{0}$ and $v$ decreases at the rate of $v$. Since, by 3) $|v-L(v)|$ increases slower than $v$, then for a sufficiently large value of $v$ (which is denote by $v_{2}$ at Fig. 4.4) one will have that $L(v)>m v+m_{0}$. As seen from Fig. 4.4, the latter implies that $P\left(\left[v_{1}, v_{2}\right]\right) \subset\left(v_{1}, v_{2}\right)$.

The contracting property of $P$ can be observed by drawing the next interation of the Poincaré map and by establishing that $P\left(P\left(\left[v_{1}, v_{2}\right]\right)\right) \subset P\left(\left(v_{1}, v_{2}\right)\right)$.

[^17]

Figure 4.4: Trajectories of equation 4.14 without impact (black); solution of equation 4.14 with the initial condition at the point $\left(\theta_{1}^{d}, v_{1}\right)$ until the first intersection with $\theta_{1}=\theta_{1}^{d}$ (black); solution of equation (4.14) with the initial condition at the point $\left(\theta_{1}^{d}, v_{2}\right)$ until the first intersection with $\theta_{1}=\theta_{1}^{d}$ (blue). The two vertical lines are $\theta=-\theta_{1}^{d}$ and $\theta=\theta_{1}^{d}$.

### 4.2 The method of Poincaré section for systems with hybrid (relay) feedback

In this section we study existence and stability of limit cycles in a switched system with Hybrid Feedback switching rule

$$
\dot{x}=f(x), \quad f(x):= \begin{cases}f^{L}(x), & H_{1}(x)>0  \tag{4.15}\\ f^{R}(x), & H_{2}(x)>0\end{cases}
$$

As earlier, this notation means that the system switches to $f^{L}$ when $H_{1}(x)>0$ and the system switches to $f^{R}(x)$, if $H_{2}(x)>0$. No switchings occur when $H_{1}(x) \leq 0$ and $H_{2}(x) \leq 0$, so that the trajectory develops along the vector field that the system switched to last time. Such a formulation requires indication of the vector field which is active at the initial time moment. We will often omit making this indication as it doesn't change the analysis.

I use strict inequalities to (4.15) to make the switched system defined also when the sets

$$
L_{1}=\left\{x: H_{1}(x)=0\right\}, \quad L_{2}=\left\{x: H_{2}(x)=0\right\}
$$

coincide. In this case 4.15 is a Filippov system, where sliding motions are possible.

### 4.2.1 An example where the Poincaré map can be obtained in a closed form: clock with a balance-wheel

This material just follows [70, Ch. III, §5]. The scanned page of my notes is available here:
http://www.utdallas.edu/ makarenkov/clock_scan.pdf

The clock model takes form (4.15 by putting
$f^{L}(x)=\binom{x_{2}}{-F \operatorname{sign} x_{2}+1}, f^{R}(x)=\binom{x_{2}}{-F \operatorname{sign} x_{2}-1}, H_{1}(x)=-1-x_{1}, H_{2}(x)=x_{1}-1$.

### 4.2.2 Stability of a given limit cycle in a simple impact oscillator

http://www.utdallas.edu/ ~makarenkov/rotating-cross-section.pdf

### 4.2.3 Stability of a given limit cycle in the general nonlinear case

Definition 20 Consider $\xi \in L_{1}$. If $H_{1}{ }^{\prime}(\xi) f^{R}(\xi)>0$, then the solution of (4.15) with the initial condition $x(0)=\xi$ can only be governed by $\dot{x}=f^{L}(x)$ right after the moment $t=0$. If, in addition to $H_{1}{ }^{\prime}(\xi) f^{R}(\xi)>0$, the solution $x$ reaches $L_{2}$, then we put

$$
\begin{equation*}
P^{L}(\xi)=X^{L}\left(T^{L}(\xi), \xi\right), \quad \text { where } T^{L}(\xi)=\min \left\{t: X^{L}(t, \xi) \in L_{2}\right\} \tag{4.16}
\end{equation*}
$$

and say that the point transformation $P^{L}$ from $L_{1}$ to $L_{2}$ is defined at $\xi$. The point transformation $P^{R}$ from $L_{2}$ to $L_{1}$ is defined by analogy.

If the point transformation $P^{L}$ from $L_{1}$ to $L_{2}$ is defined at $\xi$ and the point transformation $P^{R}$ from $L_{2}$ to $L_{1}$ is defined at $P^{L}(\xi)$, then the map

$$
\begin{equation*}
P(\xi)=P^{R}\left(P^{L}(\xi)\right) \tag{4.17}
\end{equation*}
$$

is defined at $\xi$ and $P(\xi) \in L^{1}$. Map 4.17) is, therefore, the Poincaré map induced by the cross-section $L_{1}$.

Proposition $12{ }^{27}$ Consider a non-singular parameterization $L_{1}(s)$ of L. Let

$$
\tilde{P}(s)=\left(L_{1}\right)^{-1}\left(P\left(L_{1}(s)\right)\right) .
$$

[^18]If $s_{0} \in \mathbb{R}^{n-1}$ is a fixed point of $\tilde{P}$ and if all the eigenvalues of the matrix $\tilde{P}^{\prime}\left(s_{0}\right)$ are strictly inside the unit circle, then the eigenvalues of $P^{\prime}\left(L_{1}\left(s_{0}\right)\right)$ are strictly inside the unit circle too. Moreover, one of the eigenvalues of $P^{\prime}\left(L_{1}\left(s_{0}\right)\right)$ equals 0 .

The essential (difficult) part of the proof of Proposition 12 follows the lines of the proof of Proposition 11.

To prove the main result of this section we need the following lemma, which can of independent interest.

Lemma 9 If $t \mapsto X(t, \xi)$ is the general solution of a smooth system

$$
\dot{x}=g(x),
$$

then the derivative $t \mapsto X_{x}^{\prime}(t, \xi)$ of $X$ with respect to the second variable is the normalized fundamental matrix solution of the linear time-dependent system

$$
\dot{y}=g_{x}^{\prime}(X(t, \xi)) y .
$$

See [24, Theorem 2.1] for a proof of this lemma.
Theorem 13 Consider $f^{L}, f^{R} \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$. Assume that (4.15) admits a cycle $x_{0}$ such that system (4.15) switches exactly two times during the period $T$ of the cycle, see Fig. 4.5. By the other words, assume that there exist $T^{L}+T^{R}=T$, such that

$$
\begin{aligned}
& \dot{x}_{0}(t)=f^{L}\left(x_{0}(t)\right), \quad t \in\left(0, T^{L}\right), \\
& \dot{x}_{0}(t)=f^{R}\left(x_{0}(t)\right), \quad t \in\left(T^{L}, T^{L}+T^{R}\right) .
\end{aligned}
$$

Assume that the trajectory $x_{0}(t)$ approaches the respective switching hyperplanes transversally, i.e.

$$
\begin{equation*}
H_{2}{ }^{\prime}\left(x^{R}\right) f^{L}\left(x^{R}\right) \cdot H_{1}^{\prime}\left(x^{L}\right) f^{R}\left(x^{L}\right) \neq 0 \tag{4.18}
\end{equation*}
$$

where

$$
x^{L}=x_{0}(0), \quad x^{R}=x_{0}\left(T^{L}\right) .
$$

Then, the point transformations $P^{L}$ and $P^{R}$ are defined in little neighborhoods of $x^{L}$ and $x^{R}$ respectively. Moreover, the derivatives $P^{L^{\prime}}\left(x^{L}\right)$ and $P^{R \prime}\left(x^{R}\right)$ exist and are given by

$$
\begin{aligned}
P^{L^{\prime}}\left(x^{L}\right) & =\left(I-\frac{f^{L}\left(x^{R}\right) H_{2}{ }^{\prime}\left(x^{R}\right)}{H_{2}{ }^{\prime}\left(x^{R}\right) f^{L}\left(x^{R}\right)}\right) \exp \left(\int_{0}^{T^{L}} f^{L^{\prime}}\left(x_{0}(s)\right) d s\right) \\
P^{R \prime}\left(x^{R}\right) & =\left(I-\frac{f^{R}\left(x^{L}\right) H_{1}{ }^{\prime}\left(x^{L}\right)}{H_{1}{ }^{\prime}\left(x^{L}\right) f^{R}\left(x^{L}\right)}\right) \exp \left(\int_{0}^{T^{R}} f^{R \prime}\left(x_{0}\left(s+T^{L}\right)\right) d s\right)
\end{aligned}
$$

The cycle $x_{0}$ is orbitally asymptotically stable, if the eigenvalues of the matrix $P^{R \prime}\left(x^{R}\right) P^{L \prime}\left(x^{L}\right)$ are strictly inside the unit circle.


Figure 4.5: The cycle $x_{0}$ of 4.15 and a sample trajectory with an initial condition $\xi$ near $x_{0}(0)$.
Here $\exp \left(\int_{0}^{t} A(s) d s\right)$ stays for the normalized fundamental matrix solution of the linear time-dependent system $\dot{y}=A(t) y$.

Proof. Step 1. In order to establish orbital asymptotic stability of the limit cycle $x_{0}$, it is sufficient to show that the eigenvalues of the derivative of the Poincaré map (4.17) at $x^{L}$ are strictly inside the unit circle. Since

$$
P^{\prime}(\xi)=P^{R^{\prime}}\left(P^{L}(\xi)\right) P^{L \prime}(\xi) \quad \text { and } \quad P^{L}\left(x^{L}\right)=x^{R}
$$

we get

$$
P^{\prime}\left(x^{L}\right)=P^{R \prime}\left(x^{R}\right) P^{L \prime}\left(x^{L}\right) .
$$

It, therefore, remains to prove the formulas for $P^{R^{\prime}}\left(x^{R}\right)$ and $P^{L^{\prime}}\left(x^{L}\right)$.
Step 2. By taking the derivative of (4.16) with respect to $\xi$ and by plugging $\xi=x^{L}$, one gets

$$
P^{L \prime}\left(x^{L}\right)=X_{t}^{L \prime}\left(T^{L}\left(x^{L}\right), x^{L}\right) T^{L \prime}\left(x^{L}\right)+X_{x}^{L \prime}\left(T^{L}\left(x^{L}\right), x^{L}\right)
$$

where $T^{L}\left(x^{L}\right)=T^{L}$ by the definitions of $T^{L}(\xi)$ and $T^{L}$,

$$
X_{x}^{L \prime}\left(T^{L}, x^{L}\right)=\exp \left(\int_{0}^{T^{L}} f^{L^{\prime}}\left(x_{0}(s)\right) d s\right)
$$

by Lemma 9, and

$$
X_{t}^{L \prime}\left(T^{L}, x^{L}\right)=f^{L}\left(X^{L}\left(T^{L}, x^{L}\right)\right)=f^{L}\left(x^{R}\right)
$$

by the definition of $x^{R}$.
Step 3. To compute $T^{L^{\prime}}\left(x^{L}\right)$ we view $T^{L}(\xi)$ as the implicit function that solves the equation

$$
H_{2}\left(X^{L}(T, \xi)\right)=0
$$

in $T$ for all $\xi$ close to $x^{L}$. To see that such an implicit function exists, we apply the Implicit Function Theorem to the function

$$
F(T, \xi)=H_{2}\left(X^{L}(T, \xi)\right)
$$

The condition $F\left(T^{L}, x^{L}\right)=0$ holds by the definition of $T^{L}$. For the derivative $F_{T}^{\prime}\left(T^{L}, x^{L}\right)$ we have

$$
F_{T}^{\prime}\left(T^{L}, x^{L}\right)=H_{2}^{\prime}\left(X^{L}\left(T^{L}, x^{L}\right)\right) X_{t}^{L \prime}\left(T^{L}, x^{L}\right)=H_{2}^{\prime}\left(x^{R}\right) f^{L}\left(x^{R}\right) \neq 0
$$

by the transversality assumption 4.18. Therefore, the Implicit Function Theorem applies which not only shows the existence of $T^{L}(\xi)$, but also allows to compute $T^{L \prime}\left(x^{L}\right)$ as

$$
T^{L \prime}\left(x^{L}\right)=-\left(F_{T}^{\prime}\left(T^{L}, x^{L}\right)\right)^{-1} F_{\xi}^{\prime}\left(T^{L}, x^{L}\right)=-\frac{1}{H_{2}^{\prime}\left(x^{R}\right) f^{L}\left(x^{R}\right)} H_{2}{ }^{\prime}\left(x^{R}\right) X_{x}^{L \prime}\left(T^{L}, x^{L}\right) .
$$

Combining the results of Step 2 and Step 3, we get the required formula for $P^{L^{\prime}}\left(x^{L}\right)$. The formula for $P^{R^{\prime}}\left(x^{R}\right)$ can be obtained by analogy. The proof of the theorem is complete.

### 4.2.4 Existence and stability of a limit cycle in the case of linear systems

The linear version of 4.15) is given by

$$
\dot{x}=f(x), \quad \text { where } f(x):= \begin{cases}A^{L} x+b^{L}, & \text { if } c^{L} x+d^{L}<0  \tag{4.19}\\ A^{R} x+b^{R}, & \text { if } c^{R} x+d^{R}>0\end{cases}
$$

Proposition 13 Let $T^{L}>0$ and $T^{R}>0$ be two arbitrary numbers such that

$$
\text { the matrices } D_{1}=I-\mathrm{e}^{A^{R} T^{R}} \mathrm{e}^{A^{L} T^{L}} \text { and } D_{2}=I-\mathrm{e}^{A^{L} T^{L}} \mathrm{e}^{A^{R} T^{R}} \text { are invertible. }
$$

Consider

$$
x^{L}=D_{1}^{-1}\left(\mathrm{e}^{A^{R} T^{R}} F^{L}\left(T^{L}\right)+F^{R}\left(T^{R}\right)\right), \quad x^{R}=D_{2}^{-1}\left(\mathrm{e}^{A^{L} T^{L}} F^{R}\left(T^{R}\right)+F^{L}\left(T^{L}\right)\right),
$$

where

$$
F^{L}(t)=\int_{0}^{t} \mathrm{e}^{A^{L} s} b^{L} d s, \quad F^{R}(t)=\int_{0}^{t} \mathrm{e}^{A^{R} s} b^{R} d s
$$

Then, for any $c^{L}, d^{L}, c^{R}, d^{R} \in \mathbb{R}$ such that

$$
\begin{align*}
& c^{L} x^{L}+d^{L}=0, \quad c^{R} x^{R}+d^{R}=0, \\
& c^{L}\left(\mathrm{e}^{A^{R} t} x^{R}+F^{R}(t)\right)+d^{L}>0, \quad \text { for any } t \in\left[0, T^{R}\right),  \tag{4.20}\\
& c^{R}\left(\mathrm{e}^{A^{L} t} x^{L}+F^{L}(t)\right)+d^{R}<0, \quad \text { for any } t \in\left[0, T^{L}\right),  \tag{4.21}\\
& \left\langle c^{L}, f^{R}\left(x^{L}\right)\right\rangle<0, \quad\left\langle c^{R}, f^{L}\left(x^{R}\right)\right\rangle>0
\end{align*}
$$

the switched system (4.19) has a cycle $x_{0}$ that satisfies $x_{0}(0)=x^{L}$ and $x_{0}\left(T^{L}\right)=x^{R}$. The cycle $x_{0}$ satisfies the first of the equations of (4.19) on $\left[0, x^{L}\right)$ and $x_{0}$ satisfies the second of the equations of (4.19) on $\left[0, x^{R}\right)$. If, in addition, the eigenvalues of the matrix

$$
\left(I-\frac{\left(A^{R} x^{L}+b^{R}\right) c^{L}}{c^{L}\left(A^{R} x^{L}+b^{R}\right)}\right) e^{A^{R} T^{R}}\left(I-\frac{\left(A^{L} x^{L}+b^{L}\right) c^{R}}{c^{R}\left(A^{L} x^{R}+b^{L}\right)}\right) e^{A^{L} T^{L}}
$$

are strictly inside the unit circle, then the cycle $x_{0}$ is orbitally asymptotically stable.
Sketch of the proof. Proposition 13 is a corollary of Theorem 13. To draw this corollary, one needs to notice that

$$
\begin{aligned}
& H_{1}(x)=-c^{L} x-d^{L}, H_{2}(x)=c^{R} x+d^{R}, H_{1}{ }^{\prime}(x)=-\left(c^{L}\right)^{T}, H_{2}{ }^{\prime}(x)=\left(c^{R}\right)^{T} \\
& X^{L}(t, \xi)=e^{A^{L} t} \xi+F^{L}(t), X^{R}(t, \xi)=e^{A^{R}} \xi+F^{R}(t)
\end{aligned}
$$

Therefore,

$$
\begin{array}{ll}
X^{L}\left(T^{L}, x^{L}\right)=x^{R}, \quad \text { is equivalent to } & \mathrm{e}^{A^{L} T^{L}} x^{L}+F^{L}\left(T^{L}\right)=x^{R} \\
X^{R}\left(T^{R}, x^{R}\right)=x^{L} & \mathrm{e}^{A^{R} T^{R}} x^{R}+F^{R}\left(T^{R}\right)=x^{L}
\end{array}
$$

from where the formulas for $x^{L}$ and $x^{R}$ are obtained.
Assumption 4.20 ensure that $t=T^{R}$ is the first time moment $t$ when $H_{2}\left(x_{0}(t)\right)=0$. Analogously 4.21 ensure that $t=T^{L}$ is the first time moment $t \geq T^{R}$ when $H_{1}\left(x_{0}(t)\right)=0$.

Proposition 13 is a minor development of results [94, 95, 96, 28, who addressed the linear symmetric system 4.19) of the form

$$
\dot{x}=A x+b\left(\operatorname{rel}_{d}(c x)\right), \quad \text { where } \operatorname{rel}_{d}(y):= \begin{cases}1, & \text { if } y \leq-d  \tag{4.22}\\ -1, & \text { if } y \geq d\end{cases}
$$

$A$ is an $n \times n$ matrix, $b, c \in \mathbb{R}^{n}$ are given vectors.

### 4.2.5 Deriving the result of section 4.2 .2 from the general theorem of section 4.2.4

http://www.utdallas.edu/ ~makarenkov/1dim-vs-2dim.pdf
${ }^{28}$ Consider the exercise
Exercise 21 The authors of [94, 95, 96] formulate theoretical results for system 4.22), but then propose examples in terms of certain transfer functions, that I don't have time to lean about right now. I would be very interested to see how those examples can be reformulated in terms of system 4.22.).

### 4.3 The bifurcation approach

Step 3 of the proof of Theorem 13 was an introduction to the so-called perturbation approach. We didn't know the explicit formula for $T^{L}(\xi)$, but still succeeded to compute $T^{L^{\prime}}\left(x^{L}\right)$. That happened because of the following two reasons:

1) we knew the dynamics (i.e. the solution $t \mapsto X^{L}(t, \xi)$ ) for $\xi=x^{L}$ completely,
2) we knew the derivatives of $X^{L}(t, \xi)$ at $(t, \xi)=\left(T^{L}, x^{L}\right)$.

Implicit Function Theorem helped us to compute the derivative $T^{L^{\prime}}\left(x^{L}\right)$ of the implicit function $T^{L}$ at $x^{L}$. We will now expose this approach further in several typical situations.

### 4.3.1 Existence and asymptotic stability of a limit cycle in a mechanical oscillator with state-dependent impulses: transversal case

In this section we study the occurrence of limit cycles in an impact oscillator from a cycle which intersects the impact threshold transversally.
http://www.utdallas.edu/ ${ }^{\sim}$ makarenkov/perturbation-transversal.pdf

### 4.3.2 Existence and asymptotic stability of a periodic solution in a mechanical oscillator with time-periodic impulses

http://www.utdallas.edu/~makarenkov/time-periodic-impulses.pdf

### 4.3.3 Existence and finite-time stability of a limit cycle in the dry-friction oscillator with small friction characteristics

In section 2.8.2.2 we established the existence and finite-time stability of a cycle in the following equation of dry friction oscillator placed on a moving belt (Fig. 2.11)

$$
\begin{align*}
& \dot{x}_{1}=x_{2}, \\
& \dot{x}_{2}=-x_{1}-c x_{2}-F\left(x_{2}-V\right), \tag{2.54a}
\end{align*}
$$

where

$$
\begin{equation*}
F(s) \text { is close to } \operatorname{sign}(s), \text { and } c>0 \text { is small, } \tag{4.23}
\end{equation*}
$$

and

$$
\begin{equation*}
-c-F^{\prime}\left(x_{2}-V\right)>0 \quad \text { for a range of } x_{2} \tag{4.24}
\end{equation*}
$$

In this section a bifurcation approach is used to replace 4.24 with an exact condition. To use the perturbation approach we model (4.23) by introducing a small parameter $\varepsilon$ that replaces 2.54 with

$$
\begin{align*}
& \dot{x}_{1}=x_{2} \\
& \dot{x}_{2}=-x_{1}-\varepsilon c x_{2}-\operatorname{sign}\left(x_{2}-V\right)+\varepsilon f\left(x_{2}-V\right), \tag{4.25}
\end{align*}
$$

where $f$ is a Lipschitz function that vanishes at 0 .

Proposition 14 Let $f$ be a Lipschitz function that vanishes at 0. If

$$
\begin{equation*}
c V \pi<\int_{0}^{2 \pi} \cos \tau \cdot f(V \cos \tau-V) d \tau \tag{4.26}
\end{equation*}
$$

then, for all $\varepsilon>0$ sufficiently small, the Filippov solution $x$ of (4.25) with the initial condition $x(0)=(1-\varepsilon c V, V)^{T}$ reaches the point $\left(\xi_{1}, V\right)^{T}$, where $\xi_{1} \in(-1-\varepsilon c V, 1-\varepsilon c V)$. As a consequence (see Proposition 9), solution $x$ is a finite-time stable limit cycle of (4.25).

Proof. When $x_{2}<V$, system (4.25) takes the form

$$
\begin{equation*}
\dot{x}=g(x, \varepsilon), \quad \text { where } \quad g(x, \varepsilon)=\binom{x_{2}}{-x_{1}-\varepsilon c x_{2}-1+\varepsilon f\left(x_{2}-V\right)} \tag{4.27}
\end{equation*}
$$

Let $t \mapsto X(t, \xi, \varepsilon)$ be the solution $x$ of 4.27 with the initial condition $x(0)=\xi$. To prove the proposition, we have to establish the existence of a function $T(\varepsilon)$ such that

$$
X_{1}\left(T(\varepsilon),\binom{1-\varepsilon c V}{V}, \varepsilon\right)<1-\varepsilon c V \quad \text { and } \quad X_{2}\left(T(\varepsilon),\binom{1-\varepsilon c V}{V}, \varepsilon\right)=V
$$

and such that

$$
X_{2}\left(t,\binom{1-\varepsilon c V}{V}, \varepsilon\right)<V \quad \text { for all } t \in(0, T(\varepsilon))
$$

Since the trajectory $t \mapsto X\left(t,\binom{1-\varepsilon c V}{V}, \varepsilon\right)$ approaches the circle of period $2 \pi$ as $\varepsilon \rightarrow 0$, we look for the solution $T(\varepsilon)$ that approaches $2 \pi$ as $\varepsilon \rightarrow 0$.

Step 1. Consider

$$
F(a, \varepsilon)=\frac{1}{\varepsilon}\left(X_{2}\left(2 \pi+\sqrt{\varepsilon} \cdot a,\binom{1-\varepsilon c V}{V}, \varepsilon\right)-V\right) 29
$$

which is called a desingularization of the function $\varepsilon F(a, \varepsilon)$. To apply the Implicit Function Theorem, we need to find $a \in \mathbb{R}$ such that

$$
F(a, 0)=0,
$$

[^19]

Case 1


Case 2


Case 3

Figure 4.6: Three possible locations of solution $x_{\varepsilon}$ with respect to $x_{0}$.
where $F(a, 0)=\lim _{\varepsilon \rightarrow 0} F(a, \varepsilon)$ by definition. We have

$$
\begin{aligned}
F(a, 0) & \stackrel{L}{=} \lim _{\varepsilon \rightarrow 0} \frac{\partial}{\partial \varepsilon}\left(X _ { 2 } \left(\boxed{\left.\left.2 \pi+\sqrt{\varepsilon} \cdot a,\binom{1-\varepsilon c V}{V}, \varepsilon\right)-V\right)=}\right.\right. \\
& =\lim _{\varepsilon \rightarrow 0}\left(\frac{X_{2 t}^{\prime}(\square)}{2 \sqrt{\varepsilon}} a+X_{2_{x_{1}}}^{\prime}(\square) \cdot(-c V)+X_{2_{\varepsilon}}^{\prime}(\square)\right)= \\
& \stackrel{L}{=} \frac{1}{2} X_{2 t t}^{\prime \prime}\left(2 \pi,\binom{1}{V}, 0\right) a^{2}-c V \cdot X_{2_{x_{1}}^{\prime}}^{\prime}\left(2 \pi,\binom{1}{V}, 0\right)+X_{2_{\varepsilon}}^{\prime}\left(2 \pi,\binom{1}{V}, 0\right) .
\end{aligned}
$$

Since

$$
X_{2}\left(2 \pi,\binom{\xi_{1}}{V}, 0\right)=V \quad \text { for any } \xi_{1} \in \mathbb{R} \quad \Longrightarrow \quad X_{2 x_{1}}^{\prime}\left(2 \pi,\binom{1}{V}, 0\right)=0
$$

the equation

$$
F(a, 0)=0
$$

yields

$$
\begin{equation*}
a= \pm \sqrt{-2 \cdot \frac{X_{2}^{\prime}\left(2 \pi,\binom{1}{V}, 0\right)}{X_{2 t t}^{\prime \prime}\left(2 \pi,\binom{1}{V}, 0\right)}} . \tag{4.28}
\end{equation*}
$$

The two roots correspond to the two intersections of the solution

$$
x_{\varepsilon}(t)=X\left(t,\binom{1-\varepsilon c V}{V}, \varepsilon\right)
$$

with the line $x_{2}=V$, see Fig. 4.6 (case 3).
To check the condition

$$
F_{a}^{\prime}(a, 0) \neq 0
$$

of the Implicit Function Theorem, we compute

$$
F_{a}^{\prime}(a, \varepsilon)=\frac{X_{2 t}^{\prime}\left(2 \pi+\sqrt{\varepsilon} \cdot a,\binom{1-\varepsilon c V}{V}, \varepsilon\right)}{\sqrt{\varepsilon}}
$$

Therefore,

$$
F_{a}^{\prime}(a, 0) \stackrel{L}{=} X_{2 t t}^{\prime \prime}\left(2 \pi,\binom{1}{V}, 0\right)
$$

Step 2. Computing $X_{2 t t}^{\prime \prime}\left(2 \pi,\binom{1}{V}, 0\right)$. By taking the derivative of

$$
\begin{equation*}
X_{t}^{\prime}\left(t,(1, V)^{T}, \varepsilon\right)=g\left(X\left(t,(1, V)^{T}, \varepsilon\right), \varepsilon\right) \tag{4.29}
\end{equation*}
$$

in $t$, one gets

$$
\begin{aligned}
X_{2 t t}^{\prime \prime}\left(2 \pi,(1, V)^{T}, 0\right) & =g_{2}^{\prime}\left(X\left(2 \pi,(1, V)^{T}, 0\right), 0\right) X_{t}^{\prime}\left(2 \pi,(1, V)^{T}, 0\right)= \\
& =g_{2_{x}^{\prime}}^{\prime}\left(X\left(2 \pi,(1, V)^{T}, 0\right), 0\right) g\left(X\left(2 \pi,(1, V)^{T}, 0\right), 0\right)
\end{aligned}
$$

The function $t \mapsto X\left(t,(1, V)^{T}, 0\right)$ is the solution of the linear system

$$
\begin{align*}
& \dot{x}_{1}=x_{2}  \tag{4.30}\\
& \dot{x}_{2}=-x_{1}+1 .
\end{align*}
$$

Therefore, $X\left(2 \pi,(1, V)^{T}, 0\right)=X\left(0,(1, V)^{T}, 0\right)=(1, V)^{T}$ and

$$
X_{2 t t}^{\prime \prime}\left(2 \pi,\binom{1}{V}, 0\right)=(-1,0)\binom{V}{0}=-V
$$

Step 3. Computing $X_{2_{\varepsilon}}^{\prime}\left(2 \pi,\binom{1}{V}, 0\right)$. Let $y(t)=X_{\varepsilon}^{\prime}\left(t,(1, V)^{T}, 0\right)$. By taking the derivative of 4.29) in $\varepsilon$ one gets

$$
\dot{y}=g_{x}^{\prime}\left(X\left(t,(1, V)^{T}, 0\right), 0\right) y+g_{\varepsilon}^{\prime}\left(X\left(t,(1, V)^{T}, 0\right), 0\right)
$$

or

$$
\dot{y}=\left(\begin{array}{ll}
0 & 1  \tag{4.31}\\
-1 & 0
\end{array}\right) y+\binom{0}{-c X_{2}\left(t,(1, V)^{T}, 0\right)+f\left(X_{2}\left(t,(1, V)^{T}, 0\right)-V\right)} .
$$

By solving 4.30,

$$
X_{2}\left(t,(1, V)^{T}, 0\right)=V \cos (t)
$$

and the method of variation of constants yields

$$
y(t)=Z(t) y(0)+Z(t) \int_{0}^{t} Z(-\tau)\binom{0}{-c V \cos \tau+f(V \cos \tau-V)} d \tau
$$

where $Z(t)$ is the solution of the linear part of (4.31) given by

$$
Z(t)=\left(\begin{array}{ll}
\cos t & \sin t \\
-\sin t & \cos t
\end{array}\right)
$$

By noticing that $y(0)=0$, we finally conclude

$$
y_{2}(2 \pi)=\int_{0}^{2 \pi} \cos \tau(-c V \cos \tau+f(V \cos \tau-V)) d \tau=-c V \pi+\int_{0}^{2 \pi} \cos \tau f(V \cos \tau-V) d \tau
$$

Step 4. Substituting the results of Step 2 and Step 3 into 4.28, and by using assumption (4.26), we conclude the existence of $a_{-}<0$ and $a_{+}>0$ such that

$$
F\left(a_{-}, 0\right)=F\left(a_{+}, 0\right)=0 \quad \text { and } \quad F_{a}^{\prime}\left(a_{-}, 0\right) \cdot F_{a}^{\prime}\left(a_{+}, 0\right) \neq 0
$$

Therefore, by the Implicit Function Theorem, there exist

$$
a_{-}(\varepsilon) \rightarrow a_{-} \quad \text { and } \quad a_{+}(\varepsilon) \rightarrow a_{+} \quad \text { as } \quad \varepsilon \rightarrow 0
$$

such that

$$
F\left(a_{-}(\varepsilon), \varepsilon\right)=F\left(a_{+}(\varepsilon), \varepsilon\right)=0 \quad \text { for all } \varepsilon>0 \text { sufficiently small. }
$$

This implies

$$
x_{2, \varepsilon}\left(2 \pi+\sqrt{\varepsilon} a_{-}(\varepsilon)\right)=x_{2, \varepsilon}\left(2 \pi+\sqrt{\varepsilon} a_{+}(\varepsilon)\right)=V \quad \text { for all } \varepsilon>0 \text { sufficiently small. }
$$

The following three cases are possible now (see Fig. 4.6).
Case 1: $x_{1, \varepsilon}\left(2 \pi+\sqrt{\varepsilon} a_{-}(\varepsilon)\right)>1-\varepsilon c V$. This case is impossible because it implies that the solution $x_{\varepsilon}$ crosses itself on ( $0,2 \pi+\sqrt{\varepsilon} a_{-}(\varepsilon)$ ), which cannot happen because of uniqueness of solutions of (4.27).

Case 2: $x_{1, \varepsilon}\left(2 \pi+\sqrt{\varepsilon} a_{-}(\varepsilon)\right)=1-\varepsilon c V$. In this case $x_{\varepsilon}$ is a $\left(2 \pi+\sqrt{\varepsilon} a_{-}(\varepsilon)\right)$-periodic solution of (4.27) that intersects $x_{2}=V$ at only one point $(1-\varepsilon c V, V)$. This contradicts the existence of the second intersection $\left(x_{1, \varepsilon}\left(2 \pi+\sqrt{\varepsilon} a_{+}(\varepsilon)\right), V\right)$.

Case 3: $x_{1, \varepsilon}\left(2 \pi+\sqrt{\varepsilon} a_{-}(\varepsilon)\right)<1-\varepsilon c V$. This is what needed to prove.
The proof of the proposition is complete.
I just discovered that stick-slip limit cycles also occur in power electronics [93], but I didn't dive into details as yet.
4.3.3.1 The analysis of fixed points of a singular map associated to the dry friction oscillator
http://www.utdallas.edu/~ makarenkov/dry-friction-singular-map.jpg
4.3.3.2 Time map of the Poincaré map of the dry friction oscillator
http://www.utdallas.edu/~ makarenkov/T(epsilon)-grazing.pdf

### 4.3.4 Existence and stability of fixed points of a Poincaré map which is given

 as an expansion in powers of a small parameterhttp://www.utdallas.edu/~ makarenkov/singular-fixed-point-epsilon.pdf
4.3.5 Existence and asymptotic stability of a limit cycle in a mechanical oscillator with state-dependent impulses: the grazing case

In this section we consider a prototypic system where impulsive limit cycles occur as a bifurcation from a cycle which is just tangent to the switching threshold at the bifurcation value of the parameter. Specifically, we consider the following modification of the system from Section 4.3.1.

$$
\begin{align*}
& \dot{x}=y  \tag{4.32}\\
& \dot{y}=-\varepsilon b y-x \\
& \binom{x(t+0)}{y(t+0)}=\binom{x(t-0)+\sqrt{\varepsilon} \cdot l}{1-\varepsilon \cdot h}, \quad \text { if } \quad y(t-0)=1, \tag{4.33}
\end{align*}
$$

where $b, l, h$ are constants.


Figure 4.7: Illustration of the Poincare map of system 4.32- 4.33

Theorem 14 If $l^{2}<b \pi<2 l^{2}$, then for all $\varepsilon>0$ sufficiently small, the impulsive system (4.32)-(4.33) has an impulsive orbitally stable periodic solution $\left(x_{\varepsilon}(t), y_{\varepsilon}(t)\right)$ which converges to the unit circle of period $2 \pi$ as $\varepsilon \rightarrow 0$.

Proof. Step 1: Computing the expansion of the time map $T(x, \varepsilon)$. Expanding $Y\left(T,\binom{x}{1-\varepsilon h}, \varepsilon\right)$ in Taylor series in $T$ about $T=2 \pi$ one can rewrite

$$
Y\left(t,\binom{x}{1-\varepsilon h}, \varepsilon\right)=1
$$

as

$$
\begin{equation*}
c(x, \varepsilon)+b(x, \varepsilon)(T-2 \pi)+a(T, x, \varepsilon)(T-2 \pi)^{2}=0, \tag{4.34}
\end{equation*}
$$

where

$$
\begin{aligned}
c(x, \varepsilon) & =-1+Y\left(2 \pi,\binom{x}{1-\varepsilon h}, \varepsilon\right) \\
b(x, \varepsilon) & =Y_{t}^{\prime}\left(2 \pi,\binom{x}{1-\varepsilon h}, \varepsilon\right), \\
a(T, x, \varepsilon) & =\frac{1}{2} \int_{0}^{1} Y_{t t}^{\prime \prime}\left(2 \pi+\lambda(T-2 \pi),\binom{x}{1-\varepsilon h}, \varepsilon\right)(1-\lambda) d \lambda .
\end{aligned}
$$

Direct substitution shows that $T=2 \pi$ is a solution of (4.34) when $x=\varepsilon=0$. Our goal is to continue this solution when $v$ and $\varepsilon$ deviate from 0 . Since $c(0,0)=b(0,0)=0$ and $a(2 \pi, 0,0) \neq 0$, we expect that (4.34) has two solutions $T^{-}(x, \varepsilon) \leq T^{+}(x, \varepsilon)$ that converge to $2 \pi$ when $(x, \varepsilon) \rightarrow 0$.
To emphasize the structure of (4.34) we expand $c(x, \varepsilon)$ and $b(x, \varepsilon)$ further as

$$
c(x, \varepsilon)=\varepsilon \bar{c}(x, \varepsilon), \quad b(x, \varepsilon)=x \tilde{b}(x, \varepsilon)+\varepsilon \bar{b}(x, \varepsilon)
$$

where

$$
\begin{aligned}
\bar{c}(x, \varepsilon) & =\int_{0}^{1} c_{\varepsilon}^{\prime}(x, \lambda \varepsilon)(1-\lambda) d \lambda \\
\tilde{b}(x, \varepsilon) & =\int_{0}^{1} b_{x}^{\prime}(\lambda x, \lambda \varepsilon)(1-\lambda) d \lambda, \quad \bar{b}(x, \varepsilon)=\int_{0}^{1} b_{\varepsilon}^{\prime}(\lambda x, \lambda \varepsilon)(1-\lambda) d \lambda,
\end{aligned}
$$

to get

$$
\begin{equation*}
\varepsilon \bar{c}(x, \varepsilon)+(x \tilde{b}(x, \varepsilon)+\varepsilon \bar{b}(x, \varepsilon))(T-2 \pi)+a(T, x, \varepsilon)(T-2 \pi)^{2}=0 . \tag{4.35}
\end{equation*}
$$

One way to prove solvability of (4.35) in $T$ would be by dividing it by $\varepsilon$ and further applying the implicit function theorem for implicit functions that branch from the boundary of a set (Makarenkov [102, Theorem C.1]) to the set $x \leq \operatorname{const} \cdot \sqrt{\varepsilon}$. We will however offer a method which doesn't require any advanced implicit function theorems. Indeed, focusing on

$$
\begin{equation*}
x \geq 0 \tag{4.36}
\end{equation*}
$$

consider the change of the variables

$$
\begin{equation*}
x=\sqrt{r \cos \phi}, \quad \varepsilon=r \sin \phi . \tag{4.37}
\end{equation*}
$$

Then (4.35) takes the form

$$
r \overline{\bar{c}}(r, \phi) \sin \phi+(\sqrt{r} \tilde{\tilde{b}}(r, \phi) \cos \phi+r \overline{\bar{b}}(r, \phi) \sin \phi)(T-2 \pi)+\hat{\hat{a}}(T, r, \phi)(T-2 \pi)^{2}=0
$$

where $\overline{\bar{c}}(r, \phi)=\bar{c}(\sqrt{r \cos \phi}, r \sin \phi), \tilde{\tilde{b}}(r, \phi)=\tilde{b}(\sqrt{r \cos \phi}, r \sin \phi), \overline{\bar{b}}(r, \phi)=\bar{b}(\sqrt{r \cos \phi}, r \sin \phi)$, $\hat{\hat{a}}(T, r, \phi)=\hat{a}(T, \sqrt{r \cos \phi}, r \sin \phi)$. Introducing

$$
\tau=\frac{T-2 \pi}{\sqrt{r}}
$$

we get the equation $F(\tau, r, \phi)=0$,

$$
F(\tau, r, \phi)=\overline{\bar{c}}(r, \phi) \sin \phi+(\tilde{\tilde{b}}(r, \phi) \sqrt{\cos \phi}+\sqrt{r} \overline{\bar{b}}(r, \phi) \sin \phi) \tau+\hat{\hat{a}}(\sqrt{r} \cdot \tau+2 \pi, r, \phi) \tau^{2}
$$

which we will now solve in $\tau$ near $\tau=0$ for $|r|>0$ sufficiently small using the standard implicit function theorem.
The equation $F(\tau, 0, \phi)=0$ reads as

$$
\bar{c}(0,0) \sin \phi+\tilde{b}(0,0) \sqrt{\cos \phi} \cdot \tau+a(2 \pi, 0,0) \tau^{2}=0
$$

whose smaller root is

$$
\bar{\tau}(\phi)=-b_{0} \sqrt{\cos \phi}-\sqrt{b_{0}^{2} \cos \phi-c_{0} \sin \phi},
$$

if

$$
\begin{equation*}
a(2 \pi, 0,0)<0 \tag{4.38}
\end{equation*}
$$

Here $b_{0}=\tilde{b}(0,0) /(2 a(2 \pi, 0,0))$ and $c_{0}=\bar{c}(0,0) / a(2 \pi, 0,0)$. Furthermore, we have $F_{\tau}^{\prime}(\bar{\tau}(\phi), 0, \phi) \neq 0$, if

$$
\begin{equation*}
b_{0}^{2} \cos \phi-c_{0} \sin \phi>0 . \tag{4.39}
\end{equation*}
$$

By the implicit function theorem we conclude that for any $\delta>0$ there exist $r_{1}>0$ and $\tau_{1}>0$ such that for all $(r, \phi) \in\left\{(r, \phi): 0 \leq r \leq r_{1},|\sin \phi| \geq \delta\right\}$ the equation $F(\tau, r, \phi)=0$ has a unique solution $\tau(r, \phi) \in\left(-\tau_{1}, \tau_{1}\right)$. Moreover, $(r, \phi) \mapsto \tau(r, \phi)$ is continuously differentiable on $\left\{(r, \phi): 0 \leq r \leq r_{1},|\sin \phi| \geq \delta\right\}$.
Expanding $\tau(r, \phi)$ as

$$
\tau(r, \phi)=\bar{r}(\phi)+\Delta(\sqrt{r}, \phi)
$$

one can see that $\Delta_{\phi}^{\prime}(0, \phi)=0$ and, by using the formula for the derivative of the implicit function,

$$
\Delta_{r}^{\prime}(0, \phi)=-\overline{\bar{b}}(0, \phi) \sin \phi \bar{\tau}(\phi) \cdot \frac{1}{2 a(2 \pi, 0,0) \bar{\tau}(\phi)}=-b_{1} \sin \phi,
$$

where $b_{1}=\bar{b}(0,0) / 2 a(2 \pi, 0,0)$. Therefore,

$$
\Delta(r, \phi)=-b_{1} \sin \phi \cdot r+o(r, \phi)
$$

where

$$
\begin{equation*}
o_{\phi}^{\prime}(0, \phi)=o_{r}^{\prime}(0, \phi)=\lim _{r \rightarrow 0} \frac{o(r, \phi)}{r}=0 . \tag{4.40}
\end{equation*}
$$

Summing up, we obtain

$$
T=2 \pi+\bar{r}(\phi) \sqrt{r}-b_{1} \sin \phi \cdot r+o(r, \phi) \sqrt{r}
$$

or, reversing the change of the variables (4.37),

$$
T(x, \varepsilon)=2 \pi-b_{0} x-\sqrt{b_{0}^{2} x^{2}-c_{0} \varepsilon}-b_{1} \varepsilon+\tilde{o}(x, \varepsilon)
$$

where

$$
\begin{equation*}
\tilde{o}(x, \varepsilon)=o(r(x, \varepsilon), \phi(x, \varepsilon)) \sqrt{r(x, \varepsilon)}, \quad \frac{\tilde{o}(x, \varepsilon)}{\left(\sqrt{x^{4}+\varepsilon^{2}}\right)^{3}} \rightarrow 0 \quad \text { as } \quad(x, \varepsilon) \rightarrow 0 \tag{4.41}
\end{equation*}
$$

Step 2: Expanding $P_{\varepsilon}(x)=X\left(T(x, \varepsilon),\binom{x}{1-\varepsilon h}, \varepsilon\right)+\sqrt{\varepsilon} \cdot l$ in powers of $\varepsilon$. Since

$$
\begin{aligned}
X\left(T,\binom{x}{y}, \varepsilon\right)= & X_{t}^{\prime}\left(2 \pi,\binom{0}{y}, 0\right)(T-2 \pi)+X_{x}^{\prime}\left(2 \pi,\binom{0}{y}, 0\right) x+ \\
& +d(y) \varepsilon+\hat{o}\left(T-2 \pi,\binom{x}{y}, \varepsilon\right), \quad d(y)=X_{\varepsilon}^{\prime}\left(2 \pi,\binom{0}{y}, 0\right)
\end{aligned}
$$

we get

$$
\begin{aligned}
P_{\varepsilon}(x)= & -b_{0} x-\sqrt{b_{0}^{2} x^{2}-c_{0} \varepsilon}-b_{1} \varepsilon+x+d(1-\varepsilon h) \varepsilon+\sqrt{\varepsilon} \cdot l+ \\
& +\tilde{o}(x, \varepsilon)+\hat{o}\left(T(x, \varepsilon)-2 \pi,\binom{x}{1-\varepsilon h}, \varepsilon\right)
\end{aligned}
$$

Step 3: Studying the dynamics of the reduced map

$$
\bar{P}_{\varepsilon}(x)=-b_{0} x-\sqrt{b_{0}^{2} x^{2}-c_{0} \varepsilon}+x+\sqrt{\varepsilon} \cdot l .
$$

Solving $\bar{P}_{\varepsilon}(x)=x$ one gets

$$
\begin{equation*}
\bar{x}_{\varepsilon}=\frac{l^{2}+c_{0}}{2 b_{0} l} \sqrt{\varepsilon} \tag{4.42}
\end{equation*}
$$

Computing $\left(\bar{P}_{\varepsilon}\right)^{\prime}\left(\bar{x}_{\varepsilon}\right)$ gives

$$
\left(\bar{P}_{\varepsilon}\right)^{\prime}\left(\bar{x}_{\varepsilon}\right)=1+\frac{2 b_{0} l^{2}}{c_{0}-l^{2}} .
$$

Thus the fixed point $\bar{x}_{\varepsilon}$ is asymptotically stable, if

$$
\begin{equation*}
-2<\frac{2 b_{0} l^{2}}{c_{0}-l^{2}}<0 \tag{4.43}
\end{equation*}
$$

Step 4: Linking the map $\bar{P}_{\varepsilon}$ to the map $P_{\varepsilon}$. The map $P_{\varepsilon}$ will have a fixed point $x_{\varepsilon}$ close to $\bar{x}_{\varepsilon}$ with the same stability property, if

$$
\frac{\tilde{o}\left(\bar{x}_{\varepsilon}, \varepsilon\right)}{\sqrt{\varepsilon}} \rightarrow 0 \quad \text { and } \quad \frac{\tilde{o}_{x}^{\prime}\left(\bar{x}_{\varepsilon}, \varepsilon\right)}{\sqrt{\varepsilon}} \rightarrow 0 \quad \text { as } \varepsilon \rightarrow 0
$$

The first of these two relations follows from (4.41), so we focus on proving the second one. We have

$$
\begin{aligned}
\frac{\tilde{o}_{x}^{\prime}\left(\bar{x}_{\varepsilon}, \varepsilon\right)}{\sqrt{\varepsilon}}= & \frac{1}{\sqrt{\varepsilon}} \cdot \frac{d}{d x}[o(r(x, \varepsilon), \phi(x, \varepsilon)) \sqrt{r(x, \varepsilon)}]= \\
= & \left(o_{r}^{\prime}(r(x, \varepsilon), \phi(x, \varepsilon)) r_{x}^{\prime}(x, \varepsilon)+o_{\phi}^{\prime}(r(x, \varepsilon), \phi(x, \varepsilon)) \phi_{x}^{\prime}(x, \varepsilon)\right) \sqrt{\frac{r(x, \varepsilon)}{\varepsilon}}+ \\
& +\frac{o(r(x, \varepsilon), \phi(x, \varepsilon))}{2 \sqrt{\varepsilon r(x, \varepsilon)}} r_{x}^{\prime}(x, \varepsilon)
\end{aligned}
$$

and now we go over evaluating the limits of the entries of this expression.
By taking the derivatives of the two identities

$$
x=\sqrt{r(x, \varepsilon) \cos \phi(x, \varepsilon)} \quad \text { and } \quad \varepsilon=r(x, \varepsilon) \sin \phi(x, \varepsilon)
$$

in $x$, and solving for $r_{x}^{\prime}(x, \varepsilon)$ and $\phi_{x}^{\prime}(x, \varepsilon)$, we conclude

$$
r_{x}^{\prime}(x, \varepsilon)=\frac{2 x^{3}}{\sqrt{\varepsilon^{2}+x^{4}}} \quad \text { and } \quad \phi_{x}^{\prime}(x, \varepsilon)=-\frac{2 \varepsilon x}{\sqrt{\varepsilon^{2}+x^{4}}}
$$

which implies $r_{x}^{\prime}\left(\bar{x}_{\varepsilon}, \varepsilon\right) \rightarrow 0$ and $r_{\varepsilon}^{\prime}\left(\bar{x}_{\varepsilon}, \varepsilon\right) \rightarrow 0$ as $\varepsilon \rightarrow 0$.
For the last reminder, we have

$$
\frac{o(r(x, \varepsilon), \phi(x, \varepsilon))}{\sqrt{\varepsilon r(x, \varepsilon)}}=\frac{o(r, \phi)}{\sqrt{r^{2} \sin \phi}}
$$

which converges to 0 by 4.40 and by noticing that

$$
\begin{equation*}
\frac{\sin \phi\left(\bar{x}_{\varepsilon}, \varepsilon\right)}{\cos \phi\left(\bar{x}_{\varepsilon}, \varepsilon\right)}=\frac{\varepsilon}{\bar{x}_{\varepsilon}^{2}}=\left(\frac{2 b_{0} l}{l^{2}+c_{0}}\right)^{2} \tag{4.44}
\end{equation*}
$$

Step 5: Verifying conditions (4.36), 4.38, 4.39), and 4.43).
To demonstrate 4.36 we will check that both $c_{0}$ and $b_{0}$ in 4.42) are positive. Denoting $\xi=(0,1)^{T}$, we have

$$
\begin{aligned}
\tilde{b}(0,0) & =(1 / 2) b_{x}^{\prime}(0,0)=(1 / 2) Y_{t x}^{\prime \prime}(2 \pi, \xi, 0)=-1 / 2 \\
a(2 \pi, 0,0) & =(1 / 4) Y_{t t}^{\prime \prime}(2 \pi, \xi, 0)=-1 / 2 \\
\bar{c}(0,0) & =(1 / 2) c_{\varepsilon}^{\prime}(0,0)=Y_{\varepsilon}^{\prime}(2 \pi, \xi, 0)=-b \pi / 2 \\
b_{0} & =\tilde{b}(0,0) /(2 a(2 \pi, 0,0))=-(1 / 2) /(2 \cdot(-1 / 2))=1 / 2 \\
c_{0} & =\bar{c}(0,0) / a(2 \pi, 0,0)=-(b \pi / 2) /(2 \cdot(-1 / 2))=b \pi / 2
\end{aligned}
$$

which also proves (4.38).
Using (4.44) we compute

$$
b_{0}^{2} \cos \phi-c_{0} \sin \phi=b_{0}^{2} \cos \phi \frac{\left(l^{2}-c_{0}\right)^{2}}{\left(l^{2}+c_{0}\right)^{2}}>0, \quad \text { if } \quad l^{2} \neq c_{0} .
$$

Finally, (4.43) holds because (4.43) is equivalent to $\left(1-b_{0}\right) l^{2}<c_{0}<l^{2}$.
01/19/2017: Added Proposition 6 .

## References Cited

[1] E. A. Barbashin, Introduction to the theory of stability, Wolters-Noordhoff, 1970.
[2] http://www.seekic.com/circuit_diagram/Control_Circuit/single_chip_pump_controller.html
[3] W. Pasillas-Lepine, Hybrid modeling and limit cycle analysis for a class of five-phase anti-lock brake algorithms, Vehicle System Dynamics 44 (2006), no. 2, 173-188.
[4] M. Tanelli, G. Osorio, M. di Bernardo, S. Savaresi, A. Astolfi, Existence, stability and robustness analysis of limit cycles in hybrid anti-lock braking systems. Int. J. Control 82 (2009) 659-678.
[5] M. Meza, A. Bhaya, Realistic threshold policy with hysteresis to control predator-prey continuous dynamics, Theory Biosci. 128 (2009) 139-149.
[6] A. Wang, Y. Xiaoa, Global dynamics of a piece-wise epidemic model with switching vaccination strategy, Discrete and continuous dynamical systems - Series B 19 (2014), no. 9, 2915-2940.
[7] G. Tanaka, K. Tsumoto, S. Tsuji, K. Aihara, Bifurcation analysis on a hybrid systems model of intermittent hormonal therapy for prostate cancer, Physica D 237 (2008), no 20, 2616-2627.
[8] E. M. Izhikevich, Resonate-and-fire neurons, Neural Networks 14 (2001), no 6-7, 883-894.
[9] A. Tonnelier, The McKean's Caricature of the Fitzhugh-Nagumo Model I. The Space-Clamped System, SIAM J. Appl. Math. 63 (2002), no. 2, 459-484.
[10] P. Holmes, Some Joys and Trials of Mathematical Neuroscience, J. Nonlinear Sci. 24 (2014), no. 2, 201-242.
[11] O. Makarenkov, J. Lamb, Dynamics and bifurcations of nonsmooth systems: A survey, Physica D 241 (2012), no. 22, 1826-1844.
[12] A. F. Filippov, Differential equations with discontinuous righthand sides. Mathematics and its Applications (Soviet Series), 18. Kluwer Academic Publishers Group, Dordrecht, 1988.
[13] T. M. Apostol, Mathematical analysis. Second edition. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1974. xvii+492 pp.
[14] Kolmogorov, A. N.; Fomn, S. V. Introductory real analysis. Translated from the second Russian edition and edited by Richard A. Silverman. Corrected reprinting. Dover Publications, Inc., New York, 1975. xii +403 pp
[15] J. Sotomayor, M. A. Teixeira, Regularization of discontinuous vector fields. International Conference on Differential Equations (Lisboa, 1995), 207-223, World Sci. Publ., River Edge, NJ, 1998.
[16] M. Kunze, Non-smooth dynamical systems. Lecture Notes in Mathematics, 1744. Springer-Verlag, Berlin, 2000. x+228 pp.
[17] K. Deimling, Multivalued differential equations. de Gruyter Series in Nonlinear Analysis and Applications, 1. Walter de Gruyter \& Co., Berlin, 1992. xii+260 pp.
[18] J. Stewart, Calculus: Early Transcendentals, 7th Edition, Brooks/Cole, 2012.
[19] A. Goswami, B. Espiau, A. Keramane, Limit cycles in a passive compass gait biped and passivity-mimicking control laws, Autonomous Robots 4 (1997), no. 3, 273-286.
[20] T. McGeer, Passive dynamic walking, International journal of robotics research 9 (1990), no. $2,62-82$.
[21] Passive walking video https://www.youtube.com/watch?v=N64KOQkbyiI
[22] S. Mazumder, K. Acharya, Multiple Lyapunov function based reaching condition for orbital existence of switching power converters, IEEE Transactions on Power Electronics 23 (2008), no. 3, 1449-1471.
[23] T. Hu, A Nonlinear-System Approach to Analysis and Design of Power-Electronic Converters With Saturation and Bilinear Terms, IEEE Transactions on Power Electronics 26 (2011), no. 2, 399-410.
[24] M. A. Krasnoselskii, The operator of translation along the trajectories of differential equations. Translations of Mathematical Monographs, Vol. 19. Translated from the Russian by Scripta Technica. American Mathematical Society, Providence, R.I. 1968 vi+294 pp.
[25] P. Hartman, Ordinary differential equations. John Wiley \& Sons, Inc., New York-London-Sydney 1964 xiv+612 pp.
[26] W. Walter, Ordinary differential equations. Translated from the sixth German (1996) edition by Russell Thompson. Graduate Texts in Mathematics, 182. Readings in Mathematics. Springer-Verlag, New York, 1998. xii +380 pp.
[27] J. K. Hale, Ordinary differential equations. Second edition. Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1980. xvi+361 pp.
[28] Coddington, Earl A.; Levinson, Norman Theory of ordinary differential equations. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955. xii+429 pp.
[29] S. P. Bhat, D. S. Bernstein, Continuous finite-time stabilization of the translational and rotational double integrators. IEEE Trans. Automat. Control 43 (1998), no. 5, 678-682.
[30] S. P. Bhat, D. S. Bernstein, Finite-time stability of continuous autonomous systems. SIAM J. Control Optim. 38 (2000), no. 3, 751-766.
[31] B. E. Paden, S. S. Sastry, A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators. IEEE Trans. Circuits and Systems 34 (1987), no. 1, 73-82.
[32] R. Santiesteban, L. Fridman, J. Moreno, Finite-time convergence analysis for Twisting controller via a strict Lyapunov function, Proceedings of 2010 11th International Workshop on Variable Structure Systems, Mexico City, Mexico, June 26-28, 2010,
[33] Y. Orlov, Finite time stability and robust control synthesis of uncertain switched systems. SIAM J. Control Optim. 43 (2004/05), no. 4, 1253-1271.
[34] J. W. Grizzle, G. Abba, F. Plestan, Asymptotically stable walking for biped robots: analysis via systems with impulse effects.(English summary) IEEE Trans. Automat. Control 46 (2001), no. 1, 51-64.
[35] W. E. Boyce, R. C. DiPrima, Elementary Differential Equations and Boundary Value Problems, 10th Edition, Wiley, 2012.
[36] F. H. Clarke, Optimization and nonsmooth analysis. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley \& Sons, Inc., New York,1983.
[37] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko, The mathematical theory of optimal processes. Translated from the Russian by K. N. Trirogoff; edited by L. W. Neustadt Interscience Publishers John Wiley \& Sons, Inc. New York-London 1962.
[38] P. Mason, M. Broucke, B. Piccoli, Time optimal swing-up of the planar pendulum. IEEE Trans. Automat. Control 53 (2008), no. 8, 1876-1886.
[39] C. C. Chung, J. Hauser, Nonlinear control of a swinging pendulum. Automatica J. IFAC 31 (1995), no. 6, 851-862.
[40] Time-optimal pendulum swing-up by moving the cart forth and back (movie): https://www.youtube.com/watch?v=iKvGr9IfVyE.
[41] M. Forti, P. Nistri, Global convergence of neural networks with discontinuous neuron activations. IEEE Trans. Circuits Systems I Fund. Theory Appl. 50 (2003), no. 11, 1421-1435.
[42] Q. Hui, W. Haddad, M. Wassim, S. Bhat, Semistability, finite-time stability, differential inclusions, and discontinuous dynamical systems having a continuum of equilibria. IEEE Trans. Automat. Control 54 (2009), no. 10, 2465-2470.
[43] R. Santiesteban, Time convergence estimation of a perturbed double integrator: Family of continuous sliding mode based output feedback synthesis, Proceedings of 2013 European Control Conference, 3764-3769.
[44] A. Sinha, D. Miller, Optimal Sliding Mode Control of a Flexible Spacecraft under Stochastic Disturbances, Journal of Guidance, Control, and Dynamics, 18 (1995), no. 3, 486-492.
[45] A. Machina, R. Edwards, P. van den Driessche, Singular Dynamics in Gene Network Models, SIAM J. Applied Dynamical Systems 12 (2013), no. 1,95-125.
[46] B. Veselic, C. Milosavljevic, Sliding mode based harmonic oscillator synchronization, Int. J. Electronics 90 (2003), no. 9, 553-570.
[47] V. I. Utkin, Variable Structure Systems with Sliding Modes, IEEE Transactions of Automatic Control 22 (1977), no. 2, 212-222.
[48] D. Shevitz, B. Paden, Lyapunov Stability Theory of Nonsmooth Systems, IEEE Transactions on Automatic Control 39 (1994), no. 9, 1910-1914.
[49] B. Wen, Recent development of vibration utilization engineering, Front. Mech. Eng. Chin. 3 (1) (2008) 1-9.
[50] J. A. Moreno, M. Osorio, Strict Lyapunov Functions for the Super-Twisting Algorithm, IEEE Transactions on Automatic Control 57 (2012), no. 4, 1035-1040.
[51] M. Zak, Terminal model of Newtonian dynamics. Internat. J. Theoret. Phys. 32 (1993), no. 1, 159-190.
[52] R. Leine, H. Nijmeijer, Dynamics and bifurcations of non-smooth mechanical systems, Springer, 2004.
[53] J. Huang, D. L. Turcotte, Evidence for chaotic fault interactions in the seismicity of the San Andreas fault and Nankai trough, Nature 348 (1990) 234-236.
[54] K. Popp and P. Stelter, Philosophical Transactions: Physical Sciences and Engineering 332 (1990), no. 1624, 89-105.
[55] U. Galvanetto, S. R. Bishop, Dynamics of a Simple Damped Oscillator Undergoing Stick-Slip Vibrations, Meccanica 34 (1999) 337-347.
[56] M. Kunze, T. Kuepper, Qualitative bifurcation analysis of a non-smooth friction-oscillator model, Z. angew. Math. Phys. 48 (1997) 87-101.
[57] Lefschetz, Solomon Differential equations: Geometric theory. Second edition. Pure and Applied Mathematics, Vol. VI. Interscience Publishers, a division of John Wiley \& Sons, New York-Lond on 1963 x +390 pp.
[58] Yu. I. Neimark, The method of point transformations in the theory of nonlinear oscillations, Izdat. "Nauka", Moscow, 1972. 471 pp.
[59] N. A. Fufaev, To the theory of electromagnetic relay, Automatics and Remote Control 14 (1953), no. 5, 570-587.
[60] V. A. Zorich, Mathematical analysis. I. Translated from the 2002 fourth Russian edition by Roger Cooke. Universitext. Springer-Verlag, Berlin,2004. xviii+574 pp.
[61] S. Liberzon, Switching in systems and control. Systems \& Control: Foundations \& Applications. Birkhauser Boston, Inc., Boston, MA, 2003. xiv+233 pp.
[62] M. Branicky, Multiple Lyapunov functions and other analysis tools for switched and hybrid systems.Hybrid control systems. IEEE Trans. Automat. Control 43 (1998), no. 4, 475-482.
[63] R. A. DeCarlo, M. S. Branicky, S. Pettersson, B. Lennartson, Perspectives and results on the stability and stabilizability of hybrid systems, Proceedings of the IEEE 88 (2000), no. 7, 1069-1082.
[64] M. Johansson, A. Rantzer, Computation of piecewise quadratic Lyapunov functions for hybrid systems. Hybrid control systems. IEEE Trans. Automat. Control 43 (1998), no. 4, 555-559.
[65] Yu.A. Kuznetsov, S. Rinaldi, A. Gragnani, One-parameter bifurcations in planar Filippov systems, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 13 (8) (2003) 2157-2188.
[66] M. Guardia, T.M. Seara, M.A. Teixeira, Generic bifurcations of low codimension of planar Filippov systems, J. Differential Equations 250 (4) (2011) 1967-2023.
[67] D.J.W. Simpson, J.D. Meiss, Andronov-Hopf bifurcations in planar, piecewise-smooth, continuous flows, Phys. Lett. A 371 (3) (2007) 213-220.
[68] M. di Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk, Piecewise-smooth dynamical systems. Theory and applications. Applied Mathematical Sciences, 163. Springer-Verlag London, Ltd., London, 2008. xxii +481 pp.
[69] C. Chicone, Ordinary differential equations with applications. Texts in Applied Mathematics, 34. Springer-Verlag, New York, 1999. xvi+561 pp.
[70] A. A. Andronov, A. A. Vitt, S. E. Khaikin, Theory of oscillators. Translated from the Russian by F. Immirzi; translation edited and abridged by W. Fishwick Pergamon Press, Oxford-New York-Toronto, Ont. 1966 xxxii+815 pp.
[71] A. Polyakov, A. Poznyak, Unified Lyapunov function for a finite-time stability analysis of relay second-order sliding mode control systems, IMA Journal of Mathematical Control and Information 29 (2012), no. 4, 529-550.
[72] A. Levant, Principles of 2-sliding mode design, Automatica 43 (2007) 576-586.
[73] J. L. Mancilla-Aguilara, R. A. Garcia, An extension of LaSalles invariance principle for switched systems, Systems \& Control Letters 55 (2006) 376-384.
[74] M. Arminjon and B. Bacroix, On plastic potentials for anisotropic metals and their derivation from the texture function, Acta Mechanica 88, 219-243 (1990)
[75] page 204 T. Mura, Micromechanics of Defects in Solids, 2nd edition, Springer, 1987.
[76] H. Anton, C. Rorres, Elementary Linear Algebra, 11th edition, Wiley, 2014.
[77] M. Vidyasagar, Nonlinear systems analysis. Reprint of the second (1993) edition. Classics in Applied Mathematics, 42. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. xviii+498 pp.
[78] H. K. Khalil, Nonlinear systems. Macmillan Publishing Company, New York, 1992. xii +564 pp .
[79] M. A. Wicks, P. Peleties, and R. A. DeCarlo. Switched Controller Synthesis for the Quadratic Stabilisation of a Pair of Unstable Linear Systems, European J. Control 4 (1998) 140-147.
[80] P. Bolzern, W. Spinelli, Quadratic stabilization of a switched affine system about a nonequilibrium point, Proceedings of the American Control Conference 5 (2004) 3890-3895.
doi: http://ieeexplore.ieee.org.libproxy.utdallas.edu/stamp/stamp.jsp?tp=\&arnumber=1383918
[81] M. A. Teixeira, Stability conditions for discontinuous vector fields. J. Differential Equations 88 (1990), no. 1, 15-29.
[82] A. Colombo, M. di Bernardo, E. Fossas, M. R. Jeffrey, M. R. Teixeira singularities in 3D switched feedback control systems. Systems Control Lett. 59 (2010), no. 10, 615-622.
[83] W. Hahn, Stability of motion. Translated from the German manuscript by Arne P. Baartz. Die Grundlehren der mathematischen Wissenschaften, Band 138 Springer-Verlag New York, Inc., New York 1967 xi+446 pp.
[84] V. Stramosk, L. Benadero, D. J. Pagano, E. Ponce, Sliding Mode Control of Interconnected Power Electronic Converters in DC Microgrids, Proceedings of 39th Annual Conference of the IEEE Industrial-Electronics-Society (2013) 8385-8390. http://ieeexplore.ieee.org.libproxy.utdallas.edu/stamp/stamp.jsp?tp=\&arnumber=6700538
[85] A. P. N. Tahim, D. J. Pagano, E. Ponce, Nonlinear Control of dc-dc Bidirectional Converters in Stand-alone dc Microgrids, Proceedings of the 51st IEEE Annual Conference on Decision and Control (2012) 3068-3073. http://ieeexplore.ieee.org.libproxy.utdallas.edu/stamp/stamp.jsp?tp=\&arnumber=6426298
[86] Y. M. Lu, X. F. Huang, B. Zhang, L. Y. Yin, Hybrid Feedback Switching Control in a Buck Converter, IEEE International Conference on Automation and Logistics (2008) 207-210.
http://ieeexplore.ieee.org.libproxy.utdallas.edu/stamp/stamp.jsp?arnumber=4636147
[87] T. S. Hu, A Nonlinear-System Approach to Analysis and Design of Power-Electronic Converters With Saturation and Bilinear Terms, IEEE Transactions on Power Electronics 26 (2011), no. 2, 399-410. http://ieeexplore.ieee.org.libproxy.utdallas.edu/stamp/stamp.jsp?tp=\&arnumber=5499054
[88] S. K. Mazumder, K. Acharya, Multiple Lyapunov Function Based Reaching Condition for Orbital Existence of Switching Power Converters, IEEE Transactions on Power Electronics 23 (2008), no. 3, 1449-1471. http://ieeexplore.ieee.org.libproxy.utdallas.edu/stamp/stamp.jsp?arnumber=4483682
[89] M. Denny, The pendulum clock: a venerable dynamical system, Eur. J. Phys 23 (2002) 449-458.
[90] T. Alpcan, T. Baar, A stability result for switched systems with multiple equilibria. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 17 (2010), no. 6, 949-958.
[91] S. Mastellone, D. M. Stipanovic, M. W. Spong, Stability and Convergence for Systems with Switching Equilibria, Proceedings of the 46th IEEE Conference on Decision and Control (2007) 4013-4020.
[92] M. A. Krasnoselskii, The operator of translation along the trajectories of differential equations. Translations of Mathematical Monographs, Vol. 19. Translated from the Russian by Scripta Technica. American Mathematical Society, Providence, R.I. 1968 vi+294 pp.
[93] I.A. Hiskens, J-W. Park and V. Donde, "Dynamic embedded optimization and shooting methods for power system performance assessment", in Applied Mathematics for Deregulated Electric Power Systems: Optimization, Control, and Computational Intelligence, J. Chow, F. Wu and J. Momoh (Editors), Springer, 2004.
[94] K. J. Astrm, Oscillations in systems with relay feedback. Adaptive control, filtering, and signal processing (Minneapolis, MN, 1993), 1-25, IMA Vol. Math. Appl., 74, Springer, New York, 1995.
[95] J. M. Gonalves, A. Megretski, M. A. Dahleh, Global stability of relay feedback systems. IEEE Trans. Automat. Control 46 (2001), no. 4, 550-562.
[96] J. M. Gonalves, A. Megretski, M. A. Dahleh, Global analysis of piecewise linear systems using impact maps and surface Lyapunov functions. IEEE Trans. Automat. Control 48 (2003), no. 12, 2089-2106.
[97] J. Guckenheimer, P. Holmes, Philip Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. Revised and corrected reprint of the 1983 original. Applied Mathematical Sciences, 42. Springer-Verlag, New York, 1990. xvi+459 pp.
[98] M. Rubensson, B. Lennartson, Global Convergence Analysis for Piecewise Linear Systems applied to Limit cycles in a DC/DC Converter, Proceedings of the American Control Conference, Anchorage (2002) 1272-1277.
[99] A. Schild, J. Lunze, J. Krupar, W. Schwarz, Design of Generalized Hysteresis Controllers for DCDC Switching Power Converters, IEEE Transactions on Power Electronics 24 (2009), no. 1, 138-146.
[100] J. W. Perram, A. Shiriaev, C. Canudas de Wit, F. Grognard, Explicit formula for a general integral of motion for a class of mechanical systems subject to holonomic constraint.(English summary) Lagrangian and Hamiltonian methods for nonlinear control 2003, 8792, IFAC, Laxenburg, 2003.
[101] G. Zhai, B. Hu, K. Yasuda, A. Michel, Stability analysis of switched systems with stable and unstable subsystems: an average dwell time approach. Internat. J. Systems Sci. 32 (2001), no. 8, 1055-1061.
[102] O. Makarenkov, Bifurcation of limit cycles from a fold-fold singularity in planar switched systems, accepted to SIAM J. Appl. Dyn. Syst. arXiv:1603.03117

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[^0]:    ${ }^{1}$ Consider the exercise:
    Exercise 1 (optional): Prove proposition 1.
    ${ }^{2}$ There exists a different explicitly defined differential equation $\dot{x}=g(t, x)$ that $x$ satisfies while sliding along a discontinuity. We will touch upon this later.

[^1]:    ${ }^{3}$ Consider the exercise:
    Exercise 2 (optional): Prove property 2.9 a) and (2.9 b)
    ${ }^{4}$ If $A, B \subset \mathbb{R}^{n}$, then $A+B=\cup_{a \in A, b \in B}(a+b)$.
    ${ }^{5}$ References for the proof of Theorem 11 1) Use [12, corollary, p. 69] to conclude that $K[f]$ is an upper semicontinuous set-valued map with compact convex values. 2) Make sure that Filippov's definition of upper semicontinuity [12, bottom of p. 65] coincides with $\varepsilon-\delta$-upper semicontinuity in [17, definition 1.2] or [16, Definition 2.1.1]. Note: there is a misprint in the formula for $\beta$ in [12, p. 64]. The correct formula is $\beta(A, B)=\sup _{\alpha \in A} \rho(a, B)$ (see p. 52 in the Russian original). 3) Apply [17, Theorem 5.1 and Proposition 1.1a] or [16, Theorem 2.2.1] to complete the proof. Both the theorems assume $\sqrt{2.10}$.

[^2]:    ${ }^{6}$ The difference between -1 and $\{-1\}$ is as follows: $-1 \in \mathbb{R}$, while $\{-1\} \subset \mathbb{R}$.

[^3]:    ${ }^{7}$ Consider the exercise:
    Exercise 3 Prove formulas for $K[\phi+\psi]$ and $K[h]$ stated as 1) and 2) at this page.

[^4]:    ${ }^{8} \mathrm{~A}$ similar result for continuous $v$ is [27, Theorem 6.1]

[^5]:    ${ }^{9}$ Note, the continuous dependence holds without assuming $g \in C^{1}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ or $g \in C^{1}(\mathbb{R} \times \mathbb{R} \backslash\{0\}, \mathbb{R})$. We only need the uniqueness of the solution of the original initial value problem that we assumed in the statement of the Lemma.

[^6]:    ${ }^{10}$ This formula is achieved by the method of separation of variables, see e.g. [35, §2.2]. Consider the exercise:

    Exercise 4 Find the solution of 2.25) when $x(0)<0$. Note: equation 2.25) has to be understood as $\dot{x}=k|x|^{\alpha}$ for negative $x$ (i.e. 2.25) can be formulated as $\dot{x}=-k \operatorname{sign}(x)|x|^{\alpha}$ for arbitrary $x$ ).
    ${ }^{11}$ Consider the exercise:
    Exercise 5 Suggest an analogue of Corollary 1 for the case where $k$ is a function of time (that, say, also vanishes from time to time).

[^7]:    ${ }^{12}$ Consider the exercise:
    Exercise 6 Prove proposition 4.

[^8]:    ${ }^{13}$ Theorem 2 is a combination of results from 29, 30, 31]
    ${ }^{14}$ Consider the exercise:
    Exercise 7 Assume that $0 \in U \subset \mathbb{R}^{n}$ is bounded. Use the estimate from Corollary 1 to derive an upper bound for the time, by when each solution of 2.3) with the initial condition in $U$ reaches the origin in the settings of Theorem 2 ,

[^9]:    ${ }^{15}$ Consider the exercises:
    Exercise 8 Theorem 4 is a version of Theorem 2 for the case where not all the elements of $S$ are points of crossing. Such a version is obtained from Theorem 2 by considering $V(x)=\left|x_{1}\right|+\ldots+\left|x_{n}\right|$. Formulate and prove an analogous version of Theorem 2 when $V(x)$ is a linear combination (with positive multipliers) of functions of the form $\max _{i \in I} g_{i}(x)$. Here $I$ is a finite set of indexes and $g_{i}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ are smooth functions. The function $V$ so defined is called a max-function. The absolute value function is a particular example of a max-function. Your theorem will, therefore, be a generalization of Theorem 4.

    Exercise 9 Suggest an estimate for the settling time (i.e. the time by when all the solutions of (2.3) that originate from $U$ reach the origin) in Theorem 4.
    ${ }^{16}$ To prove the latter statement of Theorem 4 , one can notice that the Filippov solution $x(t)$ of $\sqrt{2.3}$ with the initial condition in $W$ never leaves $W$, because of the property $\dot{v}(t) \leq-k$ that we establish while proving the former statement.

[^10]:    ${ }^{17}$ Consider the exercise
    Exercise 10 Let $c: \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz function and let $b: \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous function. Prove that the function $a(t)=c(b(t))$ is absolutely continuous.

[^11]:    ${ }^{18}$ Consider the exercise:

[^12]:    Exercise 11 Attempt to follow the Example A from [31], where Theorem 4 is applied to a two-degree-of-freedom manipulator. You may make sure first that Theorem ${ }^{2}$ doesn't apply, i.e. discover that some points of $S \backslash\{0\}$ are not points of crossing. The authors of [31] do not check that and just apply Theorem 4. However, applying Theorem 4 is harder as you have to compute some convexifications. If you can prove that some points from $S \backslash\{0\}$ are points of crossing, then you don't need to compute convexification in those points. The authors of [31] state in the introduction that intersection of certain hyperplanes is formed by points which are not points of crossing, but they don't show that explicitly. Forgot to mention: in order to apply Theorem 4 in the context of example A from [31], you will have to reformulate Theorem 4 in such a way that the new theorem ensures finite-time stability of a given $\bar{x} \in \mathbb{R}^{n}$, which is not necessary the origin (as the target in [31] is not the origin).

[^13]:    ${ }^{19}$ Consider the exercise:
    Exercise 12 Attempt to follow the construction in [38] and design the control $u$ that swings up the planar pendulum

    $$
    \begin{aligned}
    & \dot{x}_{1}=x_{2} \\
    & \dot{x}_{2}=\sin x_{1}-u \cos x_{1}
    \end{aligned}
    $$

    in shortest time. The value of $u$ in this paper is deemed to be the acceleration of a cart on which the pendulum is installed, see [39] for the equations of the cart-pendulum system as a whole. To implement the control from [38] in practice, one, of course, also needs that the control stabilizes the cart itself. The paper [38] doesn't address this issue. Finite-time stabilization of the cart-pendulum system is addressed in [32] using a different control strategy (see Exercise ??). Here 40 is the movie of what one gets ultimately.

[^14]:    ${ }^{20}$ Consider the exercise:
    Exercise 13 Prove proposition 7 .
    ${ }^{21}$ Consider the exercise:
    

[^15]:    ${ }^{23}$ Consider the exercise:

[^16]:    ${ }^{24}$ Consider the exercise:

[^17]:    ${ }^{26}$ You are very welcome to conclude that from [34, formulas (49)-(50)]

[^18]:    ${ }^{27}$ Consider the exercise
    Exercise 20 Build upon the proof of Proposition 11 and prove Proposition 12.

[^19]:    ${ }^{29}$ Similar analysis can be execute by letting $\varepsilon=a \delta^{2}$ and by considering

    $$
    F(\delta, a)=\frac{1}{\delta}\left(X_{2}\left(2 \pi-\delta,\binom{1-a \delta^{2} c V}{V}, a \delta^{2}\right)-V\right) .
    $$

