Abstract

The geometric transportation problem takes as input a set of points $P$ in $d$-dimensional Euclidean space and a supply function $\mu : P \to \mathbb{R}$. The goal is to find a transportation map, a non-negative assignment $\tau : P \times P \to \mathbb{R}_{\geq 0}$, to pairs of points so the total assignment leaving each point is equal to its supply, i.e., $\sum_{r \in P} \tau(q, r) - \sum_{p \in P} \tau(p, q) = \mu(q)$ for all points $q \in P$. The goal is to minimize the weighted sum of Euclidean distances for the pairs, $\sum_{(p, q) \in P \times P} \tau(p, q) \cdot ||q - p||_2$.

We describe the first algorithm for this problem that returns, with high probability, a $(1 + \varepsilon)$-approximation to the optimal transportation map in $O(n \text{ poly}(1/\varepsilon) \text{ polylog } n)$ time. In contrast to the previous best algorithms for this problem, our near-linear running time bound is independent of the spread of $P$ and the magnitude of its real-valued supplies.
1 Introduction

We consider the geometric transportation problem in d-dimensional Euclidean space for any constant d. In this problem, we are given a set \( P \subset \mathbb{R}^d \) of \( n \) points. Each point is assigned a real supply \( \mu : P \to \mathbb{R} \) with the guarantee that \( \sum_{p \in P} \mu(p) = 0 \). A transportation map is a non-negative real assignment \( \tau : P \times P \to \mathbb{R}_{\geq 0} \) to pairs of points such that for all \( q \in P \) we have \( \sum_{r \in P} \tau(q, r) - \sum_{p \in P} \tau(p, q) = \mu(q) \). The cost of the transportation map is the weighted sum of Euclidean distances across all pairs, i.e., \( \sum_{(p,q) \in P \times P} \tau(p,q) \cdot ||g-p||_2 \). Our goal is to find a transportation map of minimum cost, and we denote this minimum cost as \( \text{Cost}(P, \mu) \).

This problem is often described as if the points with positive supply are piles of earth and the points with negative supplies are holes in the ground. A transportation map describes how to transfer all of the earth to the holes without overfilling any hole, and its cost is the total number of “earth-miles” used to do the transfer. Consequently, \( \text{Cost}(P, \mu) \) is often referred to as the earth mover’s distance, although it can also be described as the 1-Wasserstein distance between measures over the positively and negatively supplied points. The continuous version of the problem is sometimes called the optimal transport or Monge-Kantorovich problem, and it has been studied extensively by various mathematics communities [18]. The discrete version we study has many applications including uses in shape matching, image retrieval, and graphics [3,5–7,12,17].

Computing an optimal transportation map is easily done in polynomial time by reducing to the uncapacitated minimum cost flow problem in a complete bipartite graph between the points with positive supply and those with negative supply. The graph has as many as \( \Omega(n^2) \) edges, so this approach takes \( O(n^3 \log n) \) time using a minimum cost flow algorithm of Orlin [11]. Assuming integral supplies with absolute values summing to \( U \), we can use an algorithm of Lee and Sidford [10] instead to reduce the running time to \( O(n^{2.5} \log(n, U)) \). Taking advantage of the geometry inherit in the problem, Agarwal et al. [1] describe how to implement Orlin’s algorithm for arbitrary supplies to find the optimal transportation map in \( O(n^2 \log n) \), but only for \( d = 2 \).

We can significantly reduce these running times by accepting a small loss in optimality. Many results along this line focus on estimating just the earth mover’s distance without actually computing the associated transportation map. Indyk [8] describes an \( O(n \log n) \) time algorithm that estimates the earth mover’s distance within a constant factor assuming unit supplies. Andoni et al. [2] describe an \( O(n^{1+\varepsilon}(1)) \) time algorithm for arbitrary supplies that estimates the cost within a \( 1 + \varepsilon \) factor (the dependency on \( \varepsilon \) is hiding in the \( o(1) \)). As pointed out by Khesin, Nikolov, and Paramonov [9], a \( 1 + \varepsilon \) factor estimation of the distance is possible in \( O(n^{1+o(1)}\varepsilon^{-O(d)}) \) time (without the \( o(1) \) hiding dependencies on \( \varepsilon \)) by running an approximation algorithm for minimum cost flow by Sherman [15] on a sparse Euclidean spanner over the input points. However, it is not clear how to extract a nearly optimal transportation map using the spanner’s flow.

Indeed, finding an approximately optimal transportation map does appear to be more difficult. Sharathkumar and Agarwal [13] describe a \( (1+\varepsilon) \)-approximation algorithm for the integral supply case (i.e., an algorithm returning a map of cost at most \( (1 + \varepsilon) \cdot \text{Cost}(P, \mu) \)) in \( O(n \sqrt{U} \log(U, \varepsilon, n)) \) time. Agarwal et al. [1] describe a randomized algorithm with expected \( \log^2(1/\varepsilon) \)-approximation ratio running in \( O(n^{1+\varepsilon}) \) expected time for the general case and a deterministic \( O(n^{3/2}\varepsilon^{-d} \log(U, n)) \) time \( (1+\varepsilon) \)-approximation algorithm for the integral supply case. Very recently, Khesin et al. [9] described a randomized \( (1+\varepsilon) \)-approximation algorithm running in \( O(n\varepsilon^{-O(d)}\log^2(P)\log(n)) \) time, where \( \text{Sp}(P) \) is the spread of the point set. The spread (also called aspect ratio) of \( P \) is the ratio of the diameter of \( P \) to the smallest pairwise distance between points in \( P \). As Khesin et al. point out, one can reduce an instance with unbounded spread but integral supplies to the above algorithm to get a \( (1+\varepsilon) \)-approximation running in \( O(n\varepsilon^{-O(d)}\log^2(U)\log^2(n)) \) time, generalizing a near-linear time \( (1+\varepsilon) \)-approximation algorithm.
by Sharathkumar and Agarwal [14] for the unit supply case. However, prior work still does not include a near-linear time \((1 + \varepsilon)-\)approximation algorithm for the general case of arbitrary spread and real valued supplies.

### 1.1 Our results

We describe a randomized \((1 + \varepsilon)-\)approximation algorithm for the geometric transportation problem that runs in near-linear time irrespective of the spread of \(P\) or the supplies of its points. We say an event occurs with high probability if it occurs with probability at least \(1 - \frac{1}{n^c}\) for some constant \(c\). Our specific result is spelled out in the following theorem.

**Theorem 1.1.** There exists a randomized algorithm that, given a set of \(n\) points \(P \in \mathbb{R}^d\) and a supply function \(\mu : P \to \mathbb{R}\), runs in time \(O(n \varepsilon^{-O(d)} \log^{O(d)} n)\) and with high probability returns a transportation map with cost at most \((1 + \varepsilon) \cdot \text{Cost}(P, \mu)\).

At a high level, our algorithm follows the approach laid out by Khesin et al. [9] for the bounded spread case. However, removing the running time’s dependency on the spread introduces fundamental and technical issues to nearly every step in their approach.

Let \(\varepsilon_0\) be a function of \(\varepsilon\) and \(P\) to be specified later. Taking a cue from prior work on geometric transportation and its specializations [2,14], Khesin et al.’s algorithm begins by building a random sparse graph over \(O(n \varepsilon^{-O(d)} \log \text{Sp}(P))\) vertices including the points in \(P\). In expectation, the shortest path distance between any pair of points in \(P\) is maintained up to an \(O(\varepsilon_0 \log \text{Sp}(P))\) factor, so computing a transportation map is done by setting \(\varepsilon_0\) to \(O(\varepsilon/\log \text{Sp}(P))\) and running a minimum cost flow algorithm on the sparse graph.

The graph is constructed by first building a randomly shifted quadtree over \(P\). The quadtree is constructed by surrounding \(P\) with an axis-aligned box we refer to as a cell, partitioning it into \(2^d\) equal sized child cells, and recursively building a quadtree in each child cell; the whole tree has depth \(\log \text{Sp}(P)\). After building the quadtree, they add \(\varepsilon_0^d\) Steiner vertices within each cell. While other methods are known for constructing such a sparse graph even without Steiner vertices [4], the hierarchical structure of Khesin et al.’s construction is necessary for extracting the transportation map after a minimum cost flow is computed. Observe that not only is the quadtree’s size dependent on \(\text{Sp}(P)\), but so is the number of Steiner vertices added to each cell.

The second stage of Khesin et al.’s [9] algorithm solves the minimum cost flow problem in the sparse graph using a framework of Sherman [15]. In short, we modify the quadtree construction so that, with high probability, all points are sufficiently far away from the boundary of every quadtree cell they appear in. Assuming this condition holds, there are only a limited number of quadtree “levels” at which a pair of points can be separated, and we use this fact to show distances increase by only an \(O(\varepsilon_0 \log n)\) factor in expectation. Surprisingly, the moats continue to help later in the algorithm as we turn to computing an approximately minimum cost flow in our sparse graph.

The second stage of Khesin et al.’s [9] algorithm solves the minimum cost flow problem in the sparse graph using a framework of Sherman [15]. First, they encode the minimum cost flow
problem as finding a flow vector $f$ of minimum cost subject to linear constraints $Af = b$ where $A$ is the vertex-edge incidence matrix and $b$ is a supply vector (not necessarily equal to $\mu$). Sherman’s framework involves repeatedly finding flows $f$ of approximately optimal cost that approximately satisfy such constraints. Each iteration of this algorithm requires an application of $A$ and $A^T$ to a pair of vectors, and the number of iterations needed in this approach is polynomial in the condition number of $A$. Unfortunately, $A$ may not be well-conditioned, so Khesin et al. describe a preconditioner matrix $B$ such that $BA$ has low condition number and is still sparse. They proceed to use Sherman’s framework under the equivalent constraints $BAf = Bb$.

One interpretation of the preconditioner is that it describes a way to charge each Steiner vertex an amount based on the supply of “descendent” vertices below it so that the sum of charges bound the cost of an optimal flow from below. Consequently, both the number of non-zero entries in each column of $B$ and the condition number of $B$ are proportional to the quadtree’s depth.

The high depth of our quadtree appears to be a problem, but moats help us once again. Our preconditioner $B$ is based on essentially the same charging scheme as Khesin et al., but thanks to the moats, we prove the condition number remains proportional to $O(\varepsilon_0^{-1} \log(n/\varepsilon_0))$ instead of the quadtree depth. This charging scheme still results in a precondition $B$ that is not sparse, so a naive implementation of Sherman’s [15] framework may take quadratic time per iteration. Therefore, we describe a pair of algorithms based on the hierarchical structure of the graph that let us apply both $BA$ and its transpose in only linear time.

The final stage of the algorithm is the extraction of an approximately minimum cost transportation map from an approximately minimum cost flow in the sparse graph. Again, the high depth of our quadtree may cause issues with Khesin et al.’s [9] approach, causing it to run in quadratic time. By applying a little more care to the process and employing some relatively simple data structures, we extract the transportation map in near-linear time.

We remark that our results (and those of Khesin et al. [9]) can be extended to work with any $L_p$ metric instead of just Euclidean distance. The rest of the paper proceeds as follows. We describe our sparse graph construction, analyze its properties, and describe the reduction to minimum cost flow in Section 2. We describe our preconditioner and its use Section 3. Finally, we describe how to extract the approximately optimal transportation map from a flow on the sparse graph in Section 4.

2 Reduction to minimum cost flow in a sparse graph

In this section, we present a way to build a sparse graph $G^* = (V^*, E^*)$ based on $P$ and reduce the transportation problem to finding a minimum cost flow in this sparse graph. Our sparse graph $G^*$ is similar to one presented by Khesin et al. [9]. However, the quadtree we use is compressed under certain conditions to guarantee the number of nodes in it is nearly linear in $n$.

2.1 Construction of the sparse graph

Given a point set $P \subset \mathbb{R}^d$ of size $n$, we say two disjoint subsets $A$ and $B$ of $P$ are $s$-well separated for some $s > 0$ if $A$ and $B$ can be enclosed within two Euclidean balls of radius $r$ such that the distance between these two balls are at least $sr$. For any constant $s$, we can compute a collection of $O(n)$ distinct pairs of subsets of $P$ called an $s$-well separated pair decomposition ($s$-WSPD) of $P$ such that, every pair of subsets in this collection is $s$-well separated and every pair of points in $P \times P$ is separated in some unique pair of subsets in this $s$-WSPD [4]. The time to compute the $s$-WSPD is $O(n \log n)$.

Our sparse graph construction begins by computing a $2$–WSPD for $P$ containing $\ell = O(n)$ $s$-well separated pairs. Let $Z = \{z_1, z_2, \ldots, z_\ell\}$ be a list of distances so that the $i$th well separated
Figure 1. Left: Randomly shifting a box around $P$. Right: The quadtree cells form a hierarchy. Each cell is partitioned into $\varepsilon_0^{-d}$ sub cells, and each subcell has a single net point at its center.

pair $(A, B)$ contains two points $p \in A, q \in B$ such that $z_i = ||q - p||_2$. By definition, the distance between any pair of points separated by the $i$th pair $(A, B)$ is in $[\frac{z_i}{3}, 3z_i]$. We compute a sub-list $Z' = \{z_i : 1 \leq i < \ell, z_i > \frac{18\sqrt{\ln n}^a \varepsilon_0}{\ell_0}z_{i+1}\}$. Let $\Box_P$ be the minimum bounding square of $P$. We fix an $\varepsilon_0 = O(\varepsilon / \log n)$ such that $1 / \varepsilon_0$ is a power of 2. Suppose the side length of $\Box_P$ is $\Delta^*$. Let $\Box'$ be a square of side length $3\Delta^*$ such that $\Box_P$ and $\Box'$ are concentric. We shift $\Box$ by a vector chosen uniformly at random from $[0, \Delta^*)^d$. See Figure 1, left.

After the random shift, we build a variant of the compressed quadtree on $P$ we call a conditionally-compressed quadtree. Let $T$ denote this tree. Each node of $T$ is a square cell in $\mathbb{R}^d$. Set $\Box$ to be the root of $T$, and let $z$ be the first element in $Z'$. We recursively process each cell $C$ as follows.

1) $C$ is a leaf node if $|P'| = 1$.

2) If $|P'| > 1$ and $\Delta \geq \frac{z}{3\sqrt{d}}$, we evenly divide $C$ into $2^d$ child cells each of side length $\frac{\Delta}{2}$.

3) If $|P'| > 1$ and $\Delta < \frac{z}{3\sqrt{d}}$, we find the minimum bounding square $\Box_{P'}$ of $P'$. We recursively build a conditionally-compressed quadtree with an independently shifted root square $\Box'$ concentric to $\Box_{P'}$ with side length $3\Delta_{P'}$. We connect the root of this sub-quadtree to $T$ as a child of $C$. The value of $z$ we use during recursion is the largest $z' \in Z'$ such that $z' \leq 3\sqrt{d}\Delta_{P'}$, which can be found via binary search.

We define a simple sub-quadtree as a sub-quadtree consisting of a cell $C$ randomly shifted independently of its parent and a maximal set of descendent cells of $C$ that were not shifted independently of $C$. In other words, all cells in the sub-quadtree are determined by the same independent random shift.

For every cell $C$ in $T$, we perform a secondary subdivision on $C$. Let $\Delta_C$ denote the side length of $C$. We divide $C$ into $\varepsilon_0^{-d}$ square sub-regions with equal side length $\varepsilon_0\Delta_C$. If a sub-region of $C$ contains a point $p \in P$, we say it is a subcell $\tilde{C}$ of $C$ and we use $C^+$ to denote the set of subcells of $C$. Again, see Figure 1.

Utilizing an idea of Agarwal et al. [1], we define the moat of size $h$ around a point $p$ as an axis-parallel square of side length $h$ around $p$. Consider a randomly shifted grid with cells of side
length $\Delta$. The probability of any of the grid lines hitting a moat of size $\frac{2\Delta}{n^3}$ around any point $p \in P$ is at most $\frac{2\Delta}{n^5} \cdot n \cdot \frac{4}{2} = O(\frac{1}{n^2})$.

**Lemma 2.1.** With high probability, the conditionally-compressed quadtree $T$ has the following properties:

1. The total number of cells is $O(n \log (n/\varepsilon_0))$.

2. Suppose cell $C$ with side length $\Delta_C$ contains $p \in P$ and let $\tilde{C}$ be the subcell of $C$ that contains $p$. Then, $p$ is at least $\frac{\Delta_C}{n^2}$ distance away from any side of $C$ and is at least $\frac{\varepsilon_0 \Delta_C}{n^3}$ distance away from any side of $\tilde{C}$. In particular, the moats of $p$ with respect to the uniform grids containing $C$ and $\tilde{C}$ as cells do not touch the grid lines.

3. Let $T'$ be any simple sub-quadtree of $T$ constructed with a distance parameter $z$. Every leaf cell of $T'$ contains at most one point from any pair $p,q \in P$ where $||q-p||_2 \geq \frac{z}{3}$, and no leaf cell of $T'$ contains exactly one point from any pair $p,q \in P$ where $||q-p||_2 < \frac{z}{5}$.

4. Let $T'$ be any simple sub-quadtree of $T$, and let $C'$ be a child cell of some leaf $C$ of $T'$. Cell $C'$ lies entirely within a subcell of $C$.

**Proof:** Every path of descendent cells with one child each has length $O(\log(n/\varepsilon_0))$. We immediately get Property 1.

We prove Properties 2 through 4 together. Let $m$ be the number of cells in $T$. We will argue Property 2 holds with probability $O(m/n^2) = O((1/n) \log(n/\varepsilon_0))$ and that Property 2 implies the remaining properties.

Now, consider the simple sub-quadtree $T_0$ containing the root cell of $T$. The first part of Property 3 holds for $T_0$ by construction. Let $m_0$ be the maximum depth of any leaf in $T_0$; $m_0$ is determined entirely by $P$ and not the random shift of the root of $T_0$. Property 2 is violated for at least one cell in $T_0$ with probability at most $O(m_0/n^2)$, because its (sub)cells lie in at most that many grids in $\mathbb{R}^d$. Assume Property 2 holds for $T_0$. Let $p,q \in P$ be any pair of points where $||q-p||_2 < \frac{z}{4}$. By definition of $z$, we have $||q-p||_2 \leq 3 \cdot \frac{z\varepsilon_0}{18\sqrt{dn^3}} = \frac{z\varepsilon_0}{6\sqrt{dn^3}}$. Both points are distance at least $\frac{z\varepsilon_0}{6\sqrt{dn^3}}$ from the side of any subcell, so they are not separated by any subcell of $T_0$, implying the second part of Property 3. Finally, Property 3 holds for all pairs of points, including the ones defining the bounding boxes for simple sub-quadtrees whose roots are children of leaves in $T_0$. The points are far enough away from the subcell boundaries that even the random shift of these simple sub-quadtrees will keep them inside their subcells. Property 4 holds for $T_0$.

Let $m_1, m_2, \ldots, m_k$ be the number of cells of each sub-quadtree going all the way to leaves of $T$ whose root has a leaf of $T_0$ as a parent. We may now assume inductively that the properties are violated for at least one sub-quadtree with probability at most $\sum_{i=1}^k O(m_i/n^2)$. Adding in the probability that the properties didn’t hold for $T_0$ itself yields our desired failure probability for $T$ as a whole. \qed

We assume from here on that the properties described above do hold, but $T$ is still randomly constructed conditional on those properties. We now build the sparse graph $G^{\ast}$ based on the decomposition.

For every cell $C$, we add a net point $\nu$ at the center of every subcell of $C$, and use $N_{\tilde{C}}$ to denote the net point of a subcell $\tilde{C}$. We add $O(\varepsilon_0^{-2d})$ edges to build a clique among net points of subcells in $C^+$. Furthermore, if $C$ has a parent cell $C^p$, for each $\tilde{C} \in C^+$, there exists a $\tilde{C}^p \in C^p^+$ such that $\tilde{C}$ is totally contained in $\tilde{C}^p$, because $1/\varepsilon_0$ is power of 2. We add an edge connecting
Euclidean distance of their endpoints. Let ˜\(C(p)\) denote the smallest subcell containing \(p\).

Children subcells and children net points are defined analogously. Edges are weighted by the Euclidean distance of their endpoints. Let ˜\(C(p)\) denote the smallest subcell containing \(p\). As a last step, for every point \(p \in P\), we add an edge connecting \(p\) to \(N_{\tilde{C}(p)}\).

Let \(V^*\) be union of \(P\) and the set of all net points we just added, and let \(E^*\) be the set of edges we added above. In short, \(V^* = \cup_{C \in T}\{N_{\tilde{C}} : \tilde{C} \in C^+\} \cup P\) and \(E^* = \cup_{C \in T}\{(u,v) : u,v \in \{N_{\tilde{C}} : \tilde{C} \in C^+\}\} \cup \{pN_{\tilde{C}(p)} : p \in P\}\). The sparse graph upon which we solve minimum cost flow is denoted \(G^* = (V^*, E^*)\).

**Lemma 2.2.** The expected distance between any pair \(p, q \in P\) in \(G^*\) is at most \((1 + O(\varepsilon_0 \log n)) ||p - q||_2\).

**Proof:** Let \(\text{dist}_{G^*}(p, q)\) be the distance between \(p\) and \(q\) in \(G^*\). Points \(p\) and \(q\) must be connected through the net points of some cell containing both of them. Let \(C(p, q)\) be the lowest common ancestor cell of \(p\) and \(q\). Let \(N_{C(p, q)}(p)\) and \(N_{C(p, q)}(q)\) be the net points of subcells of \(C(p, q)\) that contains \(p\) and \(q\), respectively. Then \(\text{dist}_{G^*}(p, q) = \text{dist}_{G^*}(p, N_{C(p, q)}(p)) + \text{dist}_{G^*}(N_{C(p, q)}(p), N_{C(p, q)}(q)) + \text{dist}_{G^*}(q, N_{C(p, q)}(q))\). Value \(\text{dist}_{G^*}(p, N_{C(p, q)}(p))\) is the distance from \(N_{C(p, q)}(p)\) to \(p\) through its descendant net points. The upper bound of it is \(\sum_{i \geq 2} 2^{-i}\sqrt{\varepsilon_0 \Delta_{C(p, q)}} \leq \sqrt{\varepsilon_0 \Delta_{C(p, q)}}\), because subcell side lengths at least halve every level down in \(T\). Similarly, \(\text{dist}_{G^*}(q, N_{C(p, q)}(q)) \leq \sqrt{\varepsilon_0 \Delta_{C(p, q)}}\).

We define the extra cost to be \(\Phi_{p,q} = \text{dist}_{G^*}(p, q) - ||p - q||_2\). Then \(\Phi_{p,q} \leq 3\sqrt{\varepsilon_0 \Delta_{C(p, q)}}\), and the expectation of the extra cost \(\mathbb{E}(\Phi_{p,q}) \leq \mathbb{E}(3\sqrt{\varepsilon_0 \Delta_{C(p, q)}}) \leq 3\sqrt{\varepsilon_0 \mathbb{E}(\Delta_{C(p, q)})}\).

Assuming the properties from Lemma 2.1, we may infer that the subset of \(P\) defining the simple sub-quadtree containing \(C(p, q)\) is determined only by \(P\) itself. Let \(T_0\) be this tree. Let \(\Delta^*\) be the side length of the root cell of \(T_0\) and let \(\lambda = ||p - q||_2\). From Property 2 of Lemma 2.1, \(\Delta_{C(p, q)} \leq n^3 \lambda\), because the grid of side length \(\frac{n^3 \lambda}{2}\) cannot separate \(p\) and \(q\). Also, \(\Delta_{C(p, q)} \geq \frac{\lambda}{\sqrt{d}}\) so that \(p\) and \(q\) can fit in the same cell. Let \(x = \text{argmax}_i \{2^{-i} \Delta^* : 2^{-i} \Delta^* \leq n^3 \lambda, i \in \mathbb{N}\}\) and \(y = \text{argmax}_i \{2^{-i} \Delta^* : 2^{-i} \Delta^* \geq \frac{\lambda}{\sqrt{d}}, i \in \mathbb{N}\}\). Possible values of \(\Delta_{C(p, q)}\) are in \(\{2^{-i} \Delta^* : x \leq i \leq y, i \in \mathbb{N}\}\). Observe, the set of possible shifts of \(T_0\)’s root that don’t result in clipping any moats relative to its cells are all equally likely. In particular, \(p\) and \(q\) are separated by a grid with side length \(\Delta\) containing cells of \(T_0\) with probability at most

\[
d \cdot \frac{\Delta^*}{\Delta} \cdot \frac{\lambda}{(1 - O((1/n) \log(n/\varepsilon_0))) \Delta^*} = O\left(\frac{\lambda}{\Delta}\right).
\]

Therefore, we have

\[
\mathbb{E}(\Delta_{C(p, q)}) = \sum_{x \leq i \leq y, i \in \mathbb{N}} \mathbb{P}[p \text{ and } q \text{ are not separated by grid of size } 2^{-i} \Delta^* \text{ and } p \text{ and } q \text{ are separated by grid of size } 2^{-i-1} \Delta^* : 2^{-i} \Delta^*] 
\leq \sum_{x \leq i \leq y, i \in \mathbb{N}} \mathbb{P}[p \text{ and } q \text{ are separated by grid of size } 2^{-i-1} \Delta^* : 2^{-i} \Delta^*] 
\leq \sum_{x \leq i \leq y, i \in \mathbb{N}} O\left(\frac{\lambda}{2^{-i-1} \Delta^*} \cdot 2^{-i} \Delta^*\right) 
\leq O(\log n) \cdot \lambda.
\]
We conclude
\[ \mathbb{E}(\text{dist}_{G^*}(p, q)) = ||p - q||_2 + \mathbb{E}(\Phi_{p,q}) \leq (1 + O(\varepsilon_0 \log n)) \cdot ||p - q||_2. \]

\[ \square \]

2.2 Reduction to minimum cost flow

Having built our sparse graph, we now reduce to a minimum cost flow problem in \( G^* \). We model the minimum cost flow problem in the following way to simplify our subsequent discussions.

Let \( G = (V, E) \) be an arbitrary undirected graph. Let \( \bar{E} \) be the set of edges in \( E \) oriented arbitrarily. We call \( f \in \mathbb{R}^{\bar{E}} \) a flow vector or more simple, a flow. Let \( A \) be a \( |V| \times |\bar{E}| \) vertex-edge incidence matrix where \( \forall (u, (v, w)) \in V \times \bar{E}, A_{u,(v,w)} = 1 \) if \( u = v \), \( A_{u,(v,w)} = -1 \) if \( u = w \), and \( A_{u,(v,w)} = 0 \) otherwise. Given \( f \), we define the divergence of a vertex \( v \) as \( (Af)_v = \sum_{(v,w)} f_{(v,w)} - \sum_{(u,v)} f_{(u,v)} \). For simplicity of exposition, we may sometimes refer to \( f_{(v,u)} \) even though \( (u, v) \in \bar{E} \). In such cases, it is assumed \( f_{(v,u)} = -f_{(u,v)} \).

Let \( || \cdot ||_{\bar{E}} \) be a norm on \( \mathbb{R}^{\bar{E}} \) such that \( ||f||_{\bar{E}} = \sum_{(u,v) \in \bar{E}} |f_{(u,v)}| \cdot ||v - u||_2 \). Let \( b \in \mathbb{R}^V \) denote a set of divergences for all \( v \in V \). We define an instance of uncapacitated minimum cost flow as the pair \( (G, b) \). We seek to solve the following optimization problem over possible flow vectors \( f \).

\[ \text{Minimize } ||f||_{\bar{E}} \quad \text{(1)} \]
\[ \text{subject to } Af = b \quad \text{(2)} \]

In particular, set \( b^* \in \mathbb{R}^V \) such that \( b^*_p = \mu(p), \forall p \in P \) and \( b^*_v = 0, \forall v \in V \backslash P \). Ultimately, we will find an approximate solution to the instance \( (G^*, b^*) \). Let \( \text{Cost}(G^*, b^*) := ||f^*||_{\bar{E}} \) for some optimal solution \( f^* \) of this instance. From construction of \( G^* \) and Lemma 2.2, \( \text{Cost}(P, \mu) \leq \text{Cost}(G^*, b^*) \) and \( \mathbb{E}(\text{Cost}(G^*, b^*)) \leq (1 + O(\varepsilon_0 \log n))\text{Cost}(P, \mu) \). In particular, \( \mathbb{E}(\text{Cost}(G^*, b^*)) - \text{Cost}(P, \mu) \leq O(\varepsilon_0 \log n)\text{Cost}(P, \mu) \). We can guarantee that bound holds with high probability by doubling the constant in the big-Oh and taking the best result from \( O(\log n) \) runs of our algorithm.

2.3 Decomposition into simpler subproblems

In the sequel, we will apply Sherman’s generalized preconditioning framework [9, 15] to find an approximate solution to the minimum cost flow instance \( (G^*, b^*) \). For technical reasons, however, we cannot afford to run the framework on the entire sparse graph \( G^* \) at once. Here, we reduce finding an approximately optimal flow in \( G^* \) to finding \( O(n) \) approximately optimal flows, each within an induced subgraph defined by the net points within a single simple sub-quadtree. We must emphasize that simple sub-quadtrees may still have linear depth, so we still need to apply our own techniques to make Sherman’s framework run within our desired time bounds.

Recall, for each point \( p \in P \), \( \tilde{C}(p) \) denotes the smallest subcell containing \( p \), and \( N_{\tilde{C}} \) denotes the net point of subcell \( \tilde{C} \). Let \( f \) be the flow such that \( f_{(p,N_{\tilde{C}}(p))} = b^*_p \) for all \( p \in P \). Let \( G' = (V', E') \) and \( A' \) be the restriction of \( G^* \) and its vertex-edge incidence matrix \( A \) after removing all vertices \( p \in P \). Let \( b' \) be the restriction of \( b - Af \) to vertices of \( G' \). Every vertex \( p \in P \) of \( G^* \) has exactly one incident edge, so an optimal solution to our original minimum cost flow instance consists of \( f \) along with an optimal solution to the instance defined on \( A' \) and \( b' \). From here one, we focus on finding an approximately minimum cost flow in \( G' \).
Now, let $G_0 = (V_0, E_0)$ be the subgraph induced by the $m$ net point vertices of a single simple sub-quadtree with no descendant sub-quadtrees. Let $C$ be the root cell of the simple sub-quadtree for $G_0$, let $u$ be a net point for an arbitrary subcell of $C$, and let $v$ be the parent net point of $u$ in $G'$. In $O(m)$ time, we compute $B = \sum_{w \in V_0} b_w E_w$, the total divergence of vertices within $G_0$. We then let $f'$ be the flow in $G'$ that is 0 everywhere except for $f_{(u,v)} := B$. Finally, let $b'' = b' - Af'$.

**Lemma 2.3.** There exists a flow $f''$ in $G'$ such that $f''(w,x) = 0$ for all $w \in V_0, x \notin V_0$; $Af'' = b''$; and $\|f'' + f'|_{\bar{E}} \leq (1 + O(1/n^3)) \cdot \text{Cost}(G', b')$.

**Proof:** Let $\bar{C}$ be the subcell for which $v$ is a net point. Let $\Delta_{\bar{C}}$ be the side length of $\bar{C}$. By construction of $G'$, at least $B$ units of flow must travel to or from vertex $v$ from $G_0$ at a cost of $\Delta_{\bar{C}}$. Specifically, $G_0$ is totally inside $\bar{C}$, $v$ is the only vertex in $\bar{C}$ incident to some edge crossing the side of $\bar{C}$, and the nearest vertex $x \notin V_0$ is at least $\Delta_{\bar{C}}$ far from $v$. So $\text{Cost}(G', b') \geq \Delta_{\bar{C}} B$.

Suppose $f^*$ is a flow in $G'$ with cost $\text{Cost}(G', b')$. Let $N_C$ be the set of net points of subcells of $C$. We may assume there is no pair $y, z \in N_C$ such that $f_{(y,v)} > 0$ and $f_{(v,z)} > 0$, because we could send the flow directly between $y$ and $z$ more cheaply. We create flow $f''$ as follows starting with $f'' = f^*$. While there exists some vertex $u' \in N_C \setminus \{u\}$ with $f''_{(u',v)} \neq 0$, let $\delta = f''_{(u',v)}$. We divert flow by setting $f''_{(u',v)} = f''_{(u',v)} + \delta$, $f''_{(w,u')} = f''_{(w,u')} - \delta$, and $f''_{(w,v)} = 0$. This increases the cost by at most twice of the length of the diagonal of $C$ per diverted unit of flow. Overall, we divert at most $B$ units. The total cost increase is at most $2\sqrt{2}\Delta_{\bar{C}} B \leq O(1/\sqrt{n}) \cdot \text{Cost}(G', b')$ where $\Delta_C$ is side length of $C$, because $\Delta_C \leq O(1/n^3) \Delta_{\bar{C}}$. We have $\|f''\|_{\bar{E}} \leq (1 + O(1/n^3)) \cdot \text{Cost}(G', b')$. Finally, let $f'' = f'' - f'$.

The above lemma implies we can use the following strategy for approximating a minimum cost flow in $G'$: Let $b_0$ be the restriction of $b''$ to $V_0$. We find a flow in $G_0$ with divergences $b_0$ of cost at most $(1 + O(\varepsilon)) \cdot \text{Cost}(G_0, b_0)$ using the algorithm described in the next section. Then, we recursively apply our algorithm on $G'' = (V'', E'')$, the induced subgraph over $V'' = V' \setminus V_0$. The depth of recursion is $O(n)$, so the total cost from combining our separately computed flows is $(1 + O(\varepsilon))(1 + O(1/n^2)) \cdot \text{Cost}(G', b') = (1 + O(\varepsilon))\text{Cost}(G', b')$.

### 3 Approximating the minimum cost flow

Let $G = (V, E)$ be an induced subgraph of sparse graph $G^*$ where $V$ is the subset of net points for one simple sub-quadtree as defined above. Let $m = |E|$, and let $A$ be the vertex-edge incidence matrix for $G$. In this section, we describe the ingredients we need to provide to efficiently approximate the minimum cost flow problem in $G$ using Sherman’s generalized preconditioning framework [9, 15]. Afterward, we provide those ingredients one-by-one to achieve a near-linear time $(1 + O(\varepsilon))$-approximate solution for the minimum cost flow instance.

#### 3.1 The preconditioning framework

Consider an instance of the minimum cost flow problem in $G$ with an arbitrary divergence vector $\tilde{b} \in \mathbb{R}^V$, and let $f^*_b := \arg\min_{f \in \mathbb{R}^E, Af = \tilde{b}} \|f\|_{\bar{E}}$. A flow vector $f \in \mathbb{R}^E$ is an $(\alpha, \beta)$ **solution** to the problem if

$$
\|f\|_{\bar{E}} \leq \alpha \|f^*_b\|_{\bar{E}}
$$

$$
\|Af - \tilde{b}\|_1 \leq \beta \|A\| \|f^*_b\|_{\bar{E}}
$$
where $||A||$ is the norm of the linear map represented by $A$. An algorithm yielding an $(\alpha, \beta)$-solution is called an $(\alpha, \beta)$-solver.

By arguments in [9], we seek a preconditioner $B \in \mathbb{R}^{V \times V}$ of full column rank such that, for any $\tilde{b} \in \mathbb{R}^V$ with $\sum_{v \in V} \tilde{b}_v = 0$, it satisfies

$$||B\tilde{b}||_1 \leq \min\{||f||_{\tilde{E}} : f \in \tilde{E}, Af = \tilde{b}\} \leq \kappa||B\tilde{b}||_1$$

(3)

for some sufficiently small function $\kappa$ of $n$, $\varepsilon$, and $d$.

Let $M$ be the times it takes to multiply $BA$ and $(BA)^T$ by a vector. Then there exists a $(1 + \varepsilon, \beta)$-solver for any $\varepsilon, \beta > 0$ for this problem with running time bounded by $O(\kappa^2(|V| + |\tilde{E}| + M) \log |\tilde{E}|(\varepsilon^{-2} + \log \beta^{-1})$ [15]. Moreover, if a feasible flow $f \in \tilde{E}$ with cost $||f||_{\tilde{E}} \leq \kappa B\tilde{b}$ can be find in time $K$, there is a $(\kappa, 0)$-solver with running time $K$. By setting $\beta = \varepsilon \kappa^{-2}$ [9], the composition of these two solvers is a $(1 + 2\varepsilon, 0)$-solver with running time bounded by $O(\kappa^2(|V| + |\tilde{E}| + M) \log |\tilde{E}|(\varepsilon^{-2} + \log \kappa) + K)$.

### 3.2 Preconditioning the minimum cost flow

We present a way to construct such a preconditioner $B$ similar to the one of Khesin et al. [9] that guarantees $\kappa$ in (3) is sufficiently small for our performance objective. Our algorithm does not compute $B$ directly, because $B$ is not sparse. However, the time for individual applications of $BA$ or $(BA)^T$ is $O(|V| + |\tilde{E}|)$.

Let $\tilde{C}$ denote the set of all subcells defining the net points of $G$. For any subcell $\tilde{C} \in \tilde{C}$, let $N_{\tilde{C}}$ denote its net point and let $\Delta_{\tilde{C}}$ denote its side length.

Let $B$ be a matrix indexed by $(u, v) \in \tilde{V} \times \tilde{V}$ such that, for every net point $v$ in $\tilde{V}\setminus P$ where $v$ is the net point of some subcell $\tilde{C}$, we set $B_{uv} = \frac{\Delta_{\tilde{C}}}{\Lambda}$ for all $v \in \tilde{C}$, where $\Lambda = 18 \log(\frac{n}{\varepsilon \delta})$. $B_{uv} = 0$ for all other $v$. Matrix $B$ has full column rank, because each column specifies exactly which “ancestor” net points each vertex has in $G$.

Now, fix any $\tilde{b} \in \mathbb{R}^V$ such that $\sum_{v \in V} \tilde{b}_v = 0$. Observe,

$$||B\tilde{b}||_1 = \sum_{\tilde{C} \in \tilde{C}} \frac{\Delta_{\tilde{C}}}{\Lambda} |\sum_{v \in \tilde{C}} \tilde{b}_v|.$$

(4)

**Lemma 3.1.** We have $||B\tilde{b}||_1 \leq \min\{||f||_{\tilde{E}} : f \in \tilde{E}, Af = \tilde{b}\}$.

**Proof:** Let $f^\star_b := \arg\min_{f \in \mathbb{R}^{\tilde{E}},Af=\tilde{b}} ||f||_{\tilde{E}}$. We arbitrarily decompose $f^\star_b$ into a set of flows $F = \{f^1, f^2, \ldots\}$ with the following properties: 1) each flow follows a simple path between two vertices $u$ and $v$; 2) for each flow $f^i \in F$ and edge $(u, v) \in \tilde{E}$ either $f^i(u, v) = 0$ or its sign is equal to the sign of $f^\star_b(u, v)$; 3) for each flow $f^i \in F$ and vertex $v$, either $(Af^i)_v = 0$ or its sign is equal to $\tilde{b}_v$; and 4) for each edge $(u, v) \in \tilde{E}$, we have $f^\star_b(u, v) = \sum_{f^i \in F} f^i(u, v)$. The existence of such a decomposition is a standard part of network flow theory and one can be computed in a simple greedy manner (however, our algorithm does not actually need to compute one). From construction, we have $\sum_{f^i \in F} ||f^i||_{\tilde{E}} = ||f^\star_b||_{\tilde{E}}$. We describe a way to charge summands of $\sum_{\tilde{C} \in \tilde{C}} \Delta_{\tilde{C}} |\sum_{v \in \tilde{C}} \tilde{b}_v|$ to the summands of $\sum_{f^i \in F} ||f^i||_{\tilde{E}}$. Our charges will cover each of the former and exceed each of the latter by at most a $\Lambda$ factor. Consider a subcell $\tilde{C}$. For each vertex $u \in \tilde{C}$, for each flow $f^i$ sending

---

1We use $\lg$ to denote the logarithm with base 2.
flow to or from \( u \), we charge \( \Delta_{\tilde{C}} |(Af^i)_u| \). Clearly, we charge at least \( \Delta_{\tilde{C}} \sum_{v \in \tilde{C}} \hat{b}_v \) for each subcell \( \tilde{C} \).

It remains to prove we did not overcharge by too large a factor. Consider an arbitrary flow \( f^i \in F \) sending flow from some vertex \( u \) to some vertex \( v \). Let \( C(u, v) \) be the lowest common ancestor cell containing \( u \) and \( v \). Let \( \Delta_{\tilde{C}(u, v)} \) be its side length, and let \( C(\hat{u}, v) \) be the child cell of \( C(u, v) \) that includes \( u \). Let \( \Delta \) be the side length of \( C(\hat{u}, v) \).

Suppose there exists a descendant cell \( C' \) of \( C(\hat{u}, v) \) containing \( u \) that is at least \( 4 \log n \) levels down from \( C(\hat{u}, v) \). Its side length \( \Delta_{C'} \) is at most \( \frac{\Delta}{\alpha^3} \). Because \( C' \) contains at least one point \( u' \in P \), and from Property 2 of Lemma 2.1, \( u \) is at least \( \frac{\Delta}{n^3} - \frac{\Delta}{n^3} \geq \frac{\Delta}{2n^3} \) distance away from any side of \( C(\hat{u}, v) \) and therefore \( v \) as well. 

Because \( \Delta_{C'} \) contains at least one point \( u' \in P \), and from Property 2 of Lemma 2.1, \( u \) is at least \( \frac{\Delta}{n^3} - \frac{\Delta}{n^3} \geq \frac{\Delta}{2n^3} \) distance away from any side of \( C(\hat{u}, v) \) and therefore \( v \) as well. Therefore, we charge at most an \( \frac{\varepsilon}{2m} \) fraction of \( ||f^i||_E \) to cover \( u' \)’s subcell in \( C' \). The amounts charged by similar subcells of smaller side length containing \( u \) form a decreasing geometric series evaluating to at most that value, so all these small subcells charge at most an \( \frac{\varepsilon}{n} \) fraction total.

Now, consider the cells with larger side length. Suppose there exists an ancestor cell \( C'' \) of \( C(\hat{u}, v) \) at least \( \log \varepsilon_0^{-1} + 1 \) levels up from \( C(\hat{u}, v) \), and let \( \tilde{C}'' \) be the subcell of \( C'' \) containing \( u \). Then the side length of \( \tilde{C}'' \) is at least \( \Delta_{C(u,v)} \) and all points in \( C(u,v) \) will be included in \( \tilde{C}'' \) also.

We do not charge to \( ||f^i||_E \) for subcell \( \tilde{C}'' \), and there are at most \( 4 \log n + \log \varepsilon_0^{-1} \leq 4 \log \frac{n}{\varepsilon_0} \) subcells in addition to those handled above for which we do charge to \( ||f^i||_E \). Consider any such subcell \( \tilde{C} \). The path carrying \( f^i \) leaves \( \tilde{C} \) through an edge of length at least \( \Delta_{\tilde{C}}/2 \), so we charge at most \( 2 \cdot ||f^i||_E \) to cover \( \tilde{C} \). Summing over all \( 4 \log \frac{n}{\varepsilon_0} \) choices of \( \tilde{C} \) and accounting for the tiny cells as discussed above, we charge at most \( (8 \log \frac{n}{\varepsilon_0} + \varepsilon/n) ||f^i||_E \leq 9 \log \left( \frac{n}{\varepsilon_0} \right) \cdot ||f^i||_E \) to cover subcells containing \( u \). We also charge to \( ||f^i||_E \) to cover subcells containing \( v \), so we overcharge by a factor of at most \( 18 \log \left( \frac{n}{\varepsilon_0} \right) = \Lambda \). The lemma follows.

\[ \text{Lemma 3.2.} \text{ We have } \min \{ ||f||_E : f \in \mathbb{R}_E, Af = \hat{b} \} \leq \kappa ||B\hat{b}||_1 \text{ for some } \kappa = O(\varepsilon_0^{-1} \log (n/\varepsilon_0)). \text{ Moreover, a flow vector } f \text{ satisfying } Af = \hat{b} \text{ of cost at most } \kappa ||B\hat{b}||_1 \text{ can be computed in } O(m) \text{ time.} \]

**Proof:** We describe a greedy algorithm based on one by Khesin et al. [9] to iteratively construct a feasible flow \( f \) satisfying \( Af = \hat{b} \) with a cost \( ||f||_E \leq \kappa B\hat{b} \) in \( O(m) \) time. At any point during \( f \)'s construction, we say the **surplus** of vertex \( u \in V \) is \( \pi(u,f) = (Af)_u - \hat{b}_u \), the difference between the current and desired divergences of \( u \).

1. For every cell \( C \) in a postorder traversal of \( G \)’s simple sub-quadtrees, for every subcell \( \tilde{C} \) of \( C \), we do the following. Let \( \nu = N_{\tilde{C}} \). We choose any two child net points \( v, w \) of \( \nu \) such that \( \pi(v, f) > 0 > \pi(w, f) \). We then add min\{\( \pi(v,f), \pi(w,f) \}\} to \( f_{(w,v)} \). In doing so, we make the surplus of at least one child net point of \( \nu \) equal to 0, and we decrease the absolute values of surpluses of both \( v \) and \( w \). Therefore, after at most a number of steps equal to the number of child net points of \( \nu \), either all child net points have non-negative surplus or all child net points have non-positive surplus. Finally, for each vertex \( v \) among child net points with non-zero surplus, we set \( f_{(\nu,v)} = \pi(v,f) \). Afterward, every child net point of \( \nu \) has surplus 0. In other words, the unbalance among those child net points is collected into \( \nu \). Each net point \( \nu \) has at most \( 2^d \) child net points. Therefore, the total running time for this step is \( O(m) \).

2. After performing step 1), all net points with parents have a surplus of 0. We pick up any two net points \( u, v \) of subcells of \( T \)’s root cell with two different surplus signs as described in step
2 and add \( \min \{ |\pi(u, f)|, |\pi(v, f)| \} \) to \( f_{(u, v)} \). After \( O(\varepsilon_0^{-d}) = O(m) \) steps, all points \( v \in V \) will have surplus 0, and \( f \) is a feasible flow satisfying \( Af = \tilde{b} \).

We now analyze \( ||f||_{\bar{E}} \). Consider a subcell \( \tilde{C} \) of some cell \( C \) with net point \( \nu \). Flow does not leave or enter \( \tilde{C} \) until we move flow between \( \nu \) and either another net point in \( C \) or \( \nu \)'s parent net point. Therefore, \( \pi(\nu, f) = -\sum_{v \in \tilde{C}} \tilde{b}_v \) immediately after moving flow from \( \nu \)'s children to \( \nu \) in step 1) above. All subsequent steps moving flow to or from \( \nu \) involve an edge of length at most \( \varepsilon_0^{-1} \sqrt{\Delta_{\tilde{C}}} \) and only serve to reduce \( |\pi(\nu, f)| \).

Summing over all subcells, we get

\[
||f||_{\bar{E}} \leq \sum_{\tilde{C} \in \tilde{C}} \varepsilon_0^{-1} \sqrt{\Delta_{\tilde{C}}} \sum_{v \in \tilde{C}} \tilde{b}_v \leq \varepsilon_0^{-1} \sqrt{\Delta} ||B\tilde{b}||_1.
\]

Therefore, \( ||f||_{\bar{E}} \leq \kappa ||B\tilde{b}||_1 \), where \( \kappa = O(\varepsilon_0^{-1} \log (n/\varepsilon_0)) \).

\[\text{Lemma 3.3. Applications of } BA \text{ and } (BA)^T \text{ to arbitrary vectors } f \in \mathbb{R}^E \text{ and } \tilde{b} \in \mathbb{R}^V, \text{ respectively, can be done in } O(m) \text{ time.}\]

\[\text{Proof: } \text{Both applications can be performed using relatively simple dynamic programming algorithms.}\]

Computing \( BAf \) Let \( A' = Af \). Recall, \( \forall v \in V, A'_v \) is the divergence of \( v \) given flow \( f \). Matrix \( A \) has \( m \) non-zero entries, so \( A' \) can be computed in \( O(m) \) time.

We compute \( BAf \) by computing \( BA' \). Let \( \nu \) be any net point of \( G \), and let \( \tilde{C} \) be its subcell. From the definition of \( B \), we have \( (BA')_{\nu} = \frac{\Delta_{\nu}}{\Delta} \sum_{v \in \tilde{C}} A'_v \). Now, let \( \tilde{C}^+ \) be the (possibly empty) set of all child subcells of \( \tilde{C} \) with net points in \( G \). We have \( \sum_{v \in \tilde{C}} A'_v = A'_v + \sum_{\tilde{C}' \in \tilde{C}^+} \sum_{v \in \tilde{C}'} A'_v \). Thus, we can use dynamic programming to compute \( BA' \) in \( O(m) \) time. Each entry is filled in during a postorder traversal of the quadtree cells.

Computing \( (BA)^T \tilde{b} \) Recall, \( (BA)^T = A^TB^T \). Let \( b' = B^T\tilde{b} \). We begin by computing \( b' \). Let \( \tilde{C} \) be any subcell with a net point in \( G \), and let \( \nu = N_{\tilde{C}} \). Let \( \tilde{C}^- \) be the set of all ancestor subcells of \( \tilde{C} \) with net points in \( G \). We have \( b'_v = \frac{\Delta_{\nu}}{\Delta} \tilde{b}_v + \sum_{\tilde{C}' \in \tilde{C}^-} \frac{\Delta_{\nu}}{\Delta} \tilde{b}_{N_{\tilde{C}'}} \). Therefore, we can use dynamic programming to compute \( b' \) in \( O(m) \) time. Each entry is filled in during a preorder traversal of the quadtree cells. Finally, \( A^T \) has \( m \) non-zero entries, so \( A^TB^T\tilde{b} = A^Tb' \) can be computed in \( O(m) \) time as well.

We have shown there exists a \( (1 + 2\varepsilon, 0) \)-solver for the minimum cost flow problem on \( G \). Plugging in all the pieces, we get a running time bounded by

\[
O(m\varepsilon_0^{-2} \log^3 (n/\varepsilon_0)(\varepsilon^{-2} + \log (n/\varepsilon_0))).
\]

We set \( \varepsilon_0 \) to be a sufficiently small multiple of \( \varepsilon/\log n \) and run the preconditioning framework algorithm in each graph \( G \) as described in Section 2.3. The final running time to compute a flow in \( G^* \) of cost at most \( (1 + \varepsilon)\text{Cost}(P, \mu) \) is

\[
O(n\varepsilon^{-O(d)} \log^O(d) n).
\]

In the sequel, we describe how to recover a transportation map of \( P \) using this approximately minimum cost flow in \( G^* \).
4 Recovering a transportation map from the minimum cost flow

Let \( f \in \hat{E} \) be the approximate minimum cost flow computed for \( G^* \). The transportation map \( \tau \) contains only weighted pairs of points in \( P \). We will implicitly maintain a flow \( f \) of cost at most \( ||f||_{E^*} \), that will eventually describe our transportation map. In short, we follow the high level strategy of Khesin et al. [9] of iteratively rerouting flow going through each net point to instead go through its neighbors in \( G^* \), eventually resulting in no flow going through any net point. Nearly every pair containing a point \( p \in P \) and an ancestor net point may at some moment carry flow during this procedure. Because quadtree \( T \) has such high depth, we must take additional care. Abusing notation, we extend the definition of \( f_{(u,v)} \) to include any pair of vertices in \( G^* \). Value \( f_{(u,v)} \) is initially 0 for all \( uv \notin E^* \).

To quickly maintain these flow assignments with points in \( P \), we store two data structures \( pt(\nu) \) and \( nt(\nu) \) for each net point \( \nu \in V \setminus P \). We call these data structures the prefix split trees of \( \nu \). The prefix split tree is stored as an ordered binary tree data structure where each node has a weight. We let \( w(x) \) denote the weight of node \( x \) in a tree \( S \) and \( w(S) \) denote the total weight of all nodes in \( S \). These trees support the standard operations of insertion and deletion. They support the \( \text{MERGE}(S, S') \) operation which takes two trees \( S \) and \( S' \) and combines them into one tree with all members of \( S \) appearing in order before \( S' \). Finally, they support the \( \text{PREFIX_SPLIT}(S, t) \) operation defined as follows. Given a target value \( t \) and a prefix split tree \( S \), \( \text{PREFIX_SPLIT} \) finds a maximal prefix of \( S \)'s nodes in order where the sum of node weights in the subset is less than or equal to \( t \). If the sum is less than \( t \), it splits the next node \( x \) into two nodes \( x_1 \) and \( x_2 \) where \( w(x_1) + w(x_2) = w(x) \). The split makes sure adding \( x_1 \) to the maximal prefix subset makes the sum weight of the subset exactly equal to \( t \). The operation then splits off all members of this subset, including \( x_1 \) if a node \( x \) was split, into their own tree \( S' \) and returns it, leaving \( S \) with only the remaining nodes. We emphasize that the order of nodes within the data structure is important for defining \( \text{PREFIX_SPLIT} \), but the nodes are not “sorted” in any meaningful sense; in particular, any two trees can be merged as defined above. All those operations can be done in amortized \( O(\log m) \) time, where \( m \) is the number of nodes in the tree, by applying simple modifications to the splay tree data structure of Sleator and Tarjan [16]. We provide details on how to implement a prefix split tree in Appendix A.

In our setting, every node in \( pt(\nu) \) and \( nt(\nu) \) represents a point \( p \in P \). Thanks to our use of the \( \text{PREFIX_SPLIT} \) procedure, some points may be represented multiple times in a single tree. We use \( pt(\nu)[p] \) to denote the set nodes representing \( p \) in \( pt(\nu) \), and define \( nt(\nu)[p] \) similarly. Our algorithm implicitly maintains the invariant that for all net points \( \nu \) and points \( p \in P \),

\[
\sum_{x \in pt(\nu)[p]} w(x) - \sum_{x \in nt(\nu)[p]} w(x) = f_{(\nu,p)}.
\]

We proceed with the algorithm given in Figure 2.

**Lemma 4.1.** Our algorithm results in a transportation map of cost at most \( ||\hat{f}||_{E^*} \), and it can be implemented to run in \( O(ne_0^{-2d} \log^2(n/\eps_0)) \) time.

**Proof:** As stated, our algorithm implicitly maintains a flow \( f \) such that for all net points \( \nu \) and points \( p \in P \),

\[
\sum_{x \in pt(\nu)[p]} w(x) - \sum_{x \in nt(\nu)[p]} w(x) = f_{(\nu,p)}.
\]

One can easily verify that after every iteration of any of the while loops, the divergences among all vertices in \( G^* \) remain the same. Further, after processing any net point \( \nu \) in the inner for loop, there are no other vertices \( u \) in \( V^* \) such that \( f_{(u,v)} \neq 0 \). Observe the algorithm never changes the flow coming into or out of a net point \( \nu \) unless \( f_{(u,v)} \neq 0 \) for some vertex \( u \). Therefore, after \( \nu \) is processed, it \textit{never} has flow going into or out of it again (Khesin et al. [9] refer to this property as \( \nu \) have \textit{uniform flow parity}). Because we eventually process every net point in \( G^* \), we eventually end up with a flow \( f \) such that \( f_{(p,q)} \neq 0 \) only if \( p, q \in P \). We immediately see \( \tau \) is a transportation map.
For all net points $v \in V^* \setminus P$ and $p \in P$ where $f_{(p,v)} > 0$
  Insert a node of weight $f_{(p,v)}$ into $nt(v)$ representing $p$
For all net points $v \in V^* \setminus P$ and $p \in P$ where $f_{(v,p)} > 0$
  Insert a node of weight $f_{(v,p)}$ into $pt(v)$ representing $p$

Let $C$ be the set of all cells

For $C \in C$ in postorder

Let $N_C = \{N_C' : C \in C\}$, $N'_C = \{\text{parent of } v : v \in N_C\}$

For each $v \in N_C$

\textit{Initialize data structures.}\
For all net points $v \in V^* \setminus P$ and $p \in P$ where $f_{(p,v)} > 0$
  Insert a node of weight $f_{(p,v)}$ into $nt(v)$ representing $p$
For all net points $v \in V^* \setminus P$ and $p \in P$ where $f_{(v,p)} > 0$
  Insert a node of weight $f_{(v,p)}$ into $pt(v)$ representing $p$

\textit{Cancel flow to/from other net points.}\
While $\exists u, w \in N_C \cup N'_C : f_{(v,u)} > 0 > f_{(v,w)}$
  $\delta \leftarrow \min\{f_{(u,v)}, f_{(v,w)}\}$
  $f_{(u,v)} \leftarrow f_{(u,v)} + \delta$
  $f_{(v,u)} \leftarrow f_{(v,u)} - \delta$
  $f_{(v,w)} \leftarrow f_{(v,w)} - \delta$

\textit{Implicitly reduce $f(v,p)$ and increase $f(u,p)$ for several $p \in P$.}\
$p' \leftarrow \text{PREFIXSPLIT}(pt(v), f_{(v,u)})$
$\text{MERGE}(p', pt(u))$

\textit{Implicitly reduce $f(p,v)$ and increase $f(p,u)$ for several $p \in P$.}\
$nt' \leftarrow \text{PREFIXSPLIT}(nt(v), f_{(v,u)})$
$\text{MERGE}(nt', nt(u))$

\textit{Now, all flow to/from $v$ involves points $p \in P$.}\
While $pt(v)$ and $nt(v)$ are not empty
  Let $x \in nt(v)[p], y \in pt(v)[q]$ for some $p, q \in P$
  $\delta \leftarrow \min\{w(x), w(y)\}$
  $f_{(p,q)} \leftarrow f_{(p,q)} + \delta$
  $w(x) \leftarrow w(x) - \delta$; if $w(x) = 0$, delete $x$ from $nt(v)$
  $w(y) \leftarrow w(y) - \delta$; if $w(y) = 0$, delete $y$ from $pt(v)$

For all $(p,q) \in P \times P$ where $f_{(p,q)} > 0$

$\tau(p,q) \leftarrow f_{(p,q)}$

Figure 2. Recovering a transportation map from an approximately minimum cost flow in $G^*$. 

To analyze the cost of $\tau$, observe that after every iteration of a while loop, we replace some $\delta$ units of flow passing through $v$, possibly between multiple sources and one destination or vice versa, with $\delta$ units going directly from the source(s) to the destination(s). By the triangle inequality, this new way to route flow is cheaper, so the final flow $f$, and subsequently $\tau$ has smaller cost than $\hat{f}$.

To implement our algorithm quickly, we only explicitly store new flow values whenever we have a line "$f_{(u,v)} \leftarrow \ldots"$ for some pair of vertices $(u,v)$. Observe that every time we finish processing a cell, every one of its net points is also processed. By the above discussion, flow no longer passes through those net points. Therefore, as we process the net points for a cell $C$, we never send flow from a net point $v \in N_C$ to a net point outside $N_C \cup N'_C$. Every time we change flow going through another net point while processing a net point $v$, we decrease the number net points $u$ such that $f_{(u,v)} \neq 0$ by one. There are $O(n\varepsilon_0^{-d} \log(n/\varepsilon_0))$ net points, and $O(\varepsilon_0^{-d})$ other net points in each $N_C \cup N'_C$, so the number of iterations total in the first three while loops is $O(n\varepsilon_0^{-2d} \log(n/\varepsilon_0))$. Finally, observe that we only do \textsc{PrefixSplit} operations during these while loops, implying we create a total of $O(n\varepsilon_0^{-2d} \log(n/\varepsilon_0))$ nodes throughout all prefix split trees. Every iteration of the fourth while loop results in deleting a node from at least one of $nt(v)$ or $pt(v)$, so the number
of iterations of this while loop is $O(n \epsilon_0^{-2d} \log(n/\epsilon_0))$ as well. Finally, every while loop iteration consists of a constant number of operations in addition to a constant number of prefix split tree operations, each of which can be done in $O(\log(n/\epsilon_0))$ amortized time.

\[\square\]

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References


A near-linear time approximation scheme for geometric transportation with real supplies


A Prefix split trees

We implement our prefix split trees by modifying the splay tree data structure of Sleator and Tarjan [16]. Let $S$ be a prefix split tree. We store the weight $w(x)$ of each node $x$ directly with the node itself. Moreover, every node $x$ keeps another value $W(x)$ equal to the sum weight of all the descendants of $x$ including $x$ itself.

A splay of a node $x$ in $S$ is a sequence of double rotations (possibly followed by a standard single rotation) that move $x$ to the root of $S$. Only those nodes on the path from the root to $x$ have their children pointers updated by a splay. We can update $W(y)$ for every such node $y$ with only a constant factor overhead in the time to perform a splay. Let $s(x)$ denote the number of descendents of $x$ in its prefix split tree, and let $r(x) = \lfloor \log s(x) \rfloor$. Let $\Phi(S) = \sum_{x \in S} r(x)$. The amortized time for an operation on $S$ can be defined as the real time spent on the operation plus the net change to $\Phi(S)$ after the operation. The amortized time for a splay in an $m$-node tree is $O(\log m)$ [16].

Recall, the order of nodes within a tree is largely irrelevant outside the definition of the PrefixSplit operation. To insert a node $x$ in $S$, we add $x$ as the child of an arbitrary leaf of $S$ and splay $x$ to the root. The number of operations in the splay dominates, so the amortized cost of insertion is $O(\log m)$. To delete a node $x$, we splay $x$ to the root and delete it, resulting in two disconnected subtrees $S_1$ and $S_2$. We then perform a Merge($S_1, S_2$) in $O(\log m)$ amortized time as described below, so the whole deletion has amortized cost $O(\log m)$.

The operation Merge($S_1, S_2$) is implemented as follows. Let $x$ be the rightmost leaf of $S_1$. We splay $x$ to the root so it has exactly one child. We then make the root of $S_2$ the other child of $x$. Let $m$ be the total number of nodes in $S_1$ and $S_2$. Adding $S_2$ as a child increases $\Phi(S_1) + \Phi(S_2)$ by $O(\log m)$, so the amortized time for the Merge is $O(\log m)$.

Finally, we discuss the implementation of PrefixSplit($S, t$). We assume $t > 0$. We use the values $W(\cdot)$ to find the prefix of nodes desired. Let $y$ be the next node in order after the prefix. We splay $y$ to the root of $S$. Let $x$ be the left child of $y$ (if it exists). Suppose $W(x) < t$. We delete $y$, creating two trees $S_1$ and $S_2$ where $S_1$ contains the nodes in the prefix. We create a new node $y_1$ of weight $t - W(y)$ and make the root of $S_1$ its child so that $y_1$ is the new root. We create a node $y_2$ of weight $w(y) - w(y_1)$ and make the root of $S_2$ its child. Now, suppose instead $W(x) = t$. In this case, we simply remove the edge between $x$ and $y$ to create a subtree $S_1$ with $x$ as its root. Let $S_2$ be the remainder of $S$. Whether or not $W(x) = t$, we return $S_1$ and set $S = S_2$. The amortized time for the PrefixSplit is the amortized time for a single splay and a constant number of edge changes, implying the PrefixSplit takes $O(\log m)$ amortized time total.