Numerical Computations (review)

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1 Compute the Pressure

The solution of the wave equation is given by

$$P(r,t) = \frac{\pi W}{2c^2}(f_1(r) - if_2(r))e^{i\omega t}, \tag{1.1}$$

since, $P(r,t) = [(B_1 + c_1(r))H_0^{(1)}(kr) + (B_2 + c_2(r))H_0^{(2)}(kr)]e^{i\omega t} = \{[\mathfrak{Re}(B_1 + B_2 + c_1(r) + c_2(r))J_0(kr) + \mathfrak{Im}(B_2 - B_1 + c_2(r) - c_1(r))Y_0(kr)] - i[-\mathfrak{Im}(B_1 + B_2 + c_1(r) + c_2(r))J_0(kr) + \mathfrak{Re}(B_2 - B_1 + c_2(r) - c_1(r))Y_0(kr)]\}[\cos(\omega t) + i\sin(\omega t)] = \frac{\pi W}{2c^2}(f_1(r) - if_2(r))e^{i\omega t}$, where we have used that $B_1 = B_2 = B = -\lim_{r\to\infty} c_1(r)$, $\mathfrak{Im}(c_1(r) + c_2(r)) = 0$, $\mathfrak{Im}(c_2(r) - c_1(r)) = -2\mathfrak{Im}(c_1(r))$, $\mathfrak{Re}(c_2(r) - c_1(r)) = 0$ and $\mathfrak{Re}(c_1(r) + c_2(r)) = 2\mathfrak{Re}(c_1(r))$.

We have defined f_1 and f_2 as

$$f_1(r) = [(-\lim_{r\to\infty} \mathfrak{Re}(c_1(r)) + \mathfrak{Re}(c_1(r)))J_0(kr) - \mathfrak{Im}(c_1(r))Y_0(kr)] \tag{1.2}$$

$$f_2(r) = \lim_{r\to\infty} \mathfrak{Im}(c_1(r))J_0(kr). \tag{1.3}$$

Recall also that,

$$\mathfrak{Re}(c_1(r)) = \int_0^r sY_0(ks)e^{-\frac{s^2}{2\sigma^2}}ds \tag{1.4}$$

$$\mathfrak{Im}(c_1(r)) = \int_0^r sJ_0(ks)e^{-\frac{s^2}{2\sigma^2}}ds. \tag{1.5}$$

For simplicity, denote

$$\alpha := -\lim_{r\to\infty} \mathfrak{Re}(c_1(r)) = -\lim_{r\to\infty} \int_0^r sY_0(ks)e^{-\frac{s^2}{2\sigma^2}}ds \tag{1.6}$$

and

$$\beta := -\lim_{r\to\infty} \mathfrak{Im}(c_1(r)) = -\lim_{r\to\infty} \int_0^r sJ_0(ks)e^{-\frac{s^2}{2\sigma^2}}ds. \tag{1.7}$$

Then, we compute in MATLAB

$$f_1(r) = \{[\alpha + \mathfrak{Re}(c_1(r))]J_0(kr) - \mathfrak{Im}(c_1(r))Y_0(kr)\} \tag{1.8}$$

$$f_2(r) = -\beta J_0(kr). \tag{1.9}$$

The real part of the solution is given by

$$P(r,t) = \frac{\pi W}{2c^2}[f_1(r)\cos(\omega t) + f_2(r)\sin(\omega t)] \tag{1.10}$$

and, the amplitude and phase are given by

$$A(r) = \frac{\pi W}{2c^2}\sqrt{f_1(r)^2 + f_2(r)^2}, \quad \theta(r) = \arctan\frac{f_2(r)}{f_1(r)}. \tag{1.11}$$

2 Compute the amplitude of the displacement

The force acting on one of the TF prongs will be the difference between the pressure on the outside surface of the right tine of the tuning fork minus the pressure on the inside surface of the right tine of the tuning fork

$$f(y,t) = T[P(x_0,y,t) - P(x_1,y,t)] = T[P(x_0,y) - P(x_1,y)])e^{i\omega t} := \frac{\pi W}{2c^2}f(y)e^{i\omega t} \tag{2.1}$$

where the outside surface is at $x_1 = W + g/2$ and the inside surface is at $x_0 = g/2$ and $P(x,y,t)$ is the pressure wave in the coordinate system of the beam and T is the thickness.

We express P in the tuning fork cartesian coordinate system by taking $r = \sqrt{(x-t_x)^2 + (y-t_y)^2}$ hence,

$P(x,y) = P(\sqrt{(x-t_x)^2 + (y-t_y)^2}) = \frac{\pi W}{2c^2}[f_1(\sqrt{(x-t_x)^2 + (y-t_y)^2}) - if_2(\sqrt{(x-t_x)^2 + (y-t_y)^2})]$.

Therefore, $f(y) = T[P(x_L,y) - P(x_R,y)] = \frac{\pi W}{2c^2}T\{[f_1(r_L) - if_2(r_L)] - [f_1(r_R) - if_2(r_R)] = T\{[f_1(r_L) - f_1(r_R)] - i[f_2(r_L) - f_2(r_R)]\}$, where $r_L = \sqrt{(x_L-t_x)^2 + (y-t_y)^2}$ and $r_R = \sqrt{(x_R-t_x)^2 + (y-t_y)^2}$, $x_L = g/2$, and

$x_R = W + g/2$.
And hence

$$f(y) = \frac{\pi W}{2c^2} T\{[f_1(r_L) - f_1(r_R] - i[f_2(r_L) - f_2(r_R)]\} \tag{2.2}$$

where r_L and r_R as described. By the EB-damping document, the real part of the steady state solution of the Euler-Bernoulli equation is given by

$$u(y,t) = \Phi_*(y) B(x_0, y_0) \cos(\omega t - \delta(x_0, y_0)), \tag{2.3}$$

where $B(x_0, y_0) = \frac{0.5\pi/c^2 W}{\sqrt{(\omega_*^2 - \omega^2)^2 + 4(\gamma\omega)^2}} |M_*(x_0, y_0)|$, $\tan(\delta) = \frac{2\gamma\omega\Re(M_*) - (\omega_n^2 - \omega^2)\Im(M_*)}{2\gamma\omega\Im(M_*) + (\omega_n^2 - \omega^2)\Re(M_*)}$ and $M_*(x_0, y_0) = \frac{1}{\rho A} \frac{\int_0^L f(y)\Phi_*(y)dy}{\int_0^L \Phi_*^2(y)dy}$ or $|M_*(x_0, y_0)|^2 =$

$$\left(\frac{1}{\rho A} \frac{\int_0^L T[f_1(\sqrt{(x_L - x_0)^2 + (y - y_0)^2}) - f_1(\sqrt{(x_R - x_0)^2 + (y - y_0)^2})]\Phi_0(y)dy}{\int_0^L \Phi_0^2(y)dy} \right)^2$$

$$+ \left(\frac{1}{\rho A} \frac{\int_0^L T[f_2(\sqrt{(x_L - x_0)^2 + (y - y_0)^2}) - f_2(\sqrt{(x_R - x_0)^2 + (y - y_0)^2})]\Phi_0(y)dy}{\int_0^L \Phi_0^2(y)dy} \right)^2$$

where $x_L = g/2$ and $x_R = W + g/2$.

3 Compute the current intensity

The value of the constant α_I can be obtained from measured values of the quality factor and the resistence R, *i.e.* $\alpha_I = \frac{1}{\omega}\sqrt{\frac{k\gamma}{R}}$. Using the form of u, we obtain that the velocity of the tip of each tine is given by

$$\frac{du}{dt}(L,t) = -\omega \Phi_*(L) B(x_0, y_0) \sin(\omega t - \delta(x_0, y_0)).$$

Therefore, the signal (which is a current intensity) generated by the tuning fork in response to the vibration when the laser beam is focused at (x_0, y_0) is given by

$$I(x_0, y_0) = \alpha_I \left[\frac{du_L}{dt}(L) - \frac{du_R}{dt}(L) \right]$$

where $du_L/dt(L)$ and $du_R/dt(L)$ are the amplitude of the velocity of the right and left tines, respectively. When $x_0 = 0$, $\frac{du_L}{dt}(L,t) = \frac{du_R}{dt}(L,t)$, hence in MATLAB we compute

$$I(0, y_0) = 2\alpha_I \frac{du_R}{dt}(L) = 2C|M_*(0, y_0)| \tag{3.1}$$

where

$$C = \omega \alpha_I \Phi_*(L) const1W \tag{3.2}$$

and

$$const1 = \frac{0.5\pi/c^2}{\sqrt{(\omega_*^2 - \omega^2)^2 + 4(\gamma\omega)^2}} \tag{3.3}$$

$$W = 10^{-13}(\gamma - 1)\alpha_W \omega W_L/(2\pi\sigma^2). \tag{3.4}$$

			Standard TF (STF)	Huge TF (HTF)
Comments		SI unit		
Frequency	f	Hz	32761.10	4249.6
Angular Frequency	$\omega = 2\pi f$	1/s		
Speed of sound	c	m/s	346	
Width of beam	σ	m	0.05×10^{-3}	
Thickness of tine	T	m	0.34×10^{-3}	
Width of tine	W	m	0.6×10^{-3}	
Length of tine	L	m	3.8×10^{-3}	
Gap between tines	g	m	0.3×10^{-3}	
Quality factor	Q	-	16064	
Young's modulus	E	$\mathrm{N/m^2} = \mathrm{Pa}$	7.87×10^{10}	
Moment of Inertia	$I = \frac{TW^3}{12}$	m^4	6.12×10^{-15}	
Cross-sectional area	$A = TW$	m^2	2.04×10^{-7}	
Density of quartz	ρ	$\mathrm{kg/m^3}$	2.6×10^{3}	
Resistance	R	Ω	111×10^{3}	
Absorbtion coefficient	$\alpha_{eff}(2f)$	m^{-1}	1.31×10^{-2}	
Laser Power	W_L	W	61.7×10^{-3}	
Damping coefficient	$\gamma = \frac{\omega}{2Q}$	$1/s$	7.22	
Spring constant	$k = E\frac{TW^3}{4L^3}$	N/m	2.63×10^{4}	

			STF	HTF
Comments		SI unit		
WE constant	$W = \frac{1}{2\pi\sigma^2}(\gamma - 1)\omega\frac{1}{2}\alpha_{eff}W_L$	$\frac{N}{m^2 s^2}$	2.38×10^{9}	
Piezoelectric constant	$\alpha_p = \frac{1}{\omega}\sqrt{\frac{k\gamma}{R}}$	$\frac{A}{m/s}$	5.64×10^{-6}	