Constants

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I. *Find and compute the constant factor from the right hand side of Eq. (1) i.e.* A_0

$$\frac{\partial^2 P}{\partial t^2} - c^2 \Delta P = A_0 e^{-\frac{r^2}{2\sigma^2}} e^{i\omega t}. \tag{1}$$

The SI unit for **pressure** is the pascal ($\mathbf{Pa} = N/m^2 = kg/ms^2$). The unit of $\frac{\partial^2 P}{\partial t^2}$ is $N/m^2 s^2$ and of ΔP N/m^4. Hence the unit of $c^2 \Delta P$ is $N/m^2 s^2$. So the left hand side of (1) has units of $N/m^2 s^2$.

According to [1, 12.1.3] the acoustic source term in the equation for the acoustic pressure is given by $(\gamma - 1)\frac{\partial H}{\partial t}$.

- γ represents **the ratio of specific heats**; it is 1.4 for Air [2].
- $H(r,t)$ represents **the rate of heating produced by the absorption of the laser light**. The absorption coefficient α is included by $H(r,t) = \alpha I(r,t)$, where $I(r,t)$ is **the intensity of the light**$(watt \cdot m^{-2})$.
  * α represents **the absorption coefficient**; in SI units it is measured in m^{-1}.
    The units of $H(r,t)$ is $m^{-1} \cdot watt \cdot m^{-2} = watt/m^3 = J/m^3 s$. Hence the unit of $\frac{\partial H}{\partial t}$ is $J/m^3 s = N/m^2 s^2$

According to [3] the source term of the acoustic wave equation is proportional to the deposited **heat power density**. The spatial size and shape of the source volume depend on the light-beam geometry and on the absorbtion length in the gas, while the time dependence of heat deposition is controlled by the time dependence of laser excitation. This article also says that in the modulated case the time dependence $e^{i\omega t}$ can be assumed.
Hence $H(r,t) = H(r)e^{i\omega t} = \alpha I(r) e^{i\omega t}$. Therefore

$$(\gamma - 1)\frac{\partial H}{\partial t} = i\omega\alpha(\gamma - 1) I(r) e^{i\omega t}.$$

By [3], $I(r) = W_L g(r)$, where W_L denotes the **laser power** and $g(r)$ is the **normalized intensity distribution**; its integral over the entire cross section of the beam is normalized to unity, *i.e.* $\int_0^\infty g(r) 2\pi r dr = 1$. If we let $g(r) = Be^{-r^2/2\sigma^2}$ then, B is determined by the normalization $2\pi B \int_0^\infty e^{-\frac{r^2}{2\sigma^2}} r dr = 1$. Denote the integral by $\mathfrak{I}$ and make the following change of variable $x = \frac{-r^2}{2\sigma^2}$. Then $dx = -\frac{1}{\sigma^2} r dr$ and hence $\mathfrak{I} = \lim_{n\to\infty} \int_0^{-\frac{n^2}{2\sigma^2}} e^x (-\sigma^2) dx =$

$-\sigma^2 \lim_{n\to\infty}(e^{-\frac{n^2}{2\sigma^2}}-1)=\sigma^2$. Therefore $B=\frac{1}{2\pi\sigma^2}$.
The right hand side in (1) therefore is

$$(\gamma-1)\frac{\partial H}{\partial t}=i\alpha(\gamma-1)\omega W_L B e^{-r^2/2\sigma^2}e^{i\omega t}=i(\gamma-1)\frac{\alpha}{2\pi\sigma^2}\omega W_L e^{-r^2/2\sigma^2}e^{i\omega t}. \qquad (2)$$

The units in (2) are $m^{-1}\cdot m^{-2}\cdot s^{-1}\cdot Watt = m^{-3}\cdot s^{-1}\cdot J/s = J/m^3s^2 = Nm/m^3s^2 = N/m^2s^2$ as needed.
The right hand side of the acoustic wave equation is given by $A_0 e^{-r^2/2\sigma^2}e^{i\omega t}$ therefore by (2),

$$A_0 = i(\gamma-1)\frac{\alpha}{2\pi\sigma^2}\omega W_L. \qquad (3)$$

According to Dr. Kosterev,

- α is measured, it is determined by the analyte concentration and is typically in the $10^{-7}-10^{-6}$ cm^{-1} range (low end).
- W_L is simply the total laser power, which is typically 10 to 100 mW in the experiments (units W).
- B is determined by the normalization (as we showed before) (units m^{-2}).
- $\gamma = 1.4$ for both air and nitrogen.
- $\omega = 2\pi f$, where $f = 32760$ kHz.
- Also, recall that $\sigma = 0.5\times 10^{-4}$ (m).

Therefore, $A_0 = 0.4\times\frac{10^{-5}m^{-1}}{2\pi 2.5\times 10^{-9}m^2}\times 2.06\times 10^5 s^{-1}\times 50\times 10^{-3}W = 2.62\times 10^6$ N/m^{2}s^2. (Notice that i was neglected since a multiplication by i, which can be written as $e^{i\frac{\pi}{2}}$, corresponds to only a counter-clockwise rotation by 90 degrees ($\frac{\pi}{2}$ rad)).

II. *Find and compute the proportionality constant for the piexoelectric effect*

Following Dr. Kosterevs notes on "TF Electromechanical Relations", the charge generated on one prong when it's deflected to u is $q=\alpha u$, where $\alpha=\frac{1}{\omega}\sqrt{\frac{k\gamma}{R}}$ $(\frac{A}{m/s})$. The current I (Amps), which measures the amount of charge that passes a given point every second, can be obtained by differentiating $q=\alpha u$ with respect to time. Hence

$$I=\alpha\frac{du}{dt}. \qquad (4)$$

When $f=32.8$ kHz, $\omega = 2\pi\times 32.8\times 10^3$ 1/s $= 2.06\times 10^5$ (1/s), $k=\frac{ETW^3}{4L^3}$, where for quartz $E=7.87\times 10^{10}$ N/m^2, hence $k=2.63\times 10^4$ (N/m). Also $R=119\times 10^3$ (Ω) and $\gamma=\frac{\omega}{2Q}=\frac{2.06\times 10^5}{2\times 13154}=7.83$ (1/sec). Therefore, $\alpha = 6.38\times 10^{-6}$ (A s/m).

References

[1] P. M. Morse and K. U. Ingard, *Theoretical Acoustics*, New Jersey: Princeton University Press, 1986.

[2] S. H. Yönak and D. R. Dowling, "Gas-phase generation of photoacoustic sound in an open environment," *J. Acoust. Soc. Am.*, vol. 114, no. 6, pp. 3167, 2003.

[3] A. Miklos, P. Hess, and Z. Bozoki, "Application of acoustic resonators in photoacoustic trace gas analysis and metrology," *Review of Sci. Instruments*, vol. 72, no. 4, pp. 1939, 2001.