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THE STIEFEL-WHITNEY SPARK

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Abstract. In this paper de Rham currents and geometric measure theory are used to study the mod 2 and integer Stiefel-Whitney classes. The paper is part of a larger program of Harvey and his collaborators to develop a theory of primary and secondary characteristic currents.

In a previous paper we proved that, in the case of the integer Stiefel-Whitney classes, to each "atomic"collection of sections of a real vector bundle there is associated a *linear dependency* current, with rectifiability properties, which is supported on the linear depenency set of the collection. The integer cohomology class of a linear dependency current is an integer Stiefel-Whitney class.

In this paper a locally Lebesgue integrable current, called the *Stiefel-Whitney spark*, is associated to each atomic collection of sections. The Stiefel-Whitney spark satisfies the local current equation that its exterior derivative is equal to the linear dependency current. This equation is the natural analogue for the integer Stiefel-Whitney classes of Harvey and Lawson's local Gauss-Bonnet-Chern formula for the Euler class. (A similar current equation is derived for the mod 2 Stiefel-Whitney classes.) Consequently, the Stiefel-Whitney spark plays the same role for the Stiefel-Whitney class that Chern's spherical potential (or transgression) plays for the Euler class.

An explicit local formula is derived for the Stiefel-Whitney spark, analogous to Chern's formula for the spherical potential. Furthermore, the Stiefel-Whitney spark yields a natural generalization of Eells' method of representing Stiefel-Whitney classes by pairs of forms with singularities.

1. Introduction

This paper is part of a larger program of Reese Harvey and his collaborators [12, 13, 14, 15, 16, 17, 18, 19] to develop a Chern-Weil-Cheeger-Simons theory of

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smooth maps between vector bundles with connection. In this theory primary and secondary characteristic classes are canonically represented by de Rham currents which are concentrated on the singularity sets of a smooth map between vector bundles.

In this paper the Stiefel-Whitney classes of a real vector bundle are represented by rectifiable currents and by locally integrable forms. These currents and forms are canonically defined in terms of the linear dependency sets of collections of sections of the vector bundle.

The results in this paper owe a great debt to the seminal papers of Stiefel [24] and Whitney [26] on Stiefel-Whitney classes, to Chern [5, 6] who investigated the special case of the top Stiefel-Whitney (i.e. Euler) class, to Federer [9] for fundamental results concerning currents and cohomology and to Thom [25] and Porteous [21] who constructed characteristic classes from the singularities of maps. Our results are also closely related to and inspired by the work of Cheeger-Simons [3, 4] on differential characters, and to Eells' paper [8] in which Stiefel-Whitney classes are represented by pairs of differential forms with singularities. We would also like to draw attention to the recent work [10] in which current representatives for the tangential Stiefel-Whitney homology classes of a singular variety are constructed.

This paper builds directly on two previous papers, [18] concerning Stiefel-Whitney currents, and [19] concerning the divisor and Euler spark of an atomic section.

In [18] we constructed canonically defined currents which represent the *integer* Stiefel-Whitney class, $W_q(F)$, of a real vector bundle F over a smooth manifold X. (The classes $W_q(F)$ are *odd* degree classes in $H^q(X, \mathbf{Z})$.) These *Stiefel-Whitney currents* are canonically associated to each collection, α, of the appropriate number of sections of the vector bundle and are supported on the linear dependency set of the collection. Consequently they are also called *linear dependency currents* and are denoted by $\mathbf{LD}(\alpha)$. A major advantage of our approach is that the linear dependency current, $\mathbf{LD}(\alpha)$, is defined whenever the collection of sections, α, satisfies a weak measure theoretic condition called *atomicity*, which is vastly more general than the usual general position hypothesis, (see [12, 18]). A review of the definition and properties of linear dependency currents is given in Section 3.

The aim of the present paper is to introduce and study the *Stiefel-Whitney spark*, $S_q(\alpha)$, of an atomic collection, α, of sections of a real vector bundle with metric compatible connection. The Stiefel-Whitney spark, $S_q(\alpha)$, is a canonically defined locally integrable current of degree $q-1$ on X whose exterior derivative is the negative of the linear dependency current of the collection, α. That is, the

Stiefel-Whitney spark satisfies the local equation

$$dS_q(\alpha) \; = \; -\,\mathbf{LD}(\alpha), \tag{1}$$

relating currents on X. Because of equation (1) the Stiefel-Whitney spark is called a *secondary characteristic current*, (c.f. [17]).

The fundamental current equation (1) should be regarded as the analogue for the integer Stiefel-Whitney class, W_q, of the local Gauss-Bonnet-Chern formula, equation (2) below, which concerns the top Stiefel-Whitney (i.e. Euler) class. For this reason, and to better appreciate this paper, it is necessary to review the special case of the divisor and Euler spark of an atomic section of a vector bundle, as treated in [19]. (For a more detailed review see Section 2.)

First recall that cohomology classes, and in particular characteristic classes, can be represented in a variety of ways. For example, on a smooth manifold X, it is natural to represent classes in $H^*(X, \mathbf{R})$ by utilizing smooth differential forms. The more general currents of de Rham can also be used. For integer cohomology, the classes in $H^*(X, \mathbf{Z})$ can be represented by locally finite, twisted, smooth singular chains, which are special cases of the locally rectifiable currents of Federer [9]. Allowing boundaries of locally rectifiable currents as well as locally rectifiable currents defines the *locally integrally flat currents* of Federer [9]. These locally integrally flat currents provide a broad natural way of representing classes in $H^*(X, \mathbf{Z})$, which can in fact be considered the $\mathbf{Z}$-analogue of the $\mathbf{R}$-de Rham theory.

Characteristic classes, when considered as classes in $H^*(X, \mathbf{R})$, are represented by the appropriate Chern-Weil form. For example, if F is a real rank $2n$ vector bundle with a metric compatible connection, then the smooth Chern-Euler-Pfaff form, χ, of Chern [5] represents the real Euler class of F in $H^{2n}(X, \mathbf{R})$. The integer Euler class, $e(F) \in H^{2n}(X, \mathbf{Z})$, of F is classically represented by the cycle obtained from the zero set of a section, α, of F with simple zeros. More generally, in [12] the authors defined the *zero or divisor current*, $\mathrm{Div}(\alpha)$, of an atomic section α of F. The divisor is a locally integrally flat current on X which represents the integer Euler class and is supported on the zero set of the section. The class of atomic sections is much larger than the class of those sections with simple zeros.

These two ways of representing the Euler class – by the Euler-Pfaff form, χ, and by the divisor, $\mathrm{Div}(\alpha)$, of an atomic section, α – are related by the *local Gauss-Bonnet-Chern formula*

$$d(\alpha^*\sigma) \; = \; \chi \; - \; \mathrm{Div}(\alpha) \qquad \text{on } X, \tag{2}$$

where σ is the classical *spherical potential* of Chern [5, 6] which is determined by a metric compatible connection. (In the case that the rank of F is odd the Chern-Euler-Pfaff form is defined to be zero, $\chi := 0$.) The pullback, $\alpha^*\sigma$, of Chern's potential, σ, by an atomic section, α, is well defined as a current on X with locally Lebesgue integrable coefficients. The locally integrable potentials, $\alpha^*\sigma$, are called *Euler sparks.* Note that $\alpha^*\sigma$ is smooth outside the zero set of α.

The close analogy between equations (1) and (2) should now be apparent. Notice that there is no term in equation (1) corresponding to the Euler form χ because the integer Stiefel-Whitney class, W_q, is torsion and so the real cohomology class of the linear dependency current $\mathbf{LD}(\alpha)$ is zero.[1]

The *main results* of the paper, which are presented in Section 4, can be summarized as follows.

The first main result (Theorem 4.1) is an explicit local formula for the Stiefel-Whitney spark, $S_q(\alpha)$, in term of local matrices for the collection of sections, α, and for the gauge and curvature of the given metric connection. This formula for $S_q(\alpha)$ is analogous to a beautiful local formula of Chern for the spherical Chern potential [5, 6].

The second main result (Theorem 4.2) is an analogue of the local Gauss-Bonnet-Chern formula (2) for the *even* degree, mod 2, Stiefel-Whitney class, $w_{q-1}(F) \in H^{q-1}(X, \mathbf{Z}_2)$. This formula, which also involves the Stiefel-Whitney spark, states that

$$2\, S_q(\alpha) \;-\; \mathbf{D} \;=\; dL, \tag{3}$$

where $\mathbf{D}$ is a canonically defined locally integrally flat current whose mod 2 reduction represents $w_{q-1}(F)$, and where L is a canonically defined locally integrable form. A formula for L in the case that α is an orthonormal collection of sections is given in Theorem 4.3. The fact that the Stiefel-Whitney spark, $S_q(\alpha)$, appears as it does in equations (1) and (3) implies the well known fact that the Bockstein of w_{q-1} is equal to W_q.

Eells [8] represents the Stiefel-Whitney classes of a Riemannian manifold X by pairs of smooth differential forms with singularities. Eells pairs of forms with singularities are obtained by pulling certain universal curvature and transgression forms on the frame bundles of X back to X by singular sections of these

[1]The vanishing of the real cohomology class of $\mathbf{LD}(\alpha)$ can also be seen directly from equation (1) since $\mathbf{LD}(\alpha)$ and $S_q(\alpha)$ are both locally flat currents, (i.e. they are of the form $\omega + d\eta$ where ω and η are locally integrable) and because $\mathbf{R}$-cohomology can be computed from the complex of locally flat currents.

frame bundles. As such Eells' approach to the Stiefel-Whitney classes is a natural generalization of Chern's derivation of the Gauss-Bonnet-Chern formula. In Section 5 we explain the intimate relationship between Eells' pairs of forms with singularities and the locally integrable currents in equations (1) and (3). This relationship suggests that there might be a uniqueness theorem for locally integrable forms such as the spark, $S_q(\alpha)$, satisfying equation (1) and the form, L, satisfying equation (3), (see the Remark following Proposition 5.3).

In [17, 19] we showed how certain locally integrable forms, such as the Euler spark, can be used to define differential characters. Differential characters, which are refinements of characteristic classes and are invariants of a vector bundle with connection, were first introduced in the seminal papers of Cheeger [3] and Cheeger and Simons [4]. In Section 6 we use the Stiefel-Whitney spark, $S_q(\alpha)$, to define a differential character called the qth Stiefel-Whitney differential character and explain that this differential character is essentially equal to the mod 2 Stiefel-Whitney class, $w_{q-1}(F)$.

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2. Review: Divisors and Euler Sparks of Atomic Sections

In this section we summarize the results of [12, 14, 19] concerning the divisor and Euler spark of an atomic section of a real vector bundle with metric compatible connection.

Let $F \to X$ be a real vector bundle of rank, n, and let μ be a smooth section of F. For simplicity of exposition we assume that F and X are oriented. (See the Remark in Section 4 below for the nonorientable case.) In [12] the authors defined the notion of the divisor or zero current, $\mathrm{Div}(\mu)$, of an atomic section, μ. The divisor is a d-closed current of degree $2n$ on X which is supported on the zero set and encodes the integer multiplicity of vanishing of the section. Atomicity is a weak measure theoretic condition which ensures the existence of the divisor of a section.

The divisor of an atomic section, μ, is defined, locally on X, as follows. Let $u : X \to \mathbf{R}^n$ be a local coordinate expression for the section. The *solid angle kernel*, Θ, on $\mathbf{R}^n$ is the locally Lebesgue integrable (abbr. L^1_{loc}) form on $\mathbf{R}^n$ defined by pulling back the normalized volume form on the unit sphere to $\mathbf{R}^n \sim \{0\}$ via the radial projection map. The exterior derivative of the L^1_{loc} current, Θ, is the Dirac mass at the origin: $d\Theta = [0]$. This equation motivates the definition of the divisor. The atomicity of the section, μ, ensures that the smooth form, $u^*(\Theta)$, on the complement of the zero set in X has an L^1_{loc} extension to all of X and so defines a current. Then the *divisor* of μ is locally defined by the current equation

$$\operatorname{Div}(\mu) \;:=\; d(u^*(\Theta)). \tag{4}$$

Finally a result of [12] (c.f. [19]) ensures that the divisor is globally defined on X.

If zero is a regular value of the section, μ, then μ is atomic and the divisor is the current of integration over the zero submanifold of μ. However, the set of atomic sections is much larger than the set of sections which have zero as a regular value. For example, all real analytic sections of a real rank m bundle whose zero sets have codimension at least m are atomic [12]. More generally, any smooth section which vanishes algebraically and whose zero set has Minkowski codimension strictly greater than $m-1$ is atomic.

There are two fundamental local equations relating currents on the base manifold X which are central to the theory of divisors. The first equation is the *comparison formula for divisors* [12] which states that, if μ_1 and μ_2 are two atomic sections of F, then there is a locally rectifiable current, R, so that

$$\operatorname{Div}(\mu_1) \;-\; \operatorname{Div}(\mu_2) \;=\; -dR \qquad \text{on } X. \tag{5}$$

This equation implies that the divisor of an atomic section is a d-closed locally integrally flat current.[2] Consequently a theorem of Federer [9] implies that the divisor of an atomic section determines an integer cohomology class. Furthermore equation (5) states that this cohomology class – the Euler class – is independent of the choice of atomic section of F.

Let D_F be a metric compatible connection on F and let $\chi(\Omega_F)$ denote the Chern-Euler-Pfaff form of the connection, D_F, [5, 6, 20]. (Recall that $\chi(\Omega_F) := 0$ if the rank of F is odd.) The second fundamental equation in the theory of divisors is the *local Gauss-Bonnet-Chern formula* [14] which states that, if μ is an atomic section of F, then there is a canonically defined, functorial, locally integrable form, $\sigma(\mu)$, of degree $2n-1$ so that

$$d\sigma(\mu) \;=\; \chi(\Omega_F) \;-\; \operatorname{Div}(\mu) \qquad \text{on } X. \tag{6}$$

The classical Gauss-Bonnet-Chern formula [5] is recovered from equation (6) by letting each of the currents in (6) act on the test function, 1, on a compact $2n$-dimensional manifold X.

[2] Recall that a *locally integrally flat current* is a current of the form $R + dS$ where R and S are locally rectifiable.

The L^1_{loc} form, $\sigma(\mu)$, in equation (6) is called the *"Euler spark"* of the atomic section μ, and depends on both the section, μ, and the metric connection, D_F. It is defined as follows. The *spherical Chern potential*, σ, is an L^1_{loc} form of degree $2n-1$ on F which satisfies the current equation

$$d\sigma \;=\; \pi^*\chi(\Omega_F) \;-\; [X] \qquad \text{on } F, \tag{7}$$

where $[X]$ denotes the current of integration over the zero section of $F \to X$. The potential, σ, was discovered by Chern [5, 6]. The *Euler spark*, $\sigma(\mu)$, of an atomic section, μ, of F (relative to the connection and metric) is the L^1_{loc} form on X defined, on the complement of the zero set of μ in X, by the equation $\sigma(\mu) := \mu^*(\sigma)$.[3]

A beautiful explicit formula for the spherical potential, σ, is also due to Chern [5, 6] (see equations (18) and (40) below). In particular the restriction of σ to each fibre of F is the solid angle kernel, Θ, of the fibre. One of our main results, Theorem 4.1 below, is an analogous explicit formula for the Stiefel-Whitney spark of a collection of sections of a real vector bundle.

Finally we recall the fundamental *comparison formula for Euler sparks* [18] which states that, if μ_1 and μ_2 are two atomic sections of F, then there is a locally rectifiable current, R, and an L^1_{loc} current, L, so that

$$\sigma(\mu_1) \;-\; \sigma(\mu_2) \;=\; R \;+\; dL \qquad \text{on } X. \tag{8}$$

Note that equation (5) can be obtained by taking the exterior derivative of equation (8). The significance of this comparison formula will be explained in Section 6 on Differential Characters.

3. Review: Linear Dependency Currents and Stiefel-Whitney Classes

In this section we summarize the results of [18] concerning the linear dependency current of a collection of sections of a real vector bundle.

Let α be a collection of m sections of an oriented rank n real vector bundle F. For the remainder of the paper we make the following

$$\textbf{Assumption:} \qquad q \;:=\; n-m+1 \text{ is odd and } q<n. \tag{9}$$

The linear dependency current, $\mathbf{LD}(\alpha)$, of the collection of sections α is a degree q, d-closed, locally integrally flat current on X which is supported on the linear

[3] The atomicity of the section μ ensures that the smooth form $\mu^*(\sigma)$ on $X \sim \text{Zero}(\mu)$ has a unique L^1_{loc} extension to all of X.

dependency set of the collection, α. The current, $\mathbf{LD}(\alpha)$, is defined whenever the collection, α, satisfies the condition of $\mathbf{LD}$-atomicity, as descibed below.

The *comparison formula for linear dependency currents* states that, if α_1 and α_2 are two $\mathbf{LD}$-atomic collections of sections of $F \to X$ then there is a locally rectifiable current, R, such that

$$\mathbf{LD}(\alpha_1) \ - \ \mathbf{LD}(\alpha_2) \ = \ dR \qquad \text{on } X. \tag{10}$$

Therefore the cohomology class of the integrally flat current, $\mathbf{LD}(\alpha)$, in $H^q(X, \mathbf{Z})$ is well defined independent of the choice of collection of sections, α.

The integer cohomology class of $\mathbf{LD}(\alpha)$ can be identified as follows. First recall that the qth *integer Stiefel-Whitney class*, W_q, is defined to be the Bockstein of the $(q-1)$st mod 2 Stiefel-Whitney class, w_{q-1} (see [2, 23, 26]), and note that the mod 2 reduction of W_q is the qth mod 2 Stiefel-Whitney class, w_q. In [18] we proved that the integer cohomology class of $\mathbf{LD}(\alpha)$ is $W_q(F)$. Furthermore, there is a locally rectifiable current, R, so that

$$2\,\mathbf{LD}(\alpha) \ = \ dR, \tag{11}$$

i.e. the cohomology class, $W_q(F)$, in $H^q(X, \mathbf{Z})$ is torsion of order 2.

We conclude this section by recalling the definition of the linear dependency current. This construction uses a well known method from intersection theory (see [11, 21, 25]), but in a C^∞ context. First, the collection of sections, α, defines a bundle map, $\alpha : \underline{\mathbf{R}}^m \to F$, from the trivial rank m bundle $\underline{\mathbf{R}}^m$ to F. Let $\pi : S(\underline{\mathbf{R}}^m) \to X$ be the unit sphere bundle in $\underline{\mathbf{R}}^m \to X$ and let $N \subset \underline{\mathbf{R}}^m$ be the normal line bundle to $S(\underline{\mathbf{R}}^m) \subset \underline{\mathbf{R}}^m$. The bundle map α induces a map $\widehat{\alpha} : N \subset \underline{\mathbf{R}}^m \longrightarrow \pi^* F$ and hence a section $\widehat{\alpha}$ of $\mathrm{Hom}(N, \pi^* F) \longrightarrow S(\underline{\mathbf{R}}^m)$.

With this notation we can make the following definitions. First, the collection of sections, α, is called $\mathbf{LD}$*-atomic* if the induced section, $\widehat{\alpha}$ of $\mathrm{Hom}(N, \pi^* F) \longrightarrow S(\underline{\mathbf{R}}^m)$, is atomic. Second, the *linear dependency current* of an $\mathbf{LD}$-atomic collection, α, is defined to be half the current pushforward of the divisor of the induced section, $\widehat{\alpha}$, i.e.

$$\mathbf{LD}(\alpha) \ := \ \tfrac{1}{2}\, \pi_*(\mathrm{Div}(\widehat{\alpha})) \qquad \text{on } \ X. \tag{12}$$

4. The Stiefel-Whitney Spark

In this section we define the Stiefel-Whitney spark of a collection of sections of a real vector bundle and state the main results of the paper.

Let α be an $\mathbf{LD}$-atomic collection of m sections of a rank n real vector bundle, $F \to X$, with metric connection, D_F, and suppose that $q = n - m + 1$ is odd. The *qth Stiefel-Whitney spark*, $S_q(\alpha)$, of α is a locally integrable form which satisfies the fundamental current equation

$$dS_q(\alpha) \;=\; -\mathbf{LD}(\alpha) \qquad \text{on} \;\; X. \tag{13}$$

The Stiefel-Whitney spark is defined, in analogy with definition (12) of the linear dependency current, as follows.

Definition 1. Let $\pi : S(\underline{\mathbf{R}}^m) \to X$ denote the unit sphere bundle of the trivial bundle $\underline{\mathbf{R}}^m \to X$, let N be the normal line bundle to $S(\underline{\mathbf{R}}^m)$ in $\underline{\mathbf{R}}^m$, and let $\mathrm{Hom}(N, \pi^*F) \longrightarrow S(\underline{\mathbf{R}}^m)$ denote the bundle of homomorphisms from N to the pullback π^*F of F over $S(\underline{\mathbf{R}}^m)$. Endow the bundle $\mathrm{Hom}(N, \pi^*F)$ with the induced metric connection.

Let α be an **LD**-atomic collection of m sections of $F \to X$. Let $\sigma(\widehat{\alpha})$ denote the Euler spark of the atomic section, $\widehat{\alpha}$, of $\mathrm{Hom}(N, \pi^*F) \longrightarrow S(\underline{\mathbf{R}}^m)$ induced by the collection, α, as in Section 3 above. Then the *qth Stiefel-Whitney spark*, $S_q(\alpha)$, of the collection, α, is the degree $(q-1)$ locally integrable form

$$S_q(\alpha) \;:=\; \tfrac{(-1)^{m-1}}{2}\, \pi_*(\sigma(\widehat{\alpha})) \qquad \text{on} \;\; X, \tag{14}$$

i.e. $S_q(\alpha)$ is $\frac{(-1)^{m-1}}{2}$ times the current pushforward from $S(\mathbf{R}^m) \times X \to X$ (i.e. fibre integral) of the Euler spark, $\sigma(\widehat{\alpha})$.

Note that the fundamental current equation (13) follows from the local Gauss-Bonnet-Chern formula (6) and from the definitions (12) and (14) of the linear dependency current and Stiefel-Whitney spark.

Remark. The existence and importance of the Stiefel-Whitney spark can be motivated as follows. Recall that real cohomology classes can be represented by locally flat currents, that is by currents of the form, $\omega + d\eta$, where ω and η are L^1_{loc} currents (see [9] or the Appendix to [18]). Also, by equation (11), the real cohomology class of the linear dependency current, $\mathbf{LD}(\alpha)$, is zero. These two facts imply that there exists an L^1_{loc} current, $S_q(\alpha)$, satisfying equation (13).

The Stiefel-Whitney spark is important because it is a *canonically defined, functorial* solution, $S_q(\alpha)$, of equation (13). Furthermore, the comparison formula (8) for Euler sparks immediately implies the important *comparison formula for Stiefel-Whitney sparks* which states that, if α_1 and α_2 are two **LD**-atomic collections of m sections of F then there is a locally rectifiable current, R, and an L^1_{loc} current, L, so that

$$S_q(\alpha_1) \;-\; S_q(\alpha_2) \;=\; R \;+\; dL. \tag{15}$$

This formula will be discussed further in Section 6 on the Stiefel-Whitney differential character.

We now present the first main result of the paper, which is an explicit local formula for the Stiefel-Whitney spark, $S_q(\alpha)$. This formula is analogous to Chern's formula for the spherical potential of an odd rank real vector bundle.

The formula for $S_q(\alpha)$ is in terms of the following local data. Let a_i be the i^{th} row of the matrix, a, and let ω_F and Ω_F be the local connection 1-form and curvature matrices of the connection, D_F. Finally let $Da := da + a\omega_F$ and $\lambda(f) := f_1 \wedge \cdots \wedge f_n$. With this notation we can state the first main result.

Theorem 4.1. *Let α be an* **LD***-atomic collection of m sections of a rank n vector bundle, F, and suppose that $q = n - m + 1 = 2q_0 + 1$ is odd. Then the qth Stiefel-Whitney spark, $S_q(\alpha)$, is locally given by the formula*

(16)

$$S_q(\alpha)\ \lambda(f) \ = \ -\tfrac{1}{2}(\tfrac{1}{4\pi})^{q_0}\tfrac{1}{(q_0)!}\,\frac{(a_1 f)\dots(a_m f)}{\sqrt{\det(aa^t)}}\ \left\{f^t[Da^t(aa^t)^{-1}Da - \Omega_F]f\right\}^{q_0}.$$

The proof of Theorem 4.1 is given in Section 7 below.

Remark (Special cases of Theorem 4.1). (1) In the case $q = 1$ (i.e., $m = n$) equation (16) reduces to

$$S(\alpha) \ = \ \tfrac{1}{2}\,\tfrac{\det a}{|\det a|}. \tag{17}$$

Consequently when $q = 1$ the linear dependency current of α equals the divisor of the induced section, $\det(\alpha)$, of the determinant line bundle of F (c.f. Proposition 3.12 of [18]).

(2) If $m = 1$ then $n = q = 2n_0 + 1$ is odd and equation (16) reduces to Chern's formula [6] for the Euler Spark of a section, $\mu = uf$, of an *odd* rank bundle:

$$S(\mu)\ \lambda(f) \ = \ -\tfrac{1}{2}(\tfrac{1}{4\pi})^{n_0}\tfrac{1}{(n_0)!}\frac{uf}{|u|}\left\{\frac{(Duf)^2}{|u|^2} - f^t\Omega_F f\right\}^{n_0}. \tag{18}$$

(3) Recall that the divisor of a section of a vector bundle is locally defined to be the exterior derivative of the pullback, by a local expression for the section, of the solid angle kernel on $\mathbf{R}^n$.

Similarly we could define the *linear dependency kernel*, S_q, on the manifold of all $m \times n$ matrices, a, by

$$S_q\ \lambda(f) \ := \ -\tfrac{1}{2}(\tfrac{1}{4\pi})^{q_0}\tfrac{1}{(q_0)!}\frac{(a_1 f)\dots(a_m f)}{\sqrt{\det(aa^t)}}\ \left\{f^t da^t(aa^t)^{-1}daf\right\}^{q_0}, \tag{19}$$

and locally define the linear dependency current by $\mathbf{LD}(\alpha) := -d(a^* S_q)$, whenever $\widehat{\alpha}$ is atomic.

The second main result of the paper is given in Theorem 4.2 below. This result is a refinement, or secondary version, of the current equation

$$2\,\mathbf{LD}(\alpha) \;-\; dR \;=\; 0, \tag{20}$$

where R is locally rectifiable, (i.e. of equation (11) above). Recall from [18] that the current, R, in equation (20) can be chosen so that its mod 2 reduction, $R^{\mathrm{mod}\,2}$, represents $w_{q-1}(F) \in H^{q-1}(X, \mathbb{Z}_2)$. Thus equation (20) states that the cohomology class of the current, $\mathbf{LD}(\alpha)$, is equal to the Bockstein of $w_{q-1}(F)$, which (by definition) is $W_q(F)$.

Theorem 4.2. *Let α be a collection of $m = n - q + 1$ sections of a real rank n vector bundle, $F \to X$, where $q > 1$ is odd. Then the qth Stiefel-Whitney spark, $S_q(\alpha)$, of α satisfies the local, current equation*

$$2\,S_q(\alpha) \;-\; R \;=\; dL \qquad \text{on } X, \tag{21}$$

where L is L^1_{loc} and R is a locally rectifiable current whose mod 2 reduction, $R^{\mathrm{mod}\,2}$, represents the $(q-1)$st mod 2 Stiefel-Whitney class $w_{q-1}(F) \in H^{q-1}(X, \mathbb{Z}_2)$.

Note that equation (20) can be obtained by taking the exterior derivative of equation (21), and using equation (13). In particular the rectifiable currents, R, in these two equations are the same.

Remark (Interpretations of Theorem 4.2). Equation (21) should be regarded as the natural analogue for the even degree mod 2 Stiefel-Whitney class, $w_{q-1}(F) \in H^{q-1}(X, \mathbf{Z}_2)$, of the local Gauss-Bonnet-Chern formula (6) for the integer Euler class $e(F) \in H^n(X, \mathbf{Z})$. In Section 5 below we will explain how equation (13) and Theorem 4.2 generalize formulae of Eells in which the Stiefel-Whitney classes, W_q and w_{q-1}, are represented by pairs of forms with singularities. Furthermore, in Section 6 below, we will show that Theorem 4.2 gives a simple relationship between the qth integer Stiefel-Whitney differential character and the $(q-1)$st mod 2 Stiefel-Whitney class.

Remark (Functorality of the currents L and R). For almost all $\mathbf{LD}$-atomic collections, α, of sections of F the currents L and R in Theorem 4.2 can be defined in terms of an additional section, β, of F, chosen so that the collection (β, α) is also $\mathbf{LD}$-atomic, (see equations (23) and (24) below). Consequently, for almost all α, the currents L and R are functorial with respect to vector bundle maps.

Remark (The nonorientable case). The results described above still hold if no orientation assumptions are made on the vector bundle, F, or on the base manifold, X, provided that the following modifications are made. (For a complete discussion of the nonorientable case see [18].)

Let $\mathcal{O}_F$ denote the principal $\mathbb{Z}_2$-bundle of orientations of F over X, and define the coefficient sheaf, $\widetilde{\mathbb{Z}}$, of twisted integers by $\widetilde{\mathbb{Z}} := \mathcal{O}_F \otimes_{\mathbb{Z}_2} \mathbb{Z}$. The first modification is that the integer Stiefel-Whitney class, $W_q \in H^q(X, \mathbb{Z})$, must be replaced by the $\widetilde{\mathbb{Z}}$-Stiefel-Whitney class, $\widetilde{W_q} \in H^q(X, \widetilde{\mathbb{Z}})$, which is defined by $\widetilde{W_q} := \beta(w_{q-1})$, where β is the Bockstein homomorphism associated with the short exact triple $0 \longrightarrow \widetilde{\mathbb{Z}} \xrightarrow{2} \widetilde{\mathbb{Z}} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$, and w_{q-1} is the $(q-1)$st mod 2 Stiefel-Whitney class. The second modification is that the linear dependency current, $\mathbf{LD}(\alpha)$, the Stiefel-Whitney spark, $S_q(\alpha)$, and the currents, L and R, in Theorem 4.2 are all actually $\mathcal{O}_F$-twisted currents on X, (see [18]). Locally an *$\mathcal{O}_F$-twisted current*, $\widetilde{S}$, on X can be expressed in the form, $\widetilde{S} = S \times [f]$, where S is a locally defined current on X and $[f] \in \mathcal{O}_F$ is the orientation class of a local frame for F. Consequently equation (16) also needs to be modified by tensoring it on the right hand side by the orientation class $[f]$.

Proof of Theorem 4.2. The proof is a refinement of the proof of Theorem 4.10 of [18]. First, by the comparison formula (15) for Stiefel-Whitney sparks, if equation (21) is true for one $\mathbf{LD}$-atomic collection, α, then it is true for all such collections. Therefore we may assume that an $\mathbf{LD}$-atomic collection, α, of m sections of F has been chosen so that the collection, (β, α), of $m+1$ sections of F is also $\mathbf{LD}$-atomic, for some choice of section, β. Furthermore, by Corollary 4.28 of [12], we may assume that the divisor, $\mathrm{Div}(\widehat{(\beta,\alpha)})$, of the induced section, $\widehat{(\beta,\alpha)}$, of $\mathrm{Hom}(NS(\underline{\mathbf{R}}^{m+1}), F) \to S(\underline{\mathbf{R}}^{m+1})$ is a locally rectifiable current on $S(\underline{\mathbf{R}}^{m+1})$. Let $t : S(\underline{\mathbf{R}}^{m+1}) \to \mathbf{R}$ correspond to the coordinate function whose zero set is the equatorial sphere, $S(\underline{\mathbf{R}}^{m}) \subset S(\underline{\mathbf{R}}^{m+1})$. Then $\frac{t}{|t|}\,\mathrm{Div}(\widehat{(\beta,\alpha)})$ is a well defined rectifiable current on $S(\underline{\mathbf{R}}^{m+1})$.

It is straightforward (c.f. Lemma 4.12 of [18]) to calculate that the potential, $\frac{t}{|t|}\sigma(\widehat{(\beta,\alpha)})$, satisfies the current equation,

(22)
$$d\big(\tfrac{t}{|t|}\sigma(\widehat{(\beta,\alpha)})\big) \;=\; 2\,i_*(\sigma(\widehat{\alpha})) \;-\; \tfrac{t}{|t|}\,\mathrm{Div}(\widehat{(\beta,\alpha)}) \;+\; \tfrac{t}{|t|}\;\chi(\Omega_{\mathrm{Hom}(NS(\underline{\mathbb{R}}^{m+1}),F)}),$$

on $S(\underline{\mathbb{R}}^{m+1})$.

Let $\pi : S(\underline{\mathbb{R}}^{m+1}) \to X$ denote the projection map to the base manifold, X. The locally rectifiable current, R, is defined to be the current pushforward,

$$R \ := \ (-1)^{m-1}\, \pi_*\big(\tfrac{1}{2}\tfrac{t}{|t|}\,\mathrm{Div}(\widehat{(\beta,\alpha)})\big) \qquad \text{on } X, \tag{23}$$

and the L^1_{loc} current, L, is defined to be the fibre integral,

$$L \ := \ (-1)^{m-1}\, \pi_*\big(\tfrac{1}{2}\tfrac{t}{|t|}\sigma(\widehat{(\beta,\alpha)})\big) \qquad \text{on } X. \tag{24}$$

Equation (21) now follows by taking $\frac{1}{2}\pi_*$ of equation (22) and appealing to Theorem 4.10 of [18], which states that the mod 2 reduction of the rectifiable current, R, given by equation (23) represents $w_{q-1}(F)$. ☐

It has not proved feasible to obtain an explicit local formula for the L^1_{loc} form, L, of equation (24) for a general **LD**-atomic collection of sections, (β,α). However, in the special case that (β,α) is an *orthonormal* collection of sections, Theorem 4.3 below gives an explicit local formula for L. This formula will be used in Section 5.

The formula for L is in terms of the following local data. Let α and β be as in the proof of Theorem 4.2 and let $f = (f',\beta,\alpha)^t$ be a local orthonormal frame for F whose last $(m+1)$ legs are (β,α), so that f' has length $q-2$. Observe that the matrix of (β,α) with respect to this frame is $(0\,|\,\mathrm{Id}_{m+1})$. Let ω and Ω denote the gauge and curvature matricies of the connection D_F on F with respect to the local frame, f. Let $\omega^{(q-1)}$ be the row vector of length $q-2$ defined by $\omega^{(q-1)} \ = \ (\omega_{q-1,1},\ldots,\omega_{q-1,q-2})$. Finally define a $(q-2)\times(q-2)$ matrix, $\Omega^{(q-1)}$, by

$$\Omega^{(q-1)}_{ij} \ = \ \Omega_{ij} \ + \ \sum_{k=q}^{n} \omega_{ik}\wedge\omega_{kj}. \tag{25}$$

Theorem 4.3. *Let (β,α) be a collection of $(1+m)$ orthonormal sections of F defined over an open subset of X and let $f = (f',\beta,\alpha)^t$ be a local orthonormal frame for F as above. Suppose that $q = n-m+1 = 2q_0+1$ is odd. Then the L^1_{loc} form, L, which appears in Theorem 4.2 and which is defined by equation (24) is given by the local formula*

$$L \ = \ -\big(\tfrac{-1}{4\pi}\big)^{q_0}\tfrac{1}{q_0!}\ \sum_{k=1}^{q_0}(-1)^k\tbinom{q_0}{k}\frac{2^{2k}}{\binom{2k}{k}}\Big\{(f')^t\Omega^{(q-1)}f'\Big\}^{q_0-k}\big(\omega^{(q-1)}f'\big)^{2k-1}. \tag{26}$$

The proof of Theorem 4.3 is given at the end of Section 7.

5. Representing Stiefel-Whitney Classes by Pairs of L^1_{LOC} Forms

The fundamental local current equations (13) and (21) relate Stiefel-Whitney sparks, which are canonically defined L^1_{loc} forms, to locally integrally flat currents which represent the Stiefel-Whitney classes. In fact, using results of Allendoerfer and Eells we will show that the pair of L^1_{loc} forms, $(2S_q(\alpha), L)$, in equation (21) in some sense represents the mod 2 Stiefel-Whitney class, $w_{q-1}(F)$, and that the pair of L^1_{loc} forms, $(0, S_q(\alpha))$, in equation (13) represents the integer Stiefel-Whitney class, $W_q(F)$.

In [1], Allendoerfer and Eells represented mod 2 and integer cohomology classes by certain equivalence classes of pairs of smooth differential forms with singularities, and in [8] Eells derived canonical pairs of forms with singularities whose equivalence classes represent the mod 2 and integer Stiefel-Whitney classes. The Stiefel-Whitney sparks and the forms of Eells are both closely related to the Chern-Euler form and the spherical Chern potential.

The aim of this section is to describe the precise relationship between Eells' and our approaches. Eells pairs of forms with singularities are defined using a special class of collections of sections of a real vector bundle, $F \to X$, which are orthonormal except on a closed nowhere dense subset of X. We will prove, for those **LD**-atomic collections of sections which satisfy Eells orthonormality condition, that the pairs of L^1_{loc} forms, $(2S_q(\alpha), L)$ and $(0, S_q(\alpha))$, of equations (13) and (21) are equal as differential forms to Eells' pairs of forms with singularities. Consequently the Stiefel-Whitney spark, $S_q(\alpha)$, and its counterpart, L, can be regarded as very natural generalizations of Eells' pairs of forms with singularities. In particular our pairs of forms are defined in terms of arbitrary **LD**-atomic collections of sections, rather than Eell's "almost everywhere orthonormal" collections of sections.

We begin by reviewing how Allendoerfer and Eells [1] represent $\mathbf{Z}$ and $\mathbf{Z}_2$ cohomology classes by equivalence classes of pairs of forms with singularities. In [1] degree r integer cohomology classes of X are represented using equivalence classes of $(\mathbf{Z}, r)$-pairs of forms. A *$(\mathbf{Z}, r)$-pair of forms with singularities*, (θ, ω), on an N-manifold, X, consists of (1) a smooth $(r-1)$-form, ω, defined on the complement of a smooth polyhedron, $e(\omega)$, of dimension $\leq N - r$ in X and (2) a smooth r-form, θ, defined on the complement of a smooth subpolyhedron, $e(\theta)$, of $e(\omega)$ of dimension $\leq N - r - 1$ in X such that (3) $d\omega = \theta$ in $X \sim e(\omega)$ and (4) for any smooth integral chain, c, such that $\mathrm{spt}(c) \cap e(\theta) = \emptyset$ and $\mathrm{spt}(\partial c) \cap e(\omega) = \emptyset$, the *residue* of the pair (θ, ω) relative to c,

$$h(\theta,\omega)(c) \;:=\; \int_c \theta \;-\; \int_{\partial c} \omega, \tag{27}$$

is an integer. Two $(\mathbf{Z}, r)$-pairs are said to be *equivalent* if their residues are equal, whenever both are defined. The main theorem of [1] states that the set of equivalence classes of $(\mathbf{Z}, r)$-pairs equipped with the exterior derivative operator $d(\theta, \omega) := (0, \theta)$ forms a complex whose cohomology is canonically isomorphic to the singular cohomology, $H^*(X, \mathbf{Z})$, of X. (The isomorphism is given by equation (27).)

On the other hand, Federer [9] proved that the complex of locally integrally flat currents can also be used to calculate $\mathbf{Z}$-cohomology. These two methods of representing $\mathbf{Z}$-cohomology can be related as follows. For each $\mathbf{Z}$-cohomology class there is a d-closed $(\mathbf{Z}, r)$-pair, (θ, ω), such that ω and θ are both *currents* on X and, furthermore, there is a d-closed locally integrally flat current, $\mathbf{D}$, on X such that

$$d\omega \;=\; \theta \;-\; \mathbf{D} \qquad \text{as currents on } X. \tag{28}$$

The mapping which send the equivalence class of the $(\mathbf{Z}, r)$-pair, (θ, ω), to the integrally flat current, $\mathbf{D}$, provides a natural isomorphism between the cohomology of the complexes of Allendoerfer-Eells and Federer.

Similar results hold for $\mathbf{Z}_2$-cohomology, provided that equivalent $(\mathbf{Z}, r)$-pairs are required to have the same residues mod 2, and that one works with the complex of mod 2 locally integrally flat currents.

With this background equation (13) and Theorem 4.2 easily imply the following

Proposition 5.1. *Let q be odd. There are* $\mathbf{LD}$*-atomic collections, α, of sections of $F \to X$ such that*
(1) the pair, $(0, -S_q(\alpha))$, is a $(\mathbf{Z}, q)$-pair of forms with singularities on X whose equivalence class represents the integer Stiefel-Whitney class, $W_q(F)$. That is, if c is any smooth integral q-chain with $\mathrm{spt}(\partial c) \cap \mathrm{singspt}(S_q(\alpha)) = \emptyset$*, then*

$$W_q(F)\ (c) \;=\; -\int_{\partial c} S_q(\alpha), \tag{29}$$

and
(2) the pair, $(2S_q(\alpha), L)$, is a $(\mathbf{Z}, q-1)$-pair of forms with singularities on X whose mod 2 equivalence class represents the mod 2 Stiefel-Whitney class, $w_{q-1}(F)$. That is, if c is any smooth integral $(q-1)$-chain with $\mathrm{spt}(c) \cap \mathrm{singspt}(S_q(\alpha)) = \emptyset$

and $\mathrm{spt}(\partial c) \cap \mathrm{singspt}(L) = \emptyset$*, then*

$$w_{q-1}(F)\ (c) \ \equiv \ \int_c 2S_q(\alpha) \ - \int_{\partial c} L \qquad \textit{mod 2.} \tag{30}$$

Eells constructs pairs of forms with singularities whose equivalence classes represent $w_{q-1}(F)$ and $W_q(F)$ as follows. Let $\mathrm{St}_{n-r}(F) \to X$ denote the Stiefel bundle of orthonormal $(n-r)$-frames of $F \to X$. Recall that there is a natural fibration, $\pi : \mathrm{St}_{n-r+1}(F) \to \mathrm{St}_{n-r}(F)$, whose fibres are $(r-1)$-spheres. Endow F with a metric connection, D_F. Using the gauge and curvature of the connection Eells constructs smooth r forms, $\Omega^{(r)}$ and $\Phi^{(r)}$, on $\mathrm{St}_{n-r}(F)$ such that

$$d\Phi^{(r-1)} \ = \ \pi^*\Omega^{(r)} \qquad \text{on } \mathrm{St}_{n-r+1}(F), \tag{31}$$

and $\Omega^{(r)} = 0$ for odd r.

Let q be odd and let α be a smooth orthonormal $(n-q+1)$-frame of F defined over the complement of a smooth $(N-q)$-polyhedron, $e(\alpha)$, in X, i.e. $\alpha : X \sim e(\alpha) \to \mathrm{St}_{n-q+1}(F)$. Eells proved that the qth integer Stiefel-Whitney class, W_q, is represented by the equivalence class of the $(\mathbf{Z}, q)$-pair, $(0, \alpha^*\Phi^{(q-1)})$. This $(\mathbf{Z}, q)$-pair is related to the qth Stiefel-Whitney spark as follows.

Proposition 5.2. *Let α be an* **LD***-atomic collection of $n-q+1$ sections of F which is orthonormal off of* $\mathrm{spt}(\mathbf{LD}(\alpha))$*. Then*

$$-S_q(\alpha) \ = \ \alpha^*\Phi^{(q-1)} \qquad \textit{as } L^1_{loc} \textit{ forms on } X. \tag{32}$$

Once again let q be odd. Eells represents the mod 2 Stiefel-Whitney class, $w_{q-1}(F)$, as follows. Let α and β be such that $\alpha : X \sim e(\alpha) \to \mathrm{St}_{n-q+1}(F)$ and $(\alpha, \beta) : X \sim e(\alpha,\beta) \to \mathrm{St}_{n-q+2}(F)$, where $e(\alpha)$ and $e(\alpha, \beta)$ are smooth polyhedra of dimensions $\leq N-q$ and $N-q+1$ respectively. Then $w_{q-1}(F)$ is represented by the equivalence class of the $(\mathbf{Z}, q-1)$-pair, $(\alpha^*\Omega^{(q-1)}\ ,\ (\alpha,\beta)^*\Phi^{(q-2)})$.

Proposition 5.3. *Let α be a collection of $n-q+1$ sections of F and β a section of F such that α and (α, β) are both* **LD***-atomic and are orthonormal off of their linear dependency sets. Then*

$$2S_q(\alpha) \ = \ \alpha^*\Omega^{(q-1)} \qquad \textit{and} \qquad L \ = \ (\alpha,\beta)^*\Phi^{(q-2)} \qquad \textit{in } L^1_{loc}, \tag{33}$$

where L is defined by equation (24).

Remark. There is a close analogy between Propositions 5.2, 5.3 and Theorem 5.13 of [17] which relates Cheeger-Simons' construction of the pth Chern differential character to the construction of the pth Chern differential character given in

[17] in terms of the pth Chern spark. Taken together these results suggest that there should be some sort of uniqueness theorem for certain canonical functorial L^1_{loc} solutions, S, of current equations of the form, $dS = \Omega - \mathbf{D}$, where Ω is a Chern-Weil form and where $\mathbf{D}$ is a linear dependency current, or more generally a degeneracy current of a bundle map.

Proof of Propositions 5.2 and 5.3. The proof simply entails specializing the formula for $S_q(\alpha)$ given in Theorem 4.1 to the case that α is an *orthonormal* collection of sections, (i.e. choosing $a = (0 \,|\, \mathrm{Id_m})$), and checking that the resulting formula for $S_q(\alpha)$ is equal to Eell's local expression for the forms, $-\Phi^{(q-1)}$ and $2\Omega^{(q-1)}$. Similarly it is easy to check that the formula for L given in Theorem 4.3 is equal to Eells local expression for the form, $\Phi^{(q-2)}$. The details are left to the reader.

$\square$

6. The Stiefel-Whitney Differential Character

The aim of this section is to define the qth Stiefel-Whitney differential character in terms of the qth Stiefel-Whitney spark, $S_q(\alpha)$, and to examine its properties. In particular we will show that the qth Stiefel-Whitney differential character is essentially the very familiar $(q-1)$st mod 2 Stiefel-Whitney class, a fact that was no doubt known by Cheeger and Simons.

Differential characters, which are refinements of characteristic classes and are invariants of a vector bundle with connection, were first introduced in the seminal papers of Cheeger [3] and Cheeger and Simons [4]. Cheeger and Simons defined an $\mathbf{R}/\mathbf{Z}$-valued *differential character* of degree ℓ on X to be a homomorphism, f, from the space of smooth singular ℓ-dimensional cycles on X to $\mathbf{R}/\mathbf{Z}$, with the property that there is a smooth d-closed $(\ell+1)$-form, ω, on X so that, for any $(\ell+1)$-chain, C,

$$f(\partial C) \equiv \int_C \omega \mod \mathbf{Z}. \tag{34}$$

The space of degree ℓ differential characters on X is denoted by $\widehat{H}^\ell(X, \mathbf{R}/\mathbf{Z})$.

Cheeger-Simons proved the fundamental *uniqueness theorem for differential characters* which states that the Weil homomorphism naturally factors through the space of differential characters. More precisely they proved the following result. Let $F \to X$ be a rank n real vector bundle with metric connection, D_F. Let $u \in H^*(X, \mathbf{Z})$ be an integer cohomology class and suppose that the image of u in $H^*(X, \mathbf{R})$ under the usual coefficient homomorphism is equal to the cohomology class of a characteristic form, $\phi(\Omega_F)$, of the connection, D_F. Then there is a unique functorial differential character, f_u, called the *u-differential character* such that, if S is any real cochain whose restriction to cycles agrees mod $\mathbf{Z}$ with the differential character, f_u, then the coboundary, δS, of S is of the form, $\delta S = \phi(\Omega_F) - c$, where c is an integer cochain which represents the class, u.

In particular the *Euler differential character*, $\widehat{\chi}(D_F)$, *of a metric connection*, D_F, is the unique functorial differential character associated with the integer Euler class and the Chern-Euler form, $\chi(\Omega_F)$, of the connection. In [19] we proved that the Euler differential character, $\widehat{\chi}(D_F)$, can be defined by the formula

$$\widehat{\chi}(D_F)\,(Z) \;=\; \int_Z \sigma(\mu) \quad \operatorname{mod} \mathbf{Z}, \tag{35}$$

where Z is a smooth singular $(n-1)$-cycle in X and where $\sigma(\mu)$ is the Euler spark of any atomic section, μ, of F whose zero set does not intersect the support of the cycle, Z.

Let q be odd. The *qth Stiefel-Whitney differential character*, $\widehat{W}_q(F)$, of a vector bundle, $F \to X$, is defined to be the unique differential character associated with the qth integer Stiefel-Whitney class, $W_q(F) \in H^q(X, \mathbf{Z})$, and the characteristic form 0. Since the characteristic form associated with $\widehat{W}_q(F)$ is zero we immediately conclude that $\widehat{W}_q(F)$ simply depends on F and not on a choice of connection on F. The Stiefel-Whitney differential character can be obtained from the Stiefel-Whitney spark by the formula

$$\widehat{W}_q(F)(Z) \;=\; \int_Z S_q(\alpha) \qquad \operatorname{mod} \mathbf{Z}, \tag{36}$$

where Z is a smooth singular $(q-1)$-cycle in X and where $S_q(\alpha)$ is the Stiefel-Whitney spark of any $\mathbf{LD}$-atomic collection of sections, α, of F whose linear dependency set does not intersect the support of the cycle, Z.

As in the case of the Euler differential character, the proof of equation (36) follows from the comparison formula (15) for Stiefel-Whitney sparks and from the fundamental equation (13).

Since the exterior derivative of $S_q(\alpha)$ is $-\mathbf{LD}(\alpha)$, which is an *integrally* flat current, the Stiefel-Whitney differential character actually defines an $\mathbf{R}/\mathbf{Z}$-*cohomology* *class* (see Cor 2.4 of [4]),

$$\widehat{W}_q(F) \in H^{q-1}(X, \mathbf{R}/\mathbf{Z}) \subset \widehat{H}^{q-1}(X, \mathbf{R}/\mathbf{Z}). \tag{37}$$

In addition, Theorem 4.2 easily implies that the range of $\widehat{W}_q(F)$ is $\mathbf{Z}_2 \cong \frac{1}{2}\mathbf{Z}/\mathbf{Z} \subset \mathbf{R}/\mathbf{Z}$.

Let $\psi : \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ be the homomorphism defined by $\psi([n]) = [\frac{n}{2}]$, and let $\psi^* : H^*(X, \mathbf{Z}_2) \to H^*(X, \mathbf{R}/\mathbf{Z})$ be the induced coefficient homomorphism on cohomology. The following result is a straightforward consequence of Theorem 4.2.

Proposition 6.1. *The Stiefel-Whitney differential character, $\widehat{W}_q(F)$, is related to the $(q-1)$st mod 2 Stiefel-Whitney class, $w_{q-1}(F)$, by the formula*

$$\widehat{W}_q(F) = \psi^*(w_{q-1}(F)) \qquad \text{in } H^{q-1}(X, \mathbf{R}/\mathbf{Z}) \subset \widehat{H}^{q-1}(X, \mathbf{R}/\mathbf{Z}). \tag{38}$$

7. Proof of Theorems 4.1 and 4.3

The proof of Theorem 4.1 is simply a calculation of the fibre integral

$$S_q(\alpha) = \tfrac{(-1)^{m-1}}{2} \int_{S(\mathbf{R}^m)} \sigma(\widehat{\alpha}), \tag{39}$$

where the L^1_{loc} form, $\sigma(\widehat{\alpha})$, on $S(\underline{\mathbf{R}}^m) \times X$ is the Euler spark of the induced section, $\widehat{\alpha}$, of $\text{Hom}(N, F)$.

Chern's formula for the spherical potential, σ, of a bundle, $V \to X$, depends on the parity of the rank of V. We are concerned with the spherical potential of the bundle, $H := \text{Hom}(N, F)$, whose rank is $n = \text{Rank}\, F$. Consequently we divide the proof of the theorem into two cases. The Even Case is the case that n, m are both even and the Odd Case is the case that n, m are both odd. (Recall that $n \equiv m \bmod 2$, since $q = n - m + 1$ is odd.)

7.1. **Proof of the Even Case.** Let ϵ be a local orthonormal frame for H and define a local function, $u : S(\underline{\mathbf{R}}^m) \to \mathbf{R}^n$, by $\widehat{\alpha} = u\epsilon$. Let $n = 2n_0$. In the even case Chern's formula for the spark, $\sigma(\widehat{\alpha})$, is

$$\sigma(\widehat{\alpha})\ \lambda(\epsilon) = \tfrac{1}{\pi^{n_0}} \sum_{p=0}^{n_0-1} C_{p,n_0} \frac{(u\epsilon)(Du\epsilon)^{2p+1}}{|u|^{2p+2}} (\epsilon^t \Omega_H \epsilon)^{n_0-p-1}, \tag{40}$$

where

$$C_{p,n_0} = (-1)^{n_0-p} \frac{p!}{(n_0-p-1)!(2p+1)!2^{2n_0-2p-1}}. \tag{41}$$

Let t be the coordinate on $\underline{\mathbf{R}}^m$. Then te is a global frame for N and $\epsilon = f \otimes (te)^*$ is a local orthonormal frame for H. Consequently, $u = ta$ and $\omega_H = \omega_F$. Let $m = 2m_0$ and note that $q_0 = n_0 - m_0$. Substituting these expressions into equation (40) and isolating the terms of degree $m-1$ in the fibre differentials, dt_i, yields the formula

$$S_q(\alpha)\ \lambda(f) \ = \ -\tfrac{1}{2\pi^{n_0}} \sum_{r=0}^{q_0} C_{m_0-1+r,n_0} \ \binom{m-1+2r}{2r} \tag{42}$$

$$(f^t \Omega_F f)^{q_0-r} \int_{|t|=1} \frac{(taf)(dtaf)^{m-1}(tDaf)^{2r}}{< taa^t, t >^{m_0+r}}, \tag{43}$$

where $<,>$ denotes the standard inner product on $\underline{\mathbf{R}}^m$.

Next note that $(dtaf)^m = m!\ dt_1 \wedge \cdots \wedge dt_m (a_1 f) \ldots (a_m f)$. This enables us to factor out the volume form, $\Theta(t) = t\frac{\partial}{\partial t} \llcorner dt_1 \wedge \cdots \wedge dt_m$, for the sphere as

$$(taf)(dtaf)^{m-1} \ = \ (m-1)!\ \Theta(t)\ (a_1 f) \ldots (a_m f). \tag{44}$$

Next we define

$$I_r \ := \int_{|t|=1} \frac{(tDaf)^{2r}}{< taa^t, t >^{m_0+r}} \ \ \Theta(t) \qquad \text{for} \ \ r \geq 0. \tag{45}$$

Substituting equation (44) into equation (42) yields

$$S_q(\alpha)\ \lambda(f) \ = \ -\tfrac{1}{2\pi^{n_0}}\ (a_1 f) \ldots (a_m f) \ \sum_{r=0}^{q_0} K_r\ (f^t \Omega_F f)^{q_0-r}\ I_r, \tag{46}$$

where $K_r := C_{m_0-1+r,n_0}\ (m-1)!\ \binom{m-1+2r}{2r}$.

Lemma 7.1.

$$I_r \ = \ \frac{\binom{2r}{r} \operatorname{Vol}(S^{m-1})}{\binom{m_0+r-1}{r} 2^{2r}} \ \frac{\{f^t Da^t (aa^t)^{-1} Daf\}^r}{\sqrt{\det(aa^t)}}. \tag{47}$$

In particular

$$I_0 \ = \ \frac{\operatorname{Vol}(S^{m-1})}{\sqrt{\det(aa^t)}}. \tag{48}$$

Given the Lemma the proof of the even case of the Theorem is completed by substituting equation (47) into equation (46) and noting that $\operatorname{Vol}(S^{m-1}) = \frac{2\pi^{m_0}}{(m_0-1)!}$.

PROOF OF LEMMA 7.1. Define $m \times m$ matrices, P and Q, by $P = aa^t \geq 0$ and $Q = (Daf)(Daf)^t$. Then $(tDaf)^2 = < tQ\,,\, t >$ and so, by equation (45),

$$I_r \;=\; \int_{|t|=1} \frac{< tQ, t >^r}{< tP, t >^{m_0+r}}\;\Theta(t). \tag{49}$$

We will prove the Lemma by showing, for any $m \times m$ matrices $P \geq 0$ and $Q = vv^t$ where v is a column vector, that

$$I_r \;=\; \frac{\binom{2r}{r}\,\mathrm{Vol}(S^{m-1})}{\binom{m_0+r-1}{r}\;2^{2r}}\;\frac{\{\mathrm{Tr}(P^{-1}Q)\}^r}{\sqrt{\det P}}. \tag{50}$$

We prove equation (50) by induction on r. The base case $r = 0$ of equation (50) follows by making the change of variables, $s = t\sqrt{P}$, and noting that $\Theta(s) = \det(\sqrt{P})\;\Theta(t)$, so that

$$I_0 \;=\; \frac{1}{\sqrt{\det P}}\int_{|sP^{-1/2}|=1}\frac{\Theta(s)}{|s|^m} \;=\; \frac{1}{\sqrt{\det P}}\int_{|s|=1}\frac{\Theta(s)}{|s|^m} \;=\; \frac{\mathrm{Vol}(S^{m-1})}{\sqrt{\det P}}. \tag{51}$$

The second equality in equation (51) follows by applying Stokes Theorem to the region between the sphere and the ellipsoid and noting that $\frac{\Theta(s)}{|s|^m}$ is a closed form on $\mathbb{R}^m \sim \{0\}$.

The induction step ($r > 0$) proceeds as follows. With abuse of notation let

$$Q\tfrac{\partial}{\partial P} \;:=\; \sum_{i,j=1}^{m} Q_{ij}\,\tfrac{\partial}{\partial P_{ij}}. \tag{52}$$

The induction formula

$$I_r \;=\; \tfrac{-1}{m_0+r-1}\;Q\tfrac{\partial}{\partial P}(I_{r-1}) \tag{53}$$

enables us to express I_r in terms of I_0 and so, by equation (51), we obtain the formula

$$I_r \;=\; (-1)^r\;\mathrm{Vol}(S^{m-1})\;\tfrac{(m_0-1)!}{(m_0+r-1)!}\;\left(Q\tfrac{\partial}{\partial P}\right)^r\;\frac{1}{\sqrt{\det P}}. \tag{54}$$

Lemma 7.1 now follows from equation (57) of Lemma 7.2 below. $\square$

Lemma 7.2. *Let P and Q be $m \times m$ matrices so that $P \geq 0$ and $Q = vv^t$, where v is a column vector. Then, for each integer $r \geq 0$,*

$$Q\tfrac{\partial}{\partial P}\ \mathrm{Tr}(P^{-1}Q) \ = \ -\{\mathrm{Tr}(P^{-1}Q)\}^2, \tag{55}$$

$$Q\tfrac{\partial}{\partial P}\left(\frac{\{\mathrm{Tr}(P^{-1}Q)\}^r}{\sqrt{\det P}}\right) \ = \ -\tfrac{2r+1}{2}\ \frac{\{\mathrm{Tr}(P^{-1}Q)\}^{r+1}}{\sqrt{\det P}}, \qquad \textit{and} \tag{56}$$

$$(Q\tfrac{\partial}{\partial P})^r\ \frac{1}{\sqrt{\det P}} \ = \ \tfrac{(-1)^r(2r)!}{2^{2r}r!}\ \frac{\{\mathrm{Tr}(P^{-1}Q)\}^r}{\sqrt{\det P}}. \tag{57}$$

Proof of Lemma 7.2. First note that $\frac{\partial}{\partial P_{ij}}(\det P) = \widetilde{P}_{ij}$, where $\widetilde{P}_{ij}$ denotes the (i,j)th cofactor of P. Therefore

$$Q\tfrac{\partial}{\partial P}(\det P) \ = \ \det(P)\ \mathrm{Tr}(P^{-1}Q). \tag{58}$$

Equation (56) now follows immediately from equation (55). Next note that equation (57) follows by repeated application of equation (56). So it suffices to prove equation (55).

To that end choose an orthogonal matrix, R, so that $v = R\underline{v}$ where $\underline{v} = (\lambda, 0, \ldots, 0)^t$. Let $\underline{P} = R^tPR$ and $\underline{Q} = R^tQR = \underline{v}\,\underline{v}^t$. Then, by the Chain Rule,

$$(Q\tfrac{\partial}{\partial P})\,\mathrm{Tr}(P^{-1}Q) \ = \ (\underline{Q}\tfrac{\partial}{\partial \underline{P}})\ \mathrm{Tr}(\underline{P}^{-1}\underline{Q}). \tag{59}$$

Finally $(\underline{Q}\frac{\partial}{\partial \underline{P}})\,\mathrm{Tr}(\underline{P}^{-1}\underline{Q}) = \underline{Q}_{11}^2\frac{\partial}{\partial \underline{P}_{11}}(\underline{P}_{11}^{-1}) = -\ \underline{Q}_{11}^2\underline{P}_{11}^{-2} = -\ \{\mathrm{Tr}(P^{-1}Q)\}^2$, as required, since $\underline{P}_{11}^{-1} = \frac{\widetilde{\underline{P}}_{11}}{\det \underline{P}}$ and $\frac{\partial}{\partial \underline{P}_{11}}(\widetilde{\underline{P}}_{11}) = 0$. □

7.2. **Proof of the Odd Case.** We present two proofs of the odd case of Theorem 4.1.

First Proof of the Odd Case. The first proof follows along the same lines as the even case. Let $n = 2n_0 + 1$ and $m = 2m_0 + 1$. In the odd case Chern's formula for the spark, $\sigma(\widehat{\alpha})$, is

$$\sigma(\widehat{\alpha})\ \lambda(\epsilon) \ = \ \tfrac{-1}{2\ n_0!}(\tfrac{-1}{4\pi})^{n_0}\frac{u\epsilon}{|u|}\left(\epsilon^t\Omega_H\epsilon \ - \ \frac{(Du\epsilon)^2}{|u|^2}\right)^{n_0}. \tag{60}$$

As in the derivation of equation (46) above it follows that,

$$S_q(\alpha)\ \lambda(f) \ = \ \tfrac{(-1)^{q_0+1}(m-1)!}{4(4\pi)^{n_0}n_0!}\ (a_1f)\ldots(a_mf)\ \sum_{r=0}^{q_0}K_r\ (f^t\Omega_Ff)^{q_0-r}\ J_r, \tag{61}$$

where

$$J_r \ = \ \int_{|t|=1}\frac{<tQ,t>^r}{<tP,t>^{\frac{m}{2}+r}}\ \Theta(t), \tag{62}$$

and

$$K_r \;=\; (-1)^r \binom{n_0}{m_0+r} \binom{2m_0+2r}{2m_0}. \tag{63}$$

Next, as in the derivation of equation (54), if follows that

$$J_r \;=\; \frac{(-1)^r \operatorname{Vol}(S^{m-1})\,(m+2r)\binom{m_0+r}{m_0}}{(r+1)!\binom{m+2r}{m-1}\binom{2r+1}{r}} \left(Q\frac{\partial}{\partial P}\right)^r \frac{1}{\sqrt{\det P}}, \tag{64}$$

and so, by Lemma 7.1,

$$J_r \;=\; \frac{\operatorname{Vol}(S^{m-1})\binom{m_0+r}{m_0}}{\binom{2m_0+2r}{2m_0}} \frac{\{\operatorname{Tr}(P^{-1}Q)\}^r}{\sqrt{\det P}} \qquad \text{for } \; r \geq 0. \tag{65}$$

The first proof of the odd case of the Theorem is completed by substituting equation (65) into equation (61) and noting that, when m is odd, $\operatorname{Vol}(S^{m-1}) = 2(4\pi)^{m_0}\frac{(m_0)!}{(m-1)!}$. □

Second Proof of the Odd Case. The second proof of the odd case of Theorem 4.1 is a simple corollary of the even case of the Theorem and the following Proposition.

□

Proposition 7.3. *Let $\alpha : \underline{\mathbf{R}}^m \to F$ be a bundle map, where $m \equiv n \mod 2$, let $q = n - m + 1$ and let $\operatorname{Id} : \underline{\mathbf{R}} \to \underline{\mathbf{R}}$ denote the identity map. Then*

$$S_q(\alpha \oplus \operatorname{Id}) \;=\; S_q(\alpha) \qquad \text{on } \; X, \tag{66}$$

where $\alpha \oplus \operatorname{Id} : \underline{\mathbf{R}}^m \oplus \underline{\mathbf{R}} \to F \oplus \underline{\mathbf{R}}$.

Proof of Proposition 7.3. First note that it suffices to prove the proposition in the case that $\operatorname{rank} F$ is odd. The case that $\operatorname{rank} F$ is even follows from this case since, if $\operatorname{rank} F$ is even, then $S(\alpha \oplus \operatorname{Id}) = S(\alpha \oplus \operatorname{Id} \oplus \operatorname{Id}) = S(\alpha)$, where the second equality is verified by inspecting the formula (16) we derived for $S(\alpha)$ in the even case.

To prove the proposition in the case that $\operatorname{rank} F$ is odd we need to recall Harvey and Lawson's definition of the Euler Spark, $\sigma(\nu)$, of an atomic section ν of an odd rank bundle, V, [14]. The Euler spark, $\sigma(\nu)$, is defined in terms of the spark, $\sigma(\nu \oplus \operatorname{Id})$, of the section, $\nu \oplus \operatorname{Id}$, of the *even* rank bundle, $V \oplus \mathbf{R} \to X \times \mathbf{R}$. Let $\pi : X \times \mathbf{R} \to X$ be the natural projection. The definition of the Euler spark, $\sigma(\nu)$, is motivated by the fact that ν is atomic if and only if $\nu \oplus \operatorname{Id}$ is atomic and furthermore that in this case $\operatorname{Div}(\nu) = \pi_* \operatorname{Div}(\nu \oplus \operatorname{Id})$ on X (see [18]). Also observe that the spark $\sigma(\nu \oplus \mathrm{Id})$ is integrable over the fibres of π. Consequently it is natural to define

$$\sigma(\nu) \;:=\; -\pi_*\sigma(\nu \oplus \mathrm{Id}). \tag{67}$$

Harvey and Lawson proved that $d\sigma(\nu) = -\operatorname{Div}(\nu)$ and derived a local formula for $\sigma(\nu)$ which agrees with Chern's formula (18).

Consequently, in the odd case the Stiefel-Whitney Spark, $S_q(\alpha)$, can be described as follows. Let $\widehat{\alpha} \oplus \mathrm{Id}$ be the induced section of $\mathrm{Hom}(N, F) \oplus \mathbf{R} \to S(\underline{\mathbf{R}}^m) \times \mathbf{R}$ and let $\rho : S(\underline{\mathbf{R}}^m) \times \mathbf{R} \to X$ be the projection map. Then, by equations (1) and (67),

$$S_q(\alpha) \;=\; \tfrac{(-1)^m}{2}\rho_*(\sigma(\hat{\alpha} \oplus \mathrm{Id})). \tag{68}$$

On the other hand, by equation (1)

$$S_q(\alpha \oplus \mathrm{Id}) \;=\; \tfrac{(-1)^m}{2}\eta_*(\sigma(\widehat{\alpha \oplus \mathrm{Id}})), \tag{69}$$

where $\eta : S(\underline{\mathbf{R}}^m \oplus \underline{\mathbf{R}}) \to X$. Let $\varphi : S(\underline{\mathbf{R}}^m) \times \mathbf{R} \to S(\underline{\mathbf{R}}^m \oplus \underline{\mathbf{R}}) \sim S(\underline{\mathbf{R}})$ be the orientation preserving diffeomorphism given by the restriction of the radial projection map on $\underline{\mathbf{R}}^m \oplus \underline{\mathbf{R}}$. Then the required equation, $S_q(\alpha \oplus \mathrm{Id}) = S_q(\alpha)$, is an immediate corollary of the following Lemma and the fact that $\rho = \eta \circ \varphi$. □

Lemma 7.4.

$$\varphi_*(\sigma(\widehat{\alpha} \oplus \mathrm{Id})) \;=\; \sigma(\widehat{\alpha \oplus \mathrm{Id}}) \tag{70}$$

on $S(\underline{\mathbf{R}}^m \oplus \underline{\mathbf{R}}) \sim S(\underline{\mathbf{R}})$.

Proof of Lemma 7.4. Let N_m denote the normal line bundle to $S(\underline{\mathbf{R}}^m) \subset \mathbf{R}^m$. First note that the Euler Spark is functorial under orientation preserving diffeomorphisms [14] and so

$$\varphi_*\,\sigma(\varphi^*(\widehat{\alpha \oplus \mathrm{Id}})) \;=\; \sigma(\widehat{\alpha \oplus \mathrm{Id}}), \tag{71}$$

where $\varphi^*(\widehat{\alpha \oplus \mathrm{Id}})$ is the pullback of $\widehat{\alpha \oplus \mathrm{Id}}$ to the bundle $\varphi^* \mathrm{Hom}(N_{m+1}, F \oplus \underline{\mathbf{R}}) \to S(\underline{\mathbf{R}}^m) \times \mathbf{R}$. So to prove the Lemma it suffices to verify that

$$\sigma(\widehat{\alpha} \oplus \mathrm{Id}) \;=\; \sigma(\varphi^*(\widehat{\alpha \oplus \mathrm{Id}})). \tag{72}$$

Now the Euler Spark of a section of a bundle is equal to the Euler Spark of the image of the section under a metric, orientation and connection preserving bundle isomorphism. Therefore equation (72) follows from Lemma 7.5 below. This completes the proof of Lemma 7.4. □

Lemma 7.5. *There is a metric, orientation and connection preserving bundle isomorphism*

$$\Psi : \mathrm{Hom}(N_m, F) \oplus \underline{\mathbf{R}} \longrightarrow \varphi^* \mathrm{Hom}(N_{m+1}, F \oplus \underline{\mathbf{R}}) \tag{73}$$

over $S(\underline{\mathbf{R}}^m) \times \mathbf{R}$ *so that*

$$\Psi(\widehat{\alpha} \oplus \mathrm{Id}) \;=\; \varphi^*(\widehat{\alpha \oplus \mathrm{Id}}). \tag{74}$$

Proof. First, $\varphi^* \mathrm{Hom}(N_{m+1}, F \oplus \underline{\mathbf{R}}) \;=\; \mathrm{Hom}(\varphi^* N_{m+1}, F) \oplus \mathrm{Hom}(\varphi^* N_{m+1}, \underline{\mathbf{R}})$. Let $\beta : \mathrm{Hom}(N_{m+1}, F) \to \mathrm{Hom}(N_m, F)$ be the bundle isomorphism induced by restricting the natural projection $\underline{\mathbf{R}}^m \oplus \underline{\mathbf{R}} \to \underline{\mathbf{R}}^m$ to N_{m+1}, and let $\gamma : \underline{\mathbf{R}} \to \mathrm{Hom}(\varphi^* N_{m+1}, \underline{\mathbf{R}})$ be the bundle isomorphism induced by the canonical trivialization $\varphi^* N_{m+1} \cong \underline{\mathbf{R}}$. Then the bundle isomorphism Ψ is defined by $\Psi = \beta^{-1} \oplus \gamma$. Formula (74) follows easily. $\square$

7.3. **Proof of Theorem 4.3.** Since the proof of the theorem is similar to that of Theorem 4.1 we simply outline the proof in the *even case* that $n = 2n_0$ and $m = 2m_0$.

Let (t, x) denote coordinates on $\mathbf{R} \times \mathbf{R}^m$, let Ω' denote the top left $(q-2) \times (q-2)$ block of the curvature matrix, Ω, of D_F, and let ω'' denote the bottom left $m \times (q-2)$ block of the connection 1-form, ω. Then, in analogy with equations (45) and (46), we have that

$$L\ \lambda(f') \;=\; \frac{1}{2\pi^{n_0}} \sum_{r=0}^{q_0-1} K_r \left\{(f')^t \Omega' f'\right\}^{q_0-1-r} I_r, \tag{75}$$

where $K_r = C_{m_0+r, n_0} \binom{m+2r+1}{m}\, m!$, (where C_{p,n_0} is defined by equation (41)), and where

$$I_r \;=\; \sum_{j=0}^{2r+1} \binom{2r+1}{j} \left(\omega^{(q-1)} f'\right)^{2r+1-j} J_{rj}, \tag{76}$$

with

$$J_{rj} \;:=\; \int_{S(\mathbf{R}^{m+1})} \frac{t^{2r+2-j}}{|t|} \; \frac{(x\omega'' f')^j}{(|t|^2 + |x|^2)^{m_0+r+1}}\; \Theta(t, x). \tag{77}$$

To calculate the integral J_{rj} we use the change of variables

$$\psi(t, x) \;=\; \left(t\, , \sqrt{|x|^2 - t^2}\; \frac{x}{|x|}\right) \tag{78}$$

to transform J_{rj} into an integral over the cylinder, $(-1,1) \times S(\mathbf{R}^m)$,

$$(79) \qquad J_{rj} = \int_{-1}^{1} \frac{t^{2r+2-j}}{|t|} \left(\sqrt{1-t^2}\right)^{m+j-2} dt \int_{S(\mathbf{R}^m)} (x\omega'' f')^j \; \Theta(x).$$

The t-integral in equation (79) is zero for j odd. Setting $j = 2j_0$ and applying equations (49) and (50) it follows that

$$(80) \qquad I_r = \sum_{j_0=0}^{r} \gamma_{rj_0} \left(\omega^{(q-1)} f'\right)^{2(r-j_0)+1} \{(f')^t (\omega'')^t \omega'' f'\}^{j_0},$$

where

$$(81) \qquad \gamma_{rj_0} = \frac{2\pi^{m_0} \binom{2r+1}{2j_0} \binom{2j_0}{j_0}}{2^{2j_0} (m_0+j_0)(m_0-1)! \binom{m_0+r}{r-j_0} \binom{m_0+j_0-1}{j_0}}.$$

The theorem now follows by substituting equation (80) into equation (75), noting that $\Omega^{(q-1)} = \Omega' - (\omega'')^t \omega''$, and simplifying. $\square$

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