EULER AND PONTRJAGIN CURRENTS OF A SECTION OF A COMPACTIFIED REAL BUNDLE

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The canonical family of Thom forms $\boldsymbol{\tau}_s$ for $0 < s < \infty$ constructed by Harvey and Lawson on an oriented real vector bundle $V \to X$ with metric connection D_V is shown to have an extension to the bundle of real projective spaces, $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$, which compactifies V in the fibre directions. The current limit as $r \to \infty$ and $s \to 0$ of the smooth transgression formula $\boldsymbol{\tau}_r - \boldsymbol{\tau}_s = d\boldsymbol{\sigma}_{r,s}$ is the current equation

$$\widetilde{\chi}(D_V) \ + \ [\mathbb{P}(V)] \ \mathrm{Res} \ - [X] \ = \ d\boldsymbol{\sigma}_{\infty,0} \qquad \text{on } \mathbb{P}(\underline{\mathbb{R}} \oplus V).$$

Here $\boldsymbol{\sigma}_{\infty,0}$ is an L^1_{loc} current on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ and $[X]$ is the current of integration over the zero section of V. If the rank of V is even $\widetilde{\chi}(D_V)$ is the twisted extension to $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ of the Chern–Euler form of the connection D_V on V, $[\mathbb{P}(V)]$ integrates densities over the nonorientable submanifold $\mathbb{P}(V)$ of $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ and Res is a smooth twisted form on $\mathbb{P}(V)$. If the rank of V is odd, $\widetilde{\chi}(D_V) \equiv 0$, $[\mathbb{P}(V)]$ is the current of integration over $\mathbb{P}(V)$ and Res is a smooth form on $\mathbb{P}(V)$.

This construction can be pulled back to X by an atomic section $\nu : X \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ satisfying an appropriate orientability hypothesis to obtain a current equation on X relating the zero and pole sets of ν to the topology of the bundle. Applications to the differential topology of maps into spheres and real projective spaces are discussed.

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Introduction.

In this paper we study meromorphic sections of real vector bundles. The main aim is to derive local (i.e. current) equations relating the behaviour of the section in a neighbourhood of its zero and pole sets to the topology of the bundle. Our results are analogous to those obtained in the study of meromorphic sections of complex vector bundles (see [Z]). The results build on the work of Harvey, Lawson and Semmes (see [HL], [HS]) who studied smooth sections of vector bundles (i.e. no pole set). The formulae obtained can be applied to study the differential topology of maps into spheres and real projective spaces. For instance we will prove the following result about maps, such as the Gauss map, from an n–dimensional manifold to the n–sphere.

Theorem. *Let X be a compact oriented n–dimensional manifold with boundary, ∂X, and let $f : X \to S^n$ be a continuous map into the n–sphere. Fix a north pole, south pole and an equator, S^{n-1}, on the n–sphere and let N, S and E be the subsets of X which are mapped by f to the north pole, south pole and equator. Suppose that N and S are isolated point sets and (for the sake of simplicity) that E is a smooth oriented connected $(n-1)$–dimensional submanifold of X and that $(N \cup S \cup E) \cap \partial X = \varnothing$. Then*

$$\#S \ + \ \#N \ - \ 2\,\mathrm{Deg}(f\,\big|_E) \ = \ W_+ \ - \ W_-.$$

Here $\#S$ and $\#N$ are the number of north and south poles counted with multiplicity and $\mathrm{Deg}(f\,\big|_E)$ *is the degree (or sheeting number) of the induced mapping $f : E \to S^{n-1}$. The winding numbers $W_\pm$ are defined as follows. Let $H_\pm$ denote the complement of the north and south poles in the upper and lower hemispheres of S^n. Since the equator disconnects S^n we can write $\partial X = \partial X_+ \sqcup \partial X_-$ where $\partial X_\pm$ is the union of those components of ∂X which are mapped into $H_\pm$. Let $p_\pm : H_\pm \to S^{n-1}$ be the natural projection onto the equator. Then we define $W_\pm$ to be the sum of the degrees of the induced maps $p_\pm \circ f$ from the various connected components of $\partial X_\pm$ to S^{n-1}.*

The main result of the paper can be described as follows. Let V be an oriented real vector bundle of rank n over a manifold X. We compactify V in the vertical directions by embedding it in the bundle of real projective spaces $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$. This embedding is defined by sending $v \in V_x$ to the line through $(1, v) \in \underline{\mathbb{R}} \oplus V_x$. Recall that $\mathbb{P}(\underline{\mathbb{R}} \oplus V) = V \sqcup \mathbb{P}(V)$ and that, for odd rank bundles W, $\mathbb{P}(W) \to X$ is nonorientable. Consequently smooth forms twisted by the orientation bundle of $\mathbb{P}(W)$ — rather than bona fide forms on $\mathbb{P}(W)$ — define currents by integration against smooth test forms. We study smooth sections ν of the bundle $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$. The **zero set** Z of ν is the inverse image of the zero section of V and the **pole set** P is the inverse image of spatial infinity, $\mathbb{P}(V)$. Note that generically Z has codimension n and P has codimension 1 in X.

Harvey and Semmes defined the **zero divisor**, $\mathrm{Div}_0(\nu)$, of an **atomic** section ν of the vector bundle $V \to X$ (see [HS] or Definition 1.4 below). The zero divisor is a codimension n current on X which is supported on the zero set Z of ν and encodes the multiplicity of vanishing of ν on Z. If 0 is a regular value of ν then ν is atomic and $\mathrm{Div}_0(\nu) = [Z]$ is the current of integration over the (suitably oriented) submanifold Z of X. Atomicity is a weak measure theoretic condition which ensures the existence of a zero divisor. Similarly we can define the pole divisor, $\mathrm{Div}_\infty(\nu)$, of a section $\nu : X \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ (see Definition 3.15). Note however that since the even dimensional real projective spaces are nonorientable the pole divisor will not always be a bona fide current on X. Instead of acting on test forms it may be defined to act on a certain space of twisted test forms on X. Consequently the pole divisor may have support on the nonorientable components of the pole set.

Next we endow V with a Riemannian metric and a metric compatible connection D_V. The Euler class of V is represented by the **Chern–Euler form**, $\chi(D_V)$, which is a $2n$–form constructed from the curvature of the connection with aid of the Pfaffian of a skew symmetric matrix. Note that when the rank of V is odd $\chi(D_V) \equiv 0$.

Roughly speaking the **main result** can be stated as follows (see Theorems 4.10 and

5.21 for precise statements). Let ν be an atomic section of $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$. Then, under appropriate orientability assumptions on X and ν, we have the current equation

$$(\star) \qquad \chi(D_V) \; + \; \mathrm{Div}_\infty(\nu)\, \nu^* \,\mathrm{Res} \; - \; \mathrm{Div}_0(\nu) \; = \; d\sigma \qquad \text{on } X.$$

Here σ is a canonically defined L^1_{loc} current on X which is smooth away from $Z \cup P$ and $\nu^*\,\mathrm{Res}$ is the pullback to X of a (possibly twisted) residue form Res on $\mathbb{P}(V)$. The restriction of Res to the fibres of $\mathbb{P}(V) \to X$ is the normalized volume form (or density) on real projective space.

It is instructive to compare this result to the results obtained in [Z] concerning meromorphic sections of complex vector bundles. Suppose that V is a complex line bundle over a 2–dimensional manifold X. In the complex case we compactify V by adding a single point at infinity, that is by embedding V in $\mathbb{P}(\underline{\mathbb{C}} \oplus V)$. The zero and pole sets of a section $\nu : X \to \mathbb{P}(\underline{\mathbb{C}} \oplus V)$ consist of isolated points, $Z = \{z_j\}$ and $P = \{p_i\}$. Then, if X is oriented, we have an equation between dimension zero currents of the form

$$(\star\star) \qquad c_1(D_V) \; + \; \sum_i n_i[p_i] \; - \; \sum_j m_j[z_j] \; = \; dT \qquad \text{on } X,$$

where $c_1(D_V)$ is the first Chern form of a connection D_V on V, T is an L^1_{loc} 1–form on X and n_i and m_j are integers. On the other hand we could consider the underlying real rank 2 bundle of V. In this case V is compactified by adding a circle at infinity, that is by embedding V in $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$. The zero set of a section $\nu : X \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ still consists of isolated points, $Z = \{z_j\}$. However the pole set P is (roughly speaking) a union of curves in X. Suppose (for the sake of simplicity) that P is a connected oriented 1–dimensional submanifold of X. Choose a metric compatible connection D_V on V. Then, under suitable orientability assumptions on X and ν, we have the dimension zero current equation

$$(\star\star\star) \qquad \chi(D_V) \; + \; [P]\, \nu^* \,\mathrm{Res} \; - \; \sum_j m_j[z_j] \; = \; d\sigma \qquad \text{on } X,$$

where σ, $\chi(D_V)$ and the 1–form Res are as above.

The **argument principle** of complex analysis can be recovered from $(\star\star)$ by letting the currents in that equation act on the test function $[X]$ on a compact surface X with boundary ∂X. It can be stated as follows. Let $f : X \to \mathbb{CP}^1$ be a continuous map with isolated zeros and poles of finite multiplicity, none of which lie on ∂X. Then

$$\#\,\text{Zeros} \;-\; \#\,\text{Poles} \;=\; \text{Wind}(f, \partial X),$$

where the winding number, $\text{Wind}(f, \partial X)$, is the number of times the image of ∂X winds around $0 \in \mathbb{CP}^1$. Alternatively, the sum of the local changes in argument around each zero and pole is equal to the global change in argument around ∂X. Similarly, when V is the trivial real line bundle over $\mathbb{R}$, we can let the currents in $(\star\star\star)$ act on the test function $[a, b]$ to obtain the completely elementary signum principle.

The signum principle. *Let $f : [a, b] \to [-\infty, \infty]$ be a continuous function with isolated zeros and poles. Then the total change in signum over the interval $[a, b]$ is equal to the sum of the local changes in signum at each of the zeros and poles.*

The main result is proved as follows. Harvey and Lawson constructed a family of smooth Thom forms $\boldsymbol{\tau}_s$ $(0 < s < \infty)$ on V in terms of a background metric connection D_V (see Remark 1.10). In the even rank case their construction proceeds as follows. (The construction in the odd rank case is more subtle – see Definition 5.1.) Let $\underline{\mathbb{R}}$ denote the trivial line bundle and let $\mathbf{V}$ be the pullback of V over itself. The tautological bundle map $\boldsymbol{\alpha} : \underline{\mathbb{R}} \to \mathbf{V}$ over V can be used to pushforward the trivial connection d on $\underline{\mathbb{R}}$ to a singular connection $\overrightarrow{\mathbf{D}}$ defined on $\mathbf{V}$ over $V \sim X$. Then a canonical smooth family of connections $\overrightarrow{\mathbf{D}}_s$ is defined on $\mathbf{V}$ over V with the property that $\overrightarrow{\mathbf{D}}_s \to \overrightarrow{\mathbf{D}}$ as $s \to 0$ on $V \sim X$. The Thom forms are then defined to be the Chern–Euler forms $\boldsymbol{\tau}_s := \chi(\overrightarrow{\mathbf{D}}_s)$. Furthermore there are smooth transgression forms $\boldsymbol{\sigma}_{r,s}$ for $0 < s < r < \infty$ so that $\boldsymbol{\tau}_r - \boldsymbol{\tau}_s = d\boldsymbol{\sigma}_{r,s}$ on V.

In Theorem 2.5 we will prove that the family $\overrightarrow{\mathbf{D}}_s$ on $\mathbf{V}$ over V can be compactified by a natural family of connections $D_{U^\perp,s}$ on a bundle $U^\perp$ defined over all of $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$. The bundle $U^\perp$ is the orthogonal complement of the tautological line bundle $U \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$. This result is a refinement of [HL IV 1.63]. In particular this implies (see Lemma 2.4) that, in the even rank case, the Thom and transgression forms on V extend smoothly to all of $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ as suitably twisted forms and so define currents.

Next we show that $\boldsymbol{\sigma}_{r,s} \to \boldsymbol{\sigma}$ in $L^1_{\text{loc}}(\mathbb{P}(\underline{\mathbb{R}} \oplus V))$ as $r \to \infty$ and $s \to 0$, and that the current limits, $\lim\limits_{s\to 0} \boldsymbol{\tau}_s = [X]$ and $\lim\limits_{r\to\infty} \boldsymbol{\tau}_r = [\mathbb{P}(V)]\ \mathrm{Res}$, exist. Consequently $[\mathbb{P}(V)]\ \mathrm{Res} - [X] = d\boldsymbol{\sigma}$ on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$. Note that the form $\boldsymbol{\sigma}$ was discovered by Chern [C1]. The formula $(\star)$ on X is proved by pulling back the twisted forms $\boldsymbol{\tau}_s$ and $\boldsymbol{\sigma}_{r,s}$ to X via ν. Under suitable orientability assumptions on X and ν these define currents on X, and, provided that ν is atomic, we can take the current limit as before to obtain $(\star)$.

Contents.

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1. Characteristic currents at time zero.

In this section we review Harvey and Lawson's results [HL IV] concerning the characteristic current at time zero associated with an SO–invariant polynomial and a section of a real vector bundle. We begin by recalling the definition of the zero divisor of an atomic section of a vector bundle (c.f. [HS]). Throughout this section we assume that V is an oriented bundle over an oriented manifold.

Definition 1.1. [HS 1.1] Let X be a smooth manifold and let $y = (y_1, \ldots, y_m)$ denote coordinates on $\mathbb{R}^m$.

(1) A smooth function $u : X \to \mathbb{R}$ is called **atomic** if its zero set has Lebesgue measure zero.

(2) A smooth function $u : X \to \mathbb{R}^m$ $(m > 1)$ is called **atomic** if for each form $\frac{dy^I}{|y|^p}$ on $\mathbb{R}^m$, with $p = |I| \le m-1$, the pullback $u^*\left(\frac{dy^I}{|y|^p}\right)$ to X has an L^1_{loc} extension across the zero set Z of u. Also assume that u does not vanish identically in any connected component of its domain X.

Note. Harvey and Semmes [HS 1.2] prove that if u is atomic then the zero set Z has Lebesgue measure zero in X. Therefore the L^1_{loc} extension of $u^*\left(\frac{dy^I}{|y|^p}\right)$ across Z is unique.

Definition 1.2. [HS 2.5] Let $V \to X$ be a smooth real vector bundle of rank m. A smooth section μ of V is called **atomic** if for each choice of local frame e for V the function u defined by $\mu = ue$ is atomic.

Let

$$\theta := \sum_{k=1}^{m} (-1)^{k-1} \frac{y_k dy_1 \wedge \cdots \wedge \widehat{dy_k} \wedge \cdots \wedge dy_m}{|y|^m} \tag{1.3}$$

denote the **solid angle kernel** on $\mathbb{R}^m$, and let γ_m denote the volume of the unit sphere in $\mathbb{R}^m$. Note that if $u : X \to \mathbb{R}^m$ is atomic then $u^*(\theta) \in L^1_{\text{loc}}(X)$. The fundamental current equation $d\,(\gamma_m^{-1}\theta) = [0]$ on $\mathbb{R}^m$, where $[0]$ denotes the point mass at the origin, motivates the definition of divisor.

Definition 1.4. [HS 2.6] Let μ be an atomic section of a smooth oriented real rank m vector bundle V over a smooth oriented manifold X. Let e be a positively oriented local frame for V and define u by $\mu = ue$. Harvey and Semmes [HS 2.11] prove that the **divisor**

of μ is well defined by the current equation

$$\mathrm{Div}(\mu) := d(\gamma_m^{-1}\, u^*(\theta)) \qquad \text{on } X.$$

Note. The divisor, $\mathrm{Div}(\mu)$, is a codimension m d–closed current supported on the zero set Z of μ. If 0 is a regular value of μ, then $\mathrm{Div}(\mu) = [Z]$, where [Z] is the current of integration over the (suitably oriented) submanifold Z.

Next endow $V \to X$ with a Riemannian metric and metric compatible connection D_V. Let μ be a section of $V \to X$ and $Z = \mathrm{Zero}(\mu)$. Harvey and Lawson [HL IV.1.7] define the **Riemannian singular pushforward connection** associated with μ to be the smooth metric compatible connection

$$\overrightarrow{D}\nu := D_V\nu - \frac{\langle \nu, \mu\rangle}{|\mu|^2}\, D_V\mu + \frac{\langle D_V\mu, \nu\rangle}{|\mu|^2}\, \mu \qquad \text{on } V \text{ over } X \sim Z. \tag{1.5}$$

Here $\langle\ ,\ \rangle$ denotes the inner product on V and $|\mu|^2 = \langle \mu, \mu\rangle$.

The singular connection $\overrightarrow{D}$ is approximated by the smooth family of smooth metric compatible connections $\overrightarrow{D}_s$ $(0 < s \le \infty)$ on V defined by

$$\overrightarrow{D}_s\nu := D_V\nu - \chi_s\, \frac{\langle \nu, \mu\rangle}{|\mu|^2}\, D_V\mu + \chi_s\, \frac{\langle D_V\mu, \nu\rangle}{|\mu|^2}\, \mu. \tag{1.6}$$

Here $\chi_s := \chi(\frac{|\mu|^2}{s^2})$ where $\chi : [0, \infty] \to [0, 1]$ is any choice of approximate one (c.f. [HL I.4.1]). Because of its geometrical significance (see Theorem 2.5 below) we choose to work with the **real algebraic approximation mode** χ defined by $\chi(t) := 1 - \frac{1}{\sqrt{t+1}}$.

Let ϕ be any SO–invariant polynomial. The **characteristic current at time zero** on X is defined to be the current limit

$$\phi((\overrightarrow{D}_0)) := \lim_{s\to 0} \phi(\overrightarrow{D}_s),$$

whenever this limit exists.

Before summarizing Harvey and Lawson's results we recall the definition of the Pontrjagin and Chern–Euler forms. Let the rank of V be m and set $m = 2n$ if m is even and $m = 2n+1$ if m is odd. Recall that the Pontrjagin forms of D_V can be defined as follows. Let $e = (e_1, ..., e_m)$ be a positively oriented orthonormal local frame for V and let ω_V and Ω_V denote the skew symmetric local gauge and curvature matrices of the connection D_V with respect to the frame e. The j–th Pontrjagin form, $p_j(D_V)$, is the degree $4j$ part of the **total Pontrjagin form**, $p(D_V)$, defined by

$$p(D_V) = 1 + p_1(D_V) + ... + p_n(D_V) := \det(I + \tfrac{1}{2\pi}\Omega_V). \tag{1.7}$$

In the case that $m = 2n$ is even the Chern–Euler form of the connection D_V is defined with the aid of the Pfaffian of a skew symmetric martix as follows. Set $\lambda(e) := e_1 \wedge ... \wedge e_m$. Then, regarding e as a column vector, $e^t \Omega_V e$ defines a global section of the bundle $\Lambda^2 V \otimes \Lambda^2 T^* X$ on X and $\lambda(e)$ defines a global section of the bundle $\Lambda^{2n}(V)$. So the **Chern–Euler form**, $\chi(D_V)$, of the metric compatible connection D_V is the globally defined $2n$–form on X defined by

$$\chi(D_V)\lambda(e) := \tfrac{1}{n!}\,(\tfrac{-1}{4\pi}\, e^t \Omega_V e)^n.$$

Recall that the $SO(2n+1)$–invariant polynomials are generated by $p_1, ..., p_n$ while the $SO(2n)$–invariant polynomials are generated by $p_1, ..., p_{n-1}, \chi$ and that $p_n = \chi^2$. The invariant polynomials of primary interest to us are those of the form $\phi = \chi\, \psi(p_1, ..., p_n)$ where $\psi(p_1, ..., p_n)$ is any polynomial in $p_1, ..., p_n$.

The ϕ–**transgression form** of a family of metric compatible connections D_s on V is defined by

$$T_{r,s} := \int_s^r \phi(\tfrac{\partial \omega_s}{\partial s}\,;\, \Omega_s)\, ds, \tag{1.8}$$

where ω_s and Ω_s are the local gauge and curvature matrices of the connection D_s with respect to a frame e as above and $\phi(A; B) := \frac{d}{dt}\big|_{t=0}\, \phi(B + tA)$. The fundamental transgression formula is $\phi(D_r) - \phi(D_s) = dT_{r,s}$.

Theorem 1.9. *[HL IV] Let μ be an atomic section of $V \to X$ and let T_s denote the smooth transgression form $T_{\infty,s}$ of the family $\overrightarrow{D}_s$. Then*

(1) The transgression forms T_s converge as L^1_{loc}–currents on X to the transgression current $T := \lim_{s\to 0} T_s$.

(2) Therefore the characteristic current $\phi((\overrightarrow{D}_0)) := \lim_{s\to 0} \phi(\overrightarrow{D}_s)$ exists on X and is given by

$$\phi((\overrightarrow{D}_0)) = \phi(D_F) + dT.$$

Note that $\phi((\overrightarrow{D}_0))$ is a d–closed flat current on X which represents the ϕ–characteristic class of V.

(3) Let $\phi(\overrightarrow{D}_0)$ denote the smooth form $\phi(\overrightarrow{D})$ on $X \sim Z$. Then $\phi(\overrightarrow{D}_0)$ extends across Z to a d–closed L^1_{loc} form on all of X, also denoted by $\phi(\overrightarrow{D}_0)$. Note that $\phi(\overrightarrow{D}_0)$ is called the L^1_{loc} part of the characteristic current $\phi((\overrightarrow{D}_0))$.

(4) The characteristic current $\phi((\overrightarrow{D}_0))$ is given by

$$\phi((\overrightarrow{D}_0)) = \phi(\overrightarrow{D}_0) + \mathrm{Res}_0(\phi)\ \mathrm{Div}_0(\alpha),$$

where the residue form $\mathrm{Res}_0(\phi)$ is a d–closed smooth form on X. Note that the residue form is independent of the choice of section μ and that the divisor is independent of the choice of invariant polynomial ϕ.

(5) The current equation

$$\phi(D_V) - \phi(\overrightarrow{D}_0) - \mathrm{Res}_0(\phi)\ \mathrm{Div}_0(\alpha) = dT \qquad \text{on } X$$

is the limit as $s \to 0$ of the smooth transgression formula

$$\phi(D_V) - \phi(\overrightarrow{D}_s) = dT_s \qquad \text{on } X.$$

(6) For each SO–invariant polynomial ϕ there exists an SO–invariant polynomial Φ so that

$$\mathrm{Res}_0(\phi) = \Phi(\Omega_V).$$

In particular, when the rank of V is odd, $\mathrm{Res}_0(\phi)$, being an odd degree form, is zero.

(7) If the rank of V is even and ψ is any SO–invariant polynomial then

$$\mathrm{Res}_0(\chi\psi) = \psi(\Omega_V).$$

Remark 1.10. Suppose that the rank of V is even. Let $\mathbf{V}$ be the pullback of V over itself and let $\mathbf{D_V}$ be the pullback to $\mathbf{V}$ of D_V. Let $\overrightarrow{\mathbf{D}}_s$ $(0 < s < \infty)$ be the approximating family of connections associated with the tautological section $c : V \to \mathbf{V}$. Then Harvey and Lawson show that the smooth forms $\boldsymbol{\tau}_s := \chi(\overrightarrow{\mathbf{D}}_s)$ on V are a family of **Thom forms** associated with the metric compatible connection D_V on V. In particular

(1) $d\boldsymbol{\tau}_s = 0$.

(2) $\int_{V_x} \boldsymbol{\tau}_s = 1$.

(3) $\boldsymbol{\tau}_s \big|_X = \chi(D_V)$.

(4) $\boldsymbol{\tau}_s = (\frac{1}{s})^* \boldsymbol{\tau}_1$ where $\frac{1}{s}$ acts by scalar multiplication on the fibres of V.

(5) $\lim\limits_{s\to 0} \boldsymbol{\tau}_s = [X]$ on V.

(6) The current equation

$$\chi(\mathbf{D_V}) - [X] = d\boldsymbol{\sigma} \qquad \text{on } V$$

is the current limit as $s \to 0$ of the smooth transgression formula

$$\chi(\mathbf{D_V}) - \boldsymbol{\tau}_s = d\boldsymbol{\sigma}_s.$$

2. The universal compactification.

Let V be an oriented Riemannian vector bundle over a manifold X. Unless otherwise stated we make no orientability assumptions on X. In this section we compactify the bundle $V \to X$ in the vertical directions by embedding it in the bundle of real projective spaces $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$ and study smooth sections $\nu : X \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$. On the complement of a certain pole set P the section ν defines a section μ of V. Suppose that V has even rank $m = 2n$. Let ϕ be an SO–invariant polynomial of the form $\phi = \chi\,\psi(p_1, ...p_n)$. The main aim of this section is to show that the smooth characteristic and transgression forms $\phi(\overrightarrow{D}_s)$ and $T_{r,s}$ of the section μ on $X \sim P$ extend smoothly to all of X. Note however that since the even dimensional real projective spaces are nonorientable these extensions will not be bona fide forms on X. Rather they will be smooth forms twisted by the pullback to X by ν of the orientation bundle of the bundle of tangent spaces to the fibres of $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$.

Let $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$ denote the bundle of real projective spaces whose fibre over a point $x \in X$ is the set of all one dimensional real subspaces of $\underline{\mathbb{R}} \oplus V$. The natural embedding

$$\begin{aligned} V &\hookrightarrow \mathbb{P}(\underline{\mathbb{R}} \oplus V) \\ v &\mapsto \ [1, v], \end{aligned} \tag{2.1}$$

provides a compactification of the bundle $V \to X$ in the vertical directions. The total space of the projectivized bundle $\mathbb{P}(V) \to X$ can be regarded as a codimension 1 submanifold of $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ by the inclusion

$$\begin{aligned} \mathbb{P}(V) &\hookrightarrow \mathbb{P}(\underline{\mathbb{R}} \oplus V) \\ [v] &\mapsto \ [0, v]. \end{aligned}$$

Note that $\mathbb{P}(\underline{\mathbb{R}} \oplus V) = V \sqcup \mathbb{P}(V)$. Also recall that X embeds in $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ as the image of the zero section of $V \subset \mathbb{P}(\underline{\mathbb{R}} \oplus V)$.

Let $Z = \nu^{-1}(X)$ denote the **zero set** and $P = \nu^{-1}(\mathbb{P}(V))$ the **pole set** of a section ν of $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$. Note that $Z \cap P = \varnothing$. Let $\overrightarrow{D}_s$ be the smooth family of smooth connections on V over $X \sim P$ which approximates the singular pushforward connection associasted with the induced section $\mu : X \sim P \to V$ (see (1.6)).

We will prove that the characteristic and transgression forms $\phi(\overrightarrow{D}_s)$ and $T_{r,s}$ on $X \sim P$ extend smoothly across the pole set P by constructing a smooth family of smooth metric compatible connections $D_{U^\perp,s}$ on a bundle $U^\perp \to X$ whose characteristic and transgression forms are smooth extensions to all of X of the characteristic and transgression forms of the family $\overrightarrow{D}_s$. The construction of the family $(D_{U^\perp,s}, U^\perp)$ is most easily described in the universal case.

Remark 2.2. The universal case. Let $\mathbf{V}$ be the pullback of V to $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ via the projection map $p : \mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$, and let $\mathbf{D_V}$ be the pullback of D_V to $\mathbf{V}$. The induced bundle $\mathbb{P}(\underline{\mathbb{R}} \oplus \mathbf{V}) \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ has a tautological or universal section $\boldsymbol{\nu}$ which sends a line $l \in \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ to itself. For any section ν of $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$ we have $\nu^*\mathbf{V} = V$ as bundles with connections, and $\nu^*(\boldsymbol{\nu}) = \nu$. So to prove results of a smooth nature on or over X which involve a section ν it often suffices to prove them on or over $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ for the universal section $\boldsymbol{\nu}$ and then pullback the result to X via ν.

Definition 2.3. Let V, W be isomorphic vector bundles. A family of connections $D_{V,s}$ on V is said to be **equivalent** to a family of connections $D_{W,s}$ on W $(0 < s < \infty)$ if there is a bundle isomorphism $\varphi : V \to W$ (independent of s) so that

$$D_{W,s} = \varphi \circ D_{V,s} \circ \varphi^{-1}$$

for all $s \in (0, \infty)$.

This concept is important because of the following fact.

Lemma 2.4. *Equivalent families of connections have equal characteristic and transgression forms.*

In the universal case the family $(D_{U^\perp,s}, U^\perp)$ is constructed over $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ as follows. Let $U \subset \underline{\mathbb{R}} \oplus \mathbf{V}$ be the tautological line bundle over $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ and let U_s $(0 < s < \infty)$ be the pullback of U over $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ by the flow

$$\begin{aligned} \Psi_s : \mathbb{P}(\underline{\mathbb{R}} \oplus V) &\longrightarrow \mathbb{P}(\underline{\mathbb{R}} \oplus V) \\ [a, b] &\longmapsto [sa, b]. \end{aligned}$$

Let ${U_s}^\perp$ be the orthogonal complement of U_s in $\underline{\mathbb{R}} \oplus \mathbf{V}$. We can induce a metric compatible connection $D_{{U_s}^\perp}$ on ${U_s}^\perp$ over $\mathbb{P}(\mathbb{R} \oplus V)$ by regarding ${U_s}^\perp$ as a subbundle of $\underline{\mathbb{R}} \oplus \mathbf{V}$ and defining

$$D_{{U_s}^\perp} = P_s \circ (d \oplus D_{\mathbf{V}}) \qquad \text{operating on sections of } {U_s}^\perp,$$

where $P_s : \underline{\mathbb{R}} \oplus \mathbf{V} \to {U_s}^\perp$ denotes orthogonal projection. Finally we will construct a natural smooth family of smooth isometries $\varphi_s : {U_s}^\perp \to U^\perp$ over $\mathbb{P}(\mathbb{R} \oplus V)$ and define the smooth family of smooth metric compatible connections $D_{U^\perp,s}$ on $U^\perp$ over $\mathbb{P}(\mathbb{R} \oplus V)$ by

$$D_{U^\perp,s} := \varphi_s \circ D_{{U_s}^\perp} \circ \varphi_s^{-1}.$$

The isometry φ_s is constructed as follows. Let $\Phi_s : \underline{\mathbb{R}} \oplus \mathbf{V} \to \underline{\mathbb{R}} \oplus \mathbf{V}$ $(0 < s < \infty)$ be the isometry which is the identity on ${U_s}^\perp \cap U^\perp = (U_s \oplus U)^\perp$ and is (roughly speaking) the rotation R_s of U_s onto U on $U_s \oplus U$. Then we define $\varphi_s := \Phi_s \big|_{{U_s}^\perp}$. More precisely the isometry R_s of $U_s \oplus U$ is defined as follows. When $s = 1$ define $R_1 = \mathrm{Id} : U \to U$. Also, over $\mathbb{P}(V) \cup X$, $U_s \oplus U = U$ and so we define $R_s = \mathrm{Id} : U \to U$ there. Finally for $s \neq 1$ over $V \sim X$ we orient the rank two bundle

$$U_s \oplus U = \underline{\mathbb{R}} \oplus U = \underline{\mathbb{R}} \oplus U_s$$

by choosing $\{(1,0)\,,\,(1,v) \in \underline{\mathbb{R}} \oplus \mathbf{V}_v\}$ to be a positively oriented basis over $v \in V \sim X$, and define R_s to be the unique rotation which takes U_s onto U through an acute angle. (U_s and U are never orthogonal.) Note that the maps Φ_s form a smooth family of smooth isometries of $\underline{\mathbb{R}} \oplus \mathbf{V}$.

The following theorem says that the family of connections $D_{U^\perp,s}$ on $U^\perp$ over all of $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ compactifies the universal pushforward family of connections $\overrightarrow{\mathbf{D}}_s$ on $\mathbf{V}$ over V.

Theorem 2.5. *In the real algebraic approximation mode the family* $(\overrightarrow{\mathbf{D}}_s, \mathbf{V})$ *(*$0 < s < \infty$*) is equivalent over* $V \subset \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ *to the family* $(D_{U^\perp,s}, U^\perp)$*. More specifically*

$$D_{U^\perp,s} = \gamma^{-1} \circ \overrightarrow{\mathbf{D}}_s \circ \gamma \qquad \textit{over } V$$

where $\gamma : \mathbf{V} \to U^{\perp} \subset \underline{\mathbb{R}} \oplus \mathbf{V}$ *is the isometry defined over* $v \in V$ *by*

$$\gamma(v) := (-\alpha^*(\alpha\alpha^* + 1)^{-\frac{1}{2}} v \ , \ (\alpha\alpha^* + 1)^{-\frac{1}{2}} v).$$

Here $\alpha : \underline{\mathbb{R}} \to \mathbf{V}$ *is the bundle map over* V *defined by* $\alpha(1) = c$, *where* $c : V \to \mathbf{V}$ *is the tautological section, and* $\alpha^* : \mathbf{V} \to \underline{\mathbb{R}}$ *is its adjoint.*

Proof. Let $\gamma_s : \mathbf{V} \to {U_s}^{\perp} \subset \underline{\mathbb{R}} \oplus \mathbf{V}$ be the isometry defined over $v \in V$ by

$$\gamma_s(v) := (-\alpha^*(\alpha\alpha^* + s^2)^{-\frac{1}{2}} v \ , \ s(\alpha\alpha^* + s^2)^{-\frac{1}{2}} v). \tag{2.6}$$

Harvey and Lawson [HL IV.1.63] prove that

$$D_{{U_s}^{\perp}} = \gamma_s \circ \overrightarrow{\mathbf{D}}_s \circ \gamma_s^{-1} \qquad \text{on } {U_s}^{\perp} \text{ over } V.$$

Note. Since the isometry $\gamma_s : \mathbf{V} \to {U_s}^{\perp}$ is s–dependent this fact is not sufficient to prove that the transgression forms $T_{r,s}$ of the family $\overrightarrow{\mathbf{D}}_s$ on V extend to all of $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ (c.f. Lemma 2.4).

So to prove the theorem it suffices to show that

$$\gamma = \varphi_s \circ \gamma_s \quad \text{on } \mathbf{V} \text{ over } V.$$

Now let $\Gamma_s : \underline{\mathbb{R}} \oplus \mathbf{V} \to \underline{\mathbb{R}} \oplus \mathbf{V}$ over V be the isometry which is the identity on $V \cap U^{\perp} = (\underline{\mathbb{R}} \oplus U)^{\perp}$ and is (just as in the definition of Φ_s) the rotation Q_s of $\underline{\mathbb{R}}$ onto U_s on the subbundle $\underline{\mathbb{R}} \oplus U_s = \underline{\mathbb{R}} \oplus U$.

Claim.

$$\gamma_s = \Gamma_s \big|_{\mathbf{V}} \qquad \textit{over } V \subset \mathbb{P}(\underline{\mathbb{R}} \oplus V).$$

Given the Claim the proof of the Theorem will be completed once we show that

$$\Gamma_1 = \Phi_s \circ \Gamma_s \quad \text{on } \underline{\mathbb{R}} \oplus \mathbf{V} \text{ over } V.$$

But this follows from the fact that $Q_1 = R_s \circ Q_s$ on $\underline{\mathbb{R}} \oplus \operatorname{Im}\alpha = \underline{\mathbb{R}} \oplus U$ over V. This can be seen by noting that if $s > 1$ (resp. $s < 1$) then Q_s is the rotation of $\underline{\mathbb{R}} \oplus \operatorname{Im}\alpha$ taking $\underline{\mathbb{R}}$ to U_s counterclockwise (through an acute angle) while R_s is the rotation which takes U_s to U counterclockwise (resp. clockwise). So $R_s \circ Q_s$ is the rotation taking $\underline{\mathbb{R}}$ to U counterclockwise and so equals Q_1. $\square$

Proof of Claim. First note that $\gamma_s = \mathrm{Id}$ on $\operatorname{Ker}\alpha^* = \mathbf{V} \cap U^{\perp}$. Secondly we examine γ_s on $\operatorname{Im}\alpha = (\operatorname{Ker}\alpha^*)^{\perp} = (\mathbf{V} \cap U^{\perp})^{\perp} \subset \mathbf{V}$. Now since $\gamma_s : \mathbf{V} \to {U_s}^{\perp}$, $\gamma_s : \mathbf{V} \cap U^{\perp} \to \mathbf{V} \cap U^{\perp}$ and γ_s is an isometry we can conclude that

$$\gamma_s(\operatorname{Im}\alpha) = {U_s}^{\perp} \cap (\mathbf{V} \cap U^{\perp})^{\perp} = {U_s}^{\perp} \cap (\underline{\mathbb{R}} \oplus U).$$

Now

$$\underline{\mathbb{R}} \oplus U = \underline{\mathbb{R}} \oplus \operatorname{Im}\alpha = U_s \oplus ({U_s}^{\perp} \cap (\underline{\mathbb{R}} \oplus U))$$

and since Q_s is a rotation of $\underline{\mathbb{R}} \oplus U$ taking $\underline{\mathbb{R}}$ onto U_s and $\operatorname{Im}\alpha$ is perpendicular to $\underline{\mathbb{R}}$ we also see that

$$Q_s(\operatorname{Im}\alpha) = {U_s}^{\perp} \cap (\underline{\mathbb{R}} \oplus U).$$

Since rank $(\operatorname{Im}\alpha) \leq 1$ and γ_s and Q_s are isometries this implies that $\gamma_s = \pm Q_s$ on $\operatorname{Im}\alpha$. Finally note that if e is a positively oriented orthonormal local frame for V and u is the local fibre coordinate for V defined by $\alpha(1) = v = ue$ then $\gamma_s : \operatorname{Im}\alpha \to \underline{\mathbb{R}} \oplus \mathbf{V}$ is given by

$$\gamma_s(ue) = \left(\frac{-\,|u|^2}{\sqrt{|u|^2 + s^2}} \;,\; \frac{s\,ue}{\sqrt{|u|^2 + s^2}} \right)$$

and so $\gamma_s = +Q_s$ on $\operatorname{Im}\alpha$ (draw a picture). $\square$

Combining Lemma 2.4 with Theorem 2.5 we conclude that in the universal case the characteristic and transgression forms $\phi(\overrightarrow{\mathbf{D}}_s)$ and $T_{r,s}$ extend smoothly from V to all of $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$. However if the rank of V is even and $\phi = \chi\,\psi(p_1, ...p_n)$ these extensions are actually **twisted** forms on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$. This follows from the fact that $U^{\perp} \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ is a nonorientable bundle. In order to describe this more carefully we suppose for now that $V \to X$ is a nonorientable bundle and recall some basic definitions and facts about the orientation line bundle $\mathcal{O}(V)$ and the $\mathcal{O}(V)$–twisted Chern–Euler form of V.

Let $\pi : V \to X$ be a real rank m vector bundle over an N dimensional manifold. The **bundle of orientations** of V, $Or(V) \to X$, is the principal $\mathbb{Z}_2$–bundle whose fibre over $x \in X$ consists of the two possible orientations of V_x. The **orientation bundle**, $\mathcal{O}(V) \to X$, of V is the associated real line bundle $\mathcal{O}(V) := Or(V) \times_\rho \mathbb{R}$ where $\rho : \mathbb{Z}_2 \to \mathrm{GL}(\mathbb{R})$ is the natural inclusion map. Note that we can regard $Or(V)$ as the S^0–bundle of $\mathcal{O}(V)$. An $\mathcal{O}(V)$–**twisted** k–**form** on X is defined to be a section of the bundle $\mathcal{O}(V) \otimes \Lambda^k(T^*X) \to X$. Set $\Omega^k(X, \mathcal{O}(V)) := \Gamma(\mathcal{O}(V) \otimes \Lambda^k(T^*X))$. An $\mathcal{O}(TX)$–twisted N–form on X is called a **density** on X. Recall that densities can be integrated over X.

The reason for our interest in twisted forms is that an $\mathcal{O}(TX)$–twisted k–form on X defines a current of degree k by wedging it with a degree $N-k$ test form and integrating the resulting density over X. More generally let $E \to X$ be any vector bundle. A smooth section η of E defines a continuous linear functional $\eta : \Gamma_{\mathrm{cpt}}(X, E^* \otimes \mathcal{O}(TX) \otimes \Lambda^N(T^*X)) \to \mathbb{R}$ by $\eta(\rho^* \otimes \omega) = \int\limits_X \rho^*(\eta)\,\omega$. So we define a distributional section of E to be a continuous linear functional on $\Gamma_{\mathrm{cpt}}(X, E^* \otimes \mathcal{O}(TX) \otimes \Lambda^N(T^*X))$. Let $\Gamma(E)'$ denote the distributional sections of E. Note that currents of degree k are distributional sections of $\mathcal{O}(TX) \otimes \Lambda^k(T^*X)$.

Remark 2.7. Locally we can describe an $\mathcal{O}(V)$–twisted k–form $\widetilde{\omega}$ as follows. Let e be a local frame for V and let $[e]$ be the local section of $\mathcal{O}(V) \to X$ defined by $[e](x) := [e_x]$ where $[e_x] \in Or(V) \subset \mathcal{O}(V)$ is the orientation class of e_x. Then we express $\widetilde{\omega}$ locally as $\widetilde{\omega} = [e] \otimes \omega$ where ω is a locally defined bona fide k–form on X.

Lemma 2.8.

(1) There is a natural isomorphism

$$\mathcal{O}(V) \cong \mathcal{O}(V^*)$$

defined locally in terms of a frame e for V by sending $[e]$ to $[e^*]$, where e^* is the dual frame of V^*.

(2)

$$\mathcal{O}(TV) \cong \pi^*(\mathcal{O}(V) \otimes \mathcal{O}(TX)) \qquad \text{on } V.$$

(3) Let $0 \to E \to F \to Q \to 0$ be a short exact sequence of vector bundles over X. Then

$$\mathcal{O}(F) \cong \mathcal{O}(E) \otimes \mathcal{O}(Q) \qquad \text{on } X.$$

Next we endow our nonorientable bundle V with a Riemannian metric and metric compatible connection D_V and suppose that V has even rank $m = 2n$. Let e be an orthonormal local frame for V and let ω_V and Ω_V be the corresponding gauge and curvature matrices. Let e^* denote the dual frame for V^* and set $\lambda(e^*) := e_1^* \wedge ... \wedge e_m^*$. Then, regarding e as a column vector,

$$e^t \Omega_V e \text{ defines a global section of the bundle } \Lambda^2 V \otimes \Lambda^2 T^* X \text{ on } X$$

and

$$[e] \otimes \lambda(e^*) \text{ defines a global section of the bundle } \mathcal{O}(V) \otimes \Lambda^{2n}(V^*).$$

So the $\mathcal{O}(V)$–**twisted Chern–Euler form** of the metric compatible connection D_V is defined by

$$\chi(D_V) := \tfrac{1}{n!}\, [e] \otimes \lambda(e^*) \left((\tfrac{-1}{4\pi}\, e^t \Omega_V e)^n \right) \ \in \Omega^{2n}(X, \mathcal{O}(V)). \tag{2.9}$$

Now let D_s be a family of metric compatible connections on V with local gauge ω_s and curvatute Ω_s. Just as above we define

$$\chi(\tfrac{\partial \omega_s}{\partial s}\, ; \Omega_s) := (\tfrac{-1}{4\pi})^n\, \tfrac{1}{(n-1)!}\, [e] \otimes \lambda(e^*) \left((e^t \tfrac{\partial \omega_s}{\partial s} e)(e^t \Omega_s e)^{n-1} \right) \ \in \Omega^{2n-1}(X, \mathcal{O}(V)).$$

Then the Chern–Euler $\mathcal{O}(V)$– **twisted transgression form** $\sigma_{r,s}$ of the family D_s is defined by

$$\sigma_{r,s} := \int_s^r \chi(\tfrac{\partial \omega_s}{\partial s}\, ; \Omega_s)\, ds. \tag{2.10}$$

Since $\mathcal{O}(V)$ is a flat bundle we can define $d : \Omega^k(X, \mathcal{O}(V)) \to \Omega^{k+1}(X, \mathcal{O}(V))$ locally by $d([e] \otimes \omega) := [e] \otimes d\omega$. Then we have the fundamental **transgression formula**

$$\chi(D_r) \; - \; \chi(D_s) \; = \; d\sigma_{r,s} \qquad \text{in } \Omega^{2n}(X, \mathcal{O}(V)).$$

More generally for invariant polynomials of the form $\phi = \chi\, \psi(p_1, ..., p_n)$ we have $\phi(D_V) \in \Omega^*(X, \mathcal{O}(V))$ by (1.7) and (2.9). Furthermore the $\mathcal{O}(V)$–twisted transgression form $T_{r,s}$ of the family D_s, which satisfies the transgression formula $\phi(D_r) - \phi(D_s) = dT_{r,s}$ in $\Omega^*(X, \mathcal{O}(V))$, is defined just as in (1.8). To see that $T_{r,s}$ is an $\mathcal{O}(V)$–twisted form note that $\phi(A;B) = \chi(A;B)\, \psi(B) + \chi(B)\, \psi(A;B)$.

Now we return to the study of the compactification $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$ of an oriented bundle $V \to X$.

Lemma 2.11. *Let $V \to X$ be an oriented Riemannian vector bundle over an oriented manifold. Let $L \subset V$ be the tautological line bundle over $\mathbb{P}(V)$ and let $L^\perp$ be its orthogonal complement.*

(1) Suppose that rank $V > 1$ is odd. Then

$$\mathcal{O}(L^*) \cong \mathcal{O}(L) \cong \mathcal{O}(L^\perp) \cong \mathcal{O}(T\mathbb{P}(V))$$

are nontrivial over $\mathbb{P}(V)$. Consequently $\mathcal{O}(L)$–twisted forms define currents on $\mathbb{P}(V)$.

(2) Suppose that rank of V is even. Then $\mathbb{P}(V)$ is orientable and

$$\mathcal{O}(L^*) \cong \mathcal{O}(L) \cong \mathcal{O}(L^\perp)$$

are nontrivial over $\mathbb{P}(V)$.

Definition 2.12. Let $U \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ be the tautological bundle. For the rest of the paper we set $\mathcal{O} := \mathcal{O}(U) \cong \mathcal{O}(U^\perp)$.

Proof of Lemma. First note that the mapping

$$\begin{aligned} p : \mathbb{P}(\underline{\mathbb{R}} \oplus V) \sim X &\longrightarrow \mathbb{P}(V) \\ [s, v] \qquad &\longmapsto [v] \end{aligned}$$

realizes $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \sim X$ as the total space of a real line bundle over $\mathbb{P}(V)$. In fact this line bundle is isomorphic to the dual L^* of the tautological line bundle L over $\mathbb{P}(V)$. Over a point $[v] \in \mathbb{P}(V)$ this isomorphism is given by the linear mapping

$$\begin{aligned} L^*_{[v]} &\longrightarrow (\mathbb{P}(\underline{\mathbb{R}} \oplus V) \sim X)_{[v]} \\ \beta &\longmapsto \quad [\beta(v), v]. \end{aligned} \tag{2.13}$$

So by Lemma 2.8(2) we can conclude that

$$\mathcal{O}(T(\mathbb{P}(\underline{\mathbb{R}} \oplus V) \sim X)) \cong \mathcal{O}(L^*) \otimes \mathcal{O}(T\mathbb{P}(V)).$$

The proof is completed by appealing to (1) and (3) of Lemma 2.8 and recalling that the odd (even) dimensional real projective spaces are orientable (nonorientable). $\square$

Finally we can combine Lemma 2.4 with Theorem 2.5 to obtain the following Corollary.

Corollary 2.14. *Let V be an oriented even rank $2n$ bundle over a manifold X and let ϕ be an SO–invariant polynomial of the form $\phi = \chi\, \psi(p_1, ...p_n)$.*
(1) Then the characteristic and transgression forms $\phi(\overrightarrow{\mathbf{D}}_s)$ and $T_{r,s}$ $(0 < s < r < \infty)$ of the family $(\overrightarrow{\mathbf{D}}_s, \mathbf{V})$ on V extend smoothly to all of $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ as the $\mathcal{O}$–twisted characteristic and transgression forms of the family $(D_{U^\perp,s}, U^\perp)$ over $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$. In particular if X is oriented they define smooth currents on $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$.
(2) Let ν be a smooth section of $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$ and let $P \subset X$ be its pole set. Then the characteristic and transgression forms $\phi(\overrightarrow{D}_s)$ and $T_{r,s}$ $(0 < s < r < \infty)$ of the family $(\overrightarrow{D}_s, V)$ on $X \sim P$ extend smoothly to all of X as the $\nu^*\mathcal{O}$–twisted characteristic and transgression forms of the family $(\nu^* D_{U^\perp,s}, \nu^* U^\perp)$ over X. In particular if $\nu^*\mathcal{O} \cong \mathcal{O}(TX)$ then $\phi(\overrightarrow{D}_s)$ and $T_{r,s}$ define smooth currents on X.

3. Divisors of atomic sections of a nonorientable bundle.

In this section we apply the work of Harvey and Semmes [HS] to define the notion of an atomic section of the compactification $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$ of an oriented real vector bundle $V \to X$ and define the zero and pole divisors of such a section. In order to do this we are forced to make sense of the divisor of a section of a nonorientable vector bundle. In [HS] orientability assumptions were imposed on the bundle and on the base manifold to ensure that the divisor of a section was a well defined current. We begin this section by investigating what happens if these orientation assumptions are relaxed. For example, let μ be a section of a rank n real vector bundle V over an n dimensional manifold X, and let x be an isolated zero of μ. Without any orientation assumptions on V or X we cannot define the multiplicity of vanishing of μ at x. However we can define the magnitude of vanishing by making arbitrary choices of orientation on V and X in a neighbourhood of x and computing the absolute value of the multiplicity of vanishing of a local coordinate expression for μ near x. One of the aims of this section is to define the zero divisor of an atomic section of an arbitrary real vector bundle as a distributional section of the appropriate space of twisted forms. The fact that we can only speak of the magnitude and not the multiplicity of vanishing is reflected in the fact that the divisor is forced to act on twisted forms.

Our first goal is to show how to define the divisor of an atomic section μ of a nonorientable bundle V over a nonorientable manifold X as a distributional section, $\mathrm{Div}(\mu) \in \Omega^m(X, \mathcal{O}(V))'$. In particular when $\mathcal{O}(V) = \mathcal{O}(TX)$ the divisor $\mathrm{Div}(\mu) \in \Omega^m(X, \mathcal{O}(TX))'$ is a current on X. In order to do this we need to recall the relationship between $\mathcal{O}(V)$–twisted forms on X and odd forms on the appropriate double cover.

Now we can regard the bundle of orientations $Or(V) \to X$ of V as a two sheeted covering space $p : \widetilde{X} \to X$. The pullback bundle $\widetilde{V} = p^*V$ on $\widetilde{X}$ is oriented by choosing the orientation on $\widetilde{V}_{\widetilde{x}} = V_{p(\widetilde{x})}$ to be the one determined by the point $\widetilde{x} \in \widetilde{X}$. The deck transformation

$g : \widetilde{X} \to \widetilde{X}$ lifts to a bundle map,

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{g_V} & \widetilde{V} \\ \downarrow & & \downarrow \\ \widetilde{X} & \xrightarrow{g} & \widetilde{X} \end{array}$$

which is orientation reversing on the fibres of $\widetilde{V}$. In the spirit of de Rham [R] a differential **form** ω on $\widetilde{X}$ is called **even** if $g^*\omega = \omega$ and **odd** if $g^*\omega = -\omega$. There is a 1–1 correspondence between forms on X and even forms on $\widetilde{X}$ given by the mapping

$$\begin{aligned} p^* : \Omega^*(X) &\longrightarrow \Omega^*_{\text{even}}(\widetilde{X}) \\ \omega \ &\longmapsto \ p^*\omega. \end{aligned}$$

On the other hand the odd forms on $\widetilde{X}$ are in 1–1 correspondence with the $\mathcal{O}(V)$–twisted forms on X. This correspondence,

$$\begin{aligned} \widetilde{p}^* : \Omega^*(X, \mathcal{O}(V)) &\longrightarrow \Omega^*_{\text{odd}}(\widetilde{X}) \\ \omega \ &\longmapsto \ \widetilde{\omega}, \end{aligned} \tag{3.1}$$

is defined as follows. In a local frame e for V write $\omega = [e] \otimes \psi$ where ψ is a bona fide form on X. The corresponding odd form $\widetilde{\omega}$ on $\widetilde{X}$ is defined by $\widetilde{\omega}_{[e]_x} := \psi_x$ and $\widetilde{\omega}_{g([e]_x)} := -\psi_x$. Conversely the $\mathcal{O}(V)$–twisted form ω on X corresponding to an odd form $\widetilde{\omega}$ on $\widetilde{X}$ is well defined locally by $\omega(x) := [e]_x \otimes \widetilde{\omega}_{[e]_x}$. Note that $d\,\widetilde{p}^* = \widetilde{p}^*\,d$ on $\Omega^*(X, \mathcal{O}(V))$.

Now suppose that X is oriented. A **current** S on $\widetilde{X}$ is called **even** if $g_*S = S$ and **odd** if $g_*S = -S$. Equivalently a current S is even (odd) iff $\langle S, \omega \rangle = 0$ for all odd (even) forms on $\widetilde{X}$. Consequently even (odd) currents on $\widetilde{X}$ are precisely the distributional sections of $\Omega^*_{\text{even}}(\widetilde{X})$ $(\Omega^*_{\text{odd}}(X))$. Just as for forms, the even currents on $\widetilde{X}$ are in 1–1 correspondence with currents on X, and odd currents on $\widetilde{X}$ are in 1–1 correspondence with distributional sections of $\Omega^*(X, \mathcal{O}(V))$. These correspondences are given by the mappings

$$\begin{aligned} p_* : \Omega^*_{\text{even}}(\widetilde{X})' &\longrightarrow \Omega^*(X)' \\ S \ &\longmapsto \ p_*S \end{aligned} \quad \text{and} \quad \begin{aligned} \widetilde{p}_* : \Omega^*_{\text{odd}}(\widetilde{X})' &\longrightarrow \Omega^*(X, \mathcal{O}(V))' \\ S \ &\longmapsto \ \widetilde{p}_*S \end{aligned} \tag{3.2}$$

where $\langle p_* S, \omega\rangle := \frac{1}{2}\langle S, p^*\omega\rangle$ and $\langle \widetilde{p}_* S, \omega\rangle := \frac{1}{2}\langle S, \widetilde{\omega}\rangle$. Clearly $\widetilde{p}_* d = d\widetilde{p}_*$.

Finally we gather together some elementary facts required to prove some of the results stated later on. Note that the orientation on the base manifold X induces a natural orientation on the manifold $\widetilde{X}$ and that the deck transformation $g : \widetilde{X} \to \widetilde{X}$ preserves the orientation on $\widetilde{X}$.

Lemma 3.3. *Suppose that X is oriented.*

(1) Let $\omega \in \Omega^*(\widetilde{X}) \subset \Omega^*(\widetilde{X})'$. Then $g_*\omega = g^*\omega$. Consequently an even (odd) form on $\widetilde{X}$ defines an even (odd) current on $\widetilde{X}$.

(2a) Let ω be an $L^1_{\rm loc}$ form on X. Then $p_*(p^*\omega) = \omega$ in $\Omega^*(X)'$.

(2b) Let ω be an $L^1_{\rm loc}$ $\mathcal{O}(V)$–twisted form on X. Then $\widetilde{p}_*(\widetilde{\omega}) = \omega$ in $\Omega^*(X, \mathcal{O}(V))'$.

(3) Let $\omega \in \Omega^*(\widetilde{X})$ and $S \in \Omega^*(\widetilde{X})'$. Then $g_*(\omega \wedge S) = g^*\omega \wedge g_*S$.

(4) Let ω, η be $\mathcal{O}(V)$–twisted forms on X. Then $p^*(\omega \wedge \eta) = \widetilde{\omega} \wedge \widetilde{\eta}$.

(5) Let ω be an $\mathcal{O}(V)$–twisted form on X and S an odd current on $\widetilde{X}$. Then $p_*(\widetilde{\omega} \wedge S) = \omega \wedge \widetilde{p}_*(S)$.

Now we are in a position to define the divisor of a section of a nonorientable bundle V over an oriented manifold X. Let $p : \widetilde{X} \to X$ denote the double cover $Or(V) \to X$ and let $\widetilde{V} \to \widetilde{X}$ be the pullback of $V \to X$. Now the lift $\widetilde{\mu} = \mu \circ p$ of μ from V to $\widetilde{V}$ is also atomic. Since the pullback, $\widetilde{u}^*(\theta)$, of the solid angle kernel is an odd $L^1_{\rm loc}$ form on $\widetilde{X}$ and odd forms define odd currents on $\widetilde{X}$ (Lemma 3.3 (1)), the divisor of $\widetilde{\mu}$ is an odd current on $\widetilde{X}$,

$$g_* \operatorname{Div}(\widetilde{\mu}) = -\operatorname{Div}(\widetilde{\mu}). \tag{3.4}$$

We define the **divisor** of μ to be the corresponding distributional section

$$\operatorname{Div}(\mu) := \widetilde{p}_* \operatorname{Div}(\widetilde{\mu}) \in \Omega^*(X, \mathcal{O}(V))'.$$

Next we consider the case that V and X are nonorientable and that $\mathcal{O}(V) = \mathcal{O}(TX)$. Let $p : \widetilde{X} \to X$ be the oriented double cover of X. Since the deck transformation $g : \widetilde{X} \to$

$\widetilde{X}$ reverses orientation on Xt we have that $g_*\omega = -g^*\omega$ for all forms $\omega \in \Omega^*(\widetilde{X}) \subset \Omega^*(\widetilde{X})'$. Consequently even (odd) forms on $\widetilde{X}$ define odd (even) currents. Therefore the divisor of $\widetilde{\mu}$ is an even current on $\widetilde{X}$,

$$g_* \operatorname{Div}(\widetilde{\mu}) = \operatorname{Div}(\widetilde{\mu}),$$

and so we define the **divisor** of μ to be the usual current pushforward (with a factor of a half) of the divisor of $\widetilde{\mu}$,

$$\operatorname{Div}(\mu) := p_* \operatorname{Div}(\widetilde{\mu}).$$

Note that in this case $\operatorname{Div}(\mu)$ is a current on X. Similarly we could define $\operatorname{Div}(\mu) \in \Omega^*(X, \mathcal{O}(V))'$ when V is orientable over a nonorientable manifold X or when both V and X are nonorientable and $\mathcal{O}(V) \neq \mathcal{O}(TX)$.

Harvey and Semmes [HS 4] have examined the geometric structure of the divisor of an atomic section μ of an oriented bundle over an oriented manifold. Their first result in this direction is the following proposition. Let $Z = \text{Zero}\ (\mu)$ and let

$$\operatorname{Reg} Z = \{x \in X \,/\, Z \text{ is a codimension } m \text{ Lipschitz submanifold near } x\}$$

denote the set of regular points of Z and $\operatorname{Sing} Z = Z \sim \operatorname{Reg} Z$ the set of singular points.

Proposition 3.5. *[HS 4.3] Let μ be an atomic section of $V \to X$. Suppose that either*

(1) *V and X are both oriented or*

(2) *V and X are both nonorientable and $\mathcal{O}(V) = \mathcal{O}(TX)$.*

Let $\{Z_j\}$ be the family of connected components of $\operatorname{Reg} Z$ which are orientable. Then we can choose an orientation on Z_j and integers $n_j \geq 0$ so that

$$\operatorname{Div}(\mu) = \sum n_j\, [Z_j] \qquad \textit{as currents on } X \sim \operatorname{Sing} Z.$$

Example 3.6. Let V be a nonorientable bundle over an oriented manifold X and let $\mathbf{V}$ be the pullback of V over itself. Then by Lemma 1.2, $\mathcal{O}(TV) = \pi^*\mathcal{O}(V) = \mathcal{O}(\mathbf{V})$, and the divisor of the tautological section $c : V \to \mathbf{V}$ is the current $\operatorname{Div}(c) = [X]$ on V.

Example 3.7. Let X be a nonorientable compact manifold and consider the tangent bundle $TX \to X$. Let μ be an atomic vector field on X with isolated zero set $\{z_1, \ldots, z_N\}$. By Proposition 3.5 there are integers n_j such that

$$\mathrm{Div}(\mu) = \sum_{j=1}^{N} n_j \, [z_j] \qquad \text{on } X.$$

Note that the integer n_j is the well defined multiplicity of vanishing of the vector field μ at z_j.

In the case that V is a nonorientable bundle over an orientable manifold the divisor of an atomic section can have support on the nonorientable components of the zero set Z.

Proposition 3.8. *Let μ be an atomic section of a nonorientable bundle V over an orientable manifold X. Suppose that the zero set Z of μ is a connected regular submanifold of X.*
(1) If Z is orientable then either $\mathrm{Div}(\mu) = 0$ or there is a neighbourhood U of Z in X so that $V \to U$ is orientable. Choose an orientation on Z. Then, for either choice of orientation on V over U, there is an integer $n > 0$ such that

$$\mathrm{Div}(\mu) = n[Z]$$

under the identification between $\Omega^*(U, \mathcal{O}(V))'$ and $\Omega^*(U)'$.
(2) If Z is nonorientable then either $\mathrm{Div}(\mu) = 0$ or $\mathcal{O}(TZ) = \mathcal{O}(V)\big|_Z$ and there is an integer n so that

$$\mathrm{Div}(\mu) = n[Z] \qquad \text{in } \Omega^*(X, \mathcal{O}(V))'.$$

Here the distributional section $[Z] \in \Omega^*(X, \mathcal{O}(V))$ is defined by restricting an $\mathcal{O}(V)$–twisted form to Z and integrating the resulting density over Z.

Example 3.9. Let X be a nonorientable manifold and consider the pullback, $\mathbf{TX} \to TX$, of the tangent bundle over itself. Recall that the total space of the tangent bundle is an oriented manifold (c.f. Lemma 2.8). Then the divisor of the tautological section c of $\mathbf{TX}$ is

$$\mathrm{Div}(c) = [X] \qquad \text{in } \Omega^*(TX, \mathcal{O}(\mathbf{TX}))'.$$

Example 3.10. Let $L \to \mathbb{P}(\mathbb{R}^4)$ be the (nonorientable) tautological line bundle over the oriented real projective space $\mathbb{P}(\mathbb{R}^4)$. Recall that sections of L are in 1–1 correspondence with homogeneous functions of degree -1 on $\mathbb{R}^4$. Let μ be the atomic section of L defined by the function

$$u(x_1, x_2, x_3, x_4) = \frac{x_4}{x_1^2 + x_2^2 + x_3^2 + x_4^2}.$$

The zero set of μ is the nonorientable connected submanifold $\mathbb{P}(\mathbb{R}^3)$ of $\mathbb{P}(\mathbb{R}^4)$. Then by Lemma 2.11(1) $\mathcal{O}(T\mathbb{P}(\mathbb{R}^3)) = \mathcal{O}(L)$ on $\mathbb{P}(\mathbb{R}^3)$ and $\mathrm{Div}(\mu) = [\mathbb{P}(\mathbb{R}^3)]$ in $\Omega^*(\mathbb{P}(\mathbb{R}^4), \mathcal{O}(L))'$.

Proof of Proposition 3.8. Let $p : \widetilde{X} \to X$ denote the double cover $\mathcal{O}(V) \to X$ and let $\widetilde{Z} = p^{-1}(Z)$. Suppose that $\mathrm{Div}(\mu) \neq 0$.

(1) Either choice of orientation on Z induces an orientation on $\widetilde{Z}$ and since the restriction of the deck transformation $g : \widetilde{X} \to \widetilde{X}$ to $\widetilde{Z}$ is orientation preserving we know that

$$g_*[\widetilde{Z}] = [\widetilde{Z}] \qquad \text{on } \widetilde{X}. \tag{3.11}$$

On the other hand by (3.4),

$$g_* \,\mathrm{Div}(\widetilde{\mu}) = -\,\mathrm{Div}(\widetilde{\mu}) \qquad \text{on } \widetilde{X}. \tag{3.12}$$

Now if $\widetilde{Z}$ were connected then $\mathrm{Div}(\widetilde{\mu}) = n[\widetilde{Z}]$ in which case (3.11) and (3.12) would force $n = 0$, a contradiction. Consequently $\widetilde{Z}$ must have two connected components which implies that the bundle $V \to Z$ is orientable . Finally we can use the tubular neighbourhood theorem to find an open set $U \subset X$ containing Z so that V is orientable over U.

(2) If $\widetilde{Z}$ is nonorientable then $\mathrm{Div}(\widetilde{\mu}) = 0$ and so $\mathrm{Div}(\mu) = 0$. On the other hand if $\widetilde{Z}$ is orientable then $\mathcal{O}(TZ) = \mathcal{O}(V)\big|_Z$. The result now follows by pushing the odd current equation $\mathrm{Div}(\widetilde{\mu}) = n[\widetilde{Z}]$ on $\widetilde{X}$ down to X via $\widetilde{p}_*$ and checking that $[Z] = \widetilde{p}_*[\widetilde{Z}]$ (independent of choice of orientation on $\widetilde{Z}$). $\square$

Next we recall the notion of atomicity with respect to the zero section of a bundle, as discussed in [Z].

Definition 3.13. Let $p : V \to X$ be a smooth real rank m vector bundle and M a smooth manifold. Let $f : M \to V$ be a smooth map and let $\mathbf{V} \to M$ be the pullback of $V \to X$ via $p \circ f$. Suppose that $\mathcal{O}(\mathbf{V}) \cong \mathcal{O}(TM)$.

(1) The map f is called **atomic with respect to** X if the induced section $\widetilde{f}$ of $\mathbf{V} \to M$ is atomic.

(2) Then the **divisor of** f **with respect to** X, $\mathrm{Div}_X(f)$, is defined to be the divisor of the induced section $\widetilde{f}$ of $\mathbf{V} \to M$.

Note. The divisor of f with respect to X is a current on M which is supported on the subset $f^{-1}(X)$.

Finally we are in a position to define the zero and pole divisors of a section $\nu : X \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$. From here on we assume that V is an oriented bundle over a manifold X. Unless otherwise noted we make no orientability assumptions on X.

Definition 3.14. A smooth section $\nu : X \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ is called **atomic** if the following two conditions hold.

(1) The induced section $\nu : X \sim P \to V$ is atomic.
(2) P has Lebesgue measure zero in X.

Definition 3.15. Let ν be an atomic section of the bundle $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$. Suppose that $\nu^*\mathcal{O} \cong \mathcal{O}(TX)$.

(1) The **zero divisor,** $\mathrm{Div}_0(\nu)$, of ν is defined to be the extension from $X \sim P$ to all of X of the divisor of the induced section $\nu : X \sim P \to V$.
(2) The **pole divisor,** $\mathrm{Div}_\infty(\nu)$, of ν is defined to be the extension from $X \sim Z$ to all of X of the divisor with respect to $\mathbb{P}(V)$ of the induced mapping $\nu : X \sim Z \to L^*$.

Remark 3.16.

(1) Using the orientation on V the orientation bundle $\mathcal{O}$ can be trivialized over the chart

$V \subset \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ defined by (2.1). Then since the restriction of ν to $X \sim P$ defines a section of V we conclude that $X \sim P$ is orientable. Consequently the zero divisor, $\mathrm{Div}_0(\nu)$, is a well defined current on X.

(2) Since the orientation bundle of the pullback of L^* to $X \sim Z$ via ν is equivalent to the orientation bundle of X the pole divisor, $\mathrm{Div}_\infty(\nu)$, is a well defined current on X.

Remark 3.17. In a neighbourhood $\mathcal{U}$ of a point $p \in P$ the atomic section $\nu : X \sim Z \to L^*$ is given by an atomic function $t : \mathcal{U} \to \mathbb{R}$. The atomicity of t implies that $\frac{t}{|t|} \in L^\infty_{\mathrm{loc}}(X)$ and the divisor of t is the current $\mathrm{Div}(t) := d\left(\frac{1}{2}\,\frac{t}{|t|}\right)$ on X. In particular (c.f. Proposition 3.5) if we let $\{P_i\}$ denote the orientable connected components of $\mathrm{Reg}\, P$ then we can choose an orientation on each P_i so that $\mathrm{Div}_\infty(\nu) = \sum\limits_i\,[P_i]$ on $X \sim \mathrm{Sing}\, P$.

Remark 3.18. Suppose that X is orientable.

(1) If the rank of V is even, then $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ is nonorientable and the pole divisor of the tautological section $c : \mathbb{P}(\underline{\mathbb{R}} \oplus V) \to \mathbb{P}(\underline{\mathbb{R}} \oplus \mathbf{V})$ is the current of integration over the orientable manifold $\mathbb{P}(V)$, $\mathrm{Div}_\infty(c) = [\mathbb{P}(V)]$.

(2) If the rank of V is odd then $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ is orientable and $\mathcal{O} \not\cong \mathcal{O}(T\mathbb{P}(\underline{\mathbb{R}} \oplus V))$. Nevertheless we can still define $\mathrm{Div}_\infty(c)$ and $[\mathbb{P}(V)]$ as distributional sections of $\Omega^*(\mathbb{P}(\underline{\mathbb{R}} \oplus V), \mathcal{O})$ and check that $\mathrm{Div}_\infty(c) = [\mathbb{P}(V)]$.

Example 3.19. Examples of sections $\nu : X \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ for which $\nu^*\mathcal{O} \cong \mathcal{O}(TX)$ can be constructed as follows.

(1) Suppose that X is orientable and let $p : S(\underline{\mathbb{R}} \oplus V) \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ be the usual projection map. Then for any map $\widetilde{\nu} : X \to S(\underline{\mathbb{R}} \oplus V)$, the pullback of $\mathcal{O}$ via $\nu := p \circ \widetilde{\nu}$ is trivial. For instance any map $F : \mathbb{R}^{2k} \to \mathbb{R}^{2n+1}$ which is homogeneous of even degree defines a map $\widetilde{\nu} : \mathbb{RP}^{2k-1} \to S^{2n}$.

(2) Suppose that $\nu : \mathbb{RP}^n \to \mathbb{RP}^n$ is homotopic to the identity map. Then $\nu^*\mathcal{O} \cong \mathcal{O}$. Many homogeneous functions $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of odd degree give rise to such maps. To show

that a particular F gives rise to such a ν we just need to show that for each $t \in [0,1]$ the map $G_t(x) := tF(x) + (1-t)x$ has no zeros on the unit sphere. In that case $g_t(x) := [G_t(x)]$ is a homotopy between ν and the identity. For example $F_k(x_0, ...x_n) = (x_0^{2k+1}, ..., x_n^{2k+1})$ gives rise to such a map.

4. The even rank case.

In this section we compute and study the ϕ–characteristic current at time infinity of an atomic section $\nu : X \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ in the case that the rank of V is even. Let $\mathcal{O} := \mathcal{O}(T\mathbb{P}(\underline{\mathbb{R}} \oplus V)) \cong \mathcal{O}(U)$. Throughout this section we assume that $\nu^*\mathcal{O} \cong \mathcal{O}(TX)$ and that the invariant polynomial ϕ is of the form $\phi = \chi\, \psi(p_1, ...p_n)$. This ensures that the smooth family $\phi(\overrightarrow{D}_s)$ which approximates the ϕ–characteristic current actually defines a family of currents on X (see Corollary 2.14).

Let $\mathbf{V} \to V$ be the pullback of $V \to X$ and let e be a positively oriented orthonormal local frame for $\mathbf{V}$. Let $u = (u_1, ..., u_m)$ be the local fibre coordinate on V defined by $c = ue$, where $c : V \to \mathbf{V}$ is the tautological section. Let t be the global coordinate on $\underline{\mathbb{R}}$. Let $\omega^{\mathcal{O}}$ be a smooth $\mathcal{O}$–twisted form on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ and let ω^{odd} be the corresponding smooth odd form on $(\underline{\mathbb{R}} \oplus V) \sim X$ (c.f. (3.1)). The restriction of $\omega^{\mathcal{O}}$ to $V \subset \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ (c.f. (2.1)) defines a bona fide form ω on V (see Remark 2.7 and use the frame $\gamma_1(e)$ for $U^{\perp}$ over V defined by (2.6)). Then if $\omega(u)$ denotes the expression for ω in terms of u, the expression for ω^{odd} in terms of the local coordinate (t, u) on $\underline{\mathbb{R}} \oplus V$ is

$$\omega^{\text{odd}}(t, u) = \tfrac{t}{|t|}\, \omega\left(\tfrac{u}{t}\right).$$

Now let $T^{\mathcal{O}}_{r,s}$ denote the $\mathcal{O}$–twisted transgression form on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ defined by ϕ (c.f. Corollary 2.14). Harvey and Lawson derived the following local formula for the corresponding form $T_{r,s}$ on V. Set $|u|^2 := uu^t$, $Du := du + u\omega_V$ and $Du^t := du^t - \omega_V u^t$. Then

$$T_{r,s} = \int_{\chi_r}^{\chi_s} \phi\left(\frac{u^t Du}{|u|^2} - \frac{Du^t u}{|u|^2}\,;\, A(x)\right) dx,$$

where $\chi_s = 1 - \frac{s}{\sqrt{|u|^2+s^2}}$ and

$$A(x) := \Omega_V - x\left(\frac{u^t u}{|u|^2}\,\Omega_V + \Omega_V\,\frac{u^t u}{|u|^2}\right) \;+\; 2x(1-\tfrac{x}{2})\left(\frac{u^t Du}{|u|^2} - \frac{Du^t u}{|u|^2}\right)^2 .$$

Either by direct calculation or using the analogue of Lemma 2.30 of [Z] we have that

$$T^{\rm odd}_{r,s} = \frac{t}{|t|}\int_{\chi_{r|t|}}^{\chi_{s|t|}} \phi\left(\frac{u^t Du}{|u|^2} - \frac{Du^t u}{|u|^2}\;;\;A(x)\right)dx.$$

Note that $T^{\rm odd}_{r,s}$ is the pullback to $(\underline{\mathbb{R}}\oplus V)\sim X$ of an odd form on the unit sphere bundle $S(\underline{\mathbb{R}}\oplus V)\to X$.

Proposition 4.1.

(1) The transgression current $T^{\rm odd}_{\infty,0} := \lim_{\substack{r\to\infty\\ s\to 0}} T^{\rm odd}_{r,s}$ *converges in* $L^1_{\rm loc}(S(\underline{\mathbb{R}}\oplus V))$ *and equals*

$$T^{\rm odd}_{\infty,0} = \tfrac{t}{|t|}\,T_{\infty,0}(u),$$

where

$$T_{\infty,0}(u) = \int_0^1 \phi\left(\frac{u^t Du}{|u|^2} - \frac{Du^t u}{|u|^2}\;;\;A(x)\right)dx. \tag{4.2}$$

(2) The transgression current $T^{\mathcal{O}}_{\infty,0} := \lim_{\substack{r\to\infty\\ s\to 0}} T^{\mathcal{O}}_{r,s}$ *converges in* $L^1_{\rm loc}(\mathbb{P}(\underline{\mathbb{R}}\oplus V),\mathcal{O})$.

(3) Let ν *be an atomic section of* $\mathbb{P}(\underline{\mathbb{R}}\oplus V)\to X$. *Then* $T_{\infty,0} := \lim_{\substack{r\to\infty\\ s\to 0}} \nu^* T^{\mathcal{O}}_{r,s}$ *converges in* $L^1_{\rm loc}(X,\nu^*\mathcal{O})$.

Proof. The proof follows along the same lines as that of Proposition 3.1 of [Z]. □

Note. Since the antipodal map on $S(\underline{\mathbb{R}}\oplus V)$ is orientation reversing odd forms on $S(\underline{\mathbb{R}}\oplus V)$ define even currents. In particular the current $T^{\rm odd}_{\infty,0}$ is even and its current pushforward to $\mathbb{P}(\underline{\mathbb{R}}\oplus V)$ is the $\mathcal{O}$–twisted $L^1_{\rm loc}$ form $T^{\mathcal{O}}_{\infty,0}$.

Theorem 4.3.

(1) The limit as $r\to\infty$ *and* $s\to 0$ *of the equation* $dT^{\rm odd}_{r,s} = \phi(D_{U^\perp,r}) \;-\; \phi(D_{U^\perp,s})$ *between odd smooth forms on* $S(\underline{\mathbb{R}}\oplus V)$ *is the even current equation*

$$dT^{\rm odd}_{\infty,0} \;=\; \tfrac{t}{|t|}\,\phi(D_{\mathbf{V}}) \;+\; [S(V)]\,(-2T_{\infty,0}(u)) \;-\; \psi(D_{\mathbf{V}})\,([X_+]+[X_-]) \tag{4.4}$$

on $S(\underline{\mathbb{R}} \oplus V)$. Here $[X_\pm] := (\psi^\pm)_*[X]$, where $\psi^\pm(v) := \frac{1}{\sqrt{1+|v|^2}}(\pm 1, v)$ maps V to $S(\underline{\mathbb{R}} \oplus V)$.
(2) The limit as $r \to \infty$ and $s \to 0$ of the equation $dT^{\mathcal{O}}_{r,s} = \phi(D_{U^\perp,r}) - \phi(D_{U^\perp,s})$ between odd smooth $\mathcal{O}$–twisted forms on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ is the current equation

$$dT^{\mathcal{O}}_{\infty,0} \;=\; p_*(\tfrac{t}{|t|})\,\phi(D_{\mathbf{V}}) \;+\; [\mathbb{P}(V)]\,(-2T_{\infty,0}(u)) \;-\; \psi(D_{\mathbf{V}})[X] \tag{4.5}$$

on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$. Here $p : S(\underline{\mathbb{R}} \oplus V) \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ is the covering map and $p_*(\frac{t}{|t|})$ is the current on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ corresponding to the odd L^∞_{loc} form $\frac{t}{|t|}$ on $S(\underline{\mathbb{R}} \oplus V)$ (c.f. (3.2)). Furthermore the characteristic currents, $\phi(\!(D_0)\!) := \lim_{s\to 0} \phi(D_{U^\perp,s})$ at time zero and $\phi(\!(D_\infty)\!) := \lim_{r\to\infty} \phi(D_{U^\perp,r})$ at time infinity, exist on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ and equal

$$\phi(\!(D_0)\!) \;=\; \psi(D_{\mathbf{V}})\,[X]$$

and

$$\phi(\!(D_\infty)\!) \;=\; p_*(\tfrac{t}{|t|})\,\phi(D_{\mathbf{V}}) \;+\; [\mathbb{P}(V)]\,(-2T_{\infty,0}(u)). \tag{4.6}$$

Note that

$$dT^{\mathcal{O}}_{\infty,0} \;=\; \phi(\!(D_\infty)\!) \;-\; \phi(\!(D_0)\!) \qquad \text{on } \mathbb{P}(\underline{\mathbb{R}} \oplus V).$$

Remark 4.7. Harvey and Lawson define the residue form at time zero by

$$\mathrm{Res}_0(\phi) := -\int_{\pi^{-1}} T_{\infty,0}(u) \;=\; -\,\pi_* T_{\infty,0}(u),$$

where $\pi : S(V) \to X$ is the natural projection. We can define the residue form at time infinity in a similar manner, just as in the complex case (see [Z 3.5]). Recall that $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \sim X$ is the total space of the real line bundle $L^* \to \mathbb{P}(V)$. (c.f. (2.13)). Choose a metric on L^* and let $\pi : S(L^*) \to \mathbb{P}(V)$ be the unit S^0–bundle of L^*. Since $\mathcal{O} \cong \mathcal{O}(TS(L^*))$, $T^{\mathcal{O}}_{\infty,0}$ is a smooth $\mathcal{O}(TS(L^*))$–twisted form on $S(L^*)$ and so defines a current. The **residue form at infinity** on $\mathbb{P}(V)$ is defined to be the current pushforward

$$\mathrm{Res}_\infty(\phi) \;:=\; -\,\pi_* T^{\mathcal{O}}_{\infty,0} \qquad \text{on } \mathbb{P}(V).$$

Note that $\mathrm{Res}_\infty(\phi)$ is a bona fide smooth odd degree form on the oriented manifold $\mathbb{P}(V)$. In fact it is easy to see that

$$\mathrm{Res}_\infty(\phi) \;=\; -2T_{\infty,0}(u) \qquad \text{on } \mathbb{P}(V).$$

Just as in the complex case (see [Z 4]) we can prove that

$$\int_{\mathbb{P}(V)} \mathrm{Res}_\infty(\phi) \;=\; \mathrm{Res}_0(\phi) \qquad \text{on } X$$

and that

$$\mathrm{Res}_\infty(\phi) \;=\; \int_{\pi^{-1}} \phi(D_{U^\perp}) \qquad \text{on } \mathbb{P}(V),$$

where $\pi : \mathbb{P}(\underline{\mathbb{R}} \oplus V) \sim X \to \mathbb{P}(V)$. It is an interesting exercise to check directly, using the definition of the transgression, that $\int\limits_{\pi^{-1}} \phi(D_{U^\perp}) = -2T_{\infty,0}(u)$.

Remark 4.8. The most interesting case is when the invariant polynomial is the Pfaffian, $\phi = \chi$. In that case the form $T_{\infty,0}(u)$ is the **spherical kernel $\boldsymbol{\sigma}$** which (see [C1]) is given by

$$\boldsymbol{\sigma}\lambda \;=\; \frac{1}{\pi^n}\sum_{p=0}^{n-1}(-1)^{n-p}\,\tfrac{p!}{(n-p-1)!(2p+1)!2^{2n-2p-1}}\,\frac{(ue)(Due)^{2p+1}}{|u|^{2p+2}}\,(e^t\Omega_V e)^{n-p-1},$$

where e is a positively oriented orthonormal local frame for V and $\lambda := e_1 \wedge \ldots \wedge_{2n}$. In particular the restriction of $\mathrm{Res}_\infty(\chi) = -2\boldsymbol{\sigma}$ to the fibres $\mathbb{P}(\mathbb{R}^{2n})$ of $\mathbb{P}(V)$ is the normalized volume form

$$\mathrm{Res}_\infty(\chi)\,\big|_{\mathbb{P}(V)_x} \;=\; \tfrac{1}{\mathrm{Vol}\,\mathbb{P}(\mathbb{R}^{2n})}\,\Theta.$$

Here Θ denotes the solid angle kernel (see (1.3)). Note that if the background connection D_V is flat then $d\,\mathrm{Res}_\infty(\chi) = 0$ and

$$d\boldsymbol{\sigma}^{\mathcal{O}} \;=\; [\mathbb{P}(V)]\,\mathrm{Res}_\infty(\chi) \;-\; [X] \qquad \text{on } \mathbb{P}(\underline{\mathbb{R}} \oplus V).$$

In particular when X is a point

$$d\boldsymbol{\sigma}^{\mathcal{O}} \;=\; [\mathbb{P}(\mathbb{R}^{2n})]\;\mathrm{Vol} \;-\; [0] \qquad \text{on } \mathbb{P}(\mathbb{R} \oplus \mathbb{R}^{2n}),$$

where Vol is the normalized volume form on $\mathbb{P}(\mathbb{R}^{2n})$.

Remark 4.9. In the complex case (see [Z 3.7]) the Chern current at time infinity has some nice structure. A complex vector bundle $F \to X$ is compactified by embedding it in the bundle of complex projective spaces $\mathbb{P}(\underline{\mathbb{C}} \oplus F) \to X$. The residue at time infinity, $\mathrm{Res}_\infty(\phi)$, is a d–closed even degree form supported on spatial infinity $\mathbb{P}(F)$. Its cohomology class can be computed in terms of the Chern classes of $F \to X$ and $L \to \mathbb{P}(F)$ where L is the tautological line bundle. Furthermore in the universal case the Chern current is a d–closed flat current on $\mathbb{P}(\underline{\mathbb{C}} \oplus F)$ which can be expressed as the sum of a singular term, $\mathrm{Res}_\infty(\phi)\,[\mathbb{P}(F)]$, supported at spatial infinity and an absolutely continuous L^1_{loc} term, $\phi(D_F)$, (which happens to be C^∞). The main point here is that the absolutely continuous and singular terms are both d–closed.

On the other hand in the real case the residue at time infinity is the odd degree form $-2T_{\infty,0}(u)$ on $\mathbb{P}(V)$ which is not d–closed. In fact

$$d\,\mathrm{Res}_\infty(\phi) \;=\; -2\phi(D_V) \qquad \text{on } \mathbb{P}(V).$$

The characteristic current at time infinity is the d–closed current given by (4.6). It splits as the sum of a singular term and an L^1_{loc} term. However neither piece is d–closed in general. In fact

$$d\left([\mathbb{P}(V)]\ \mathrm{Res}_\infty(\phi)\right) \;=\; [\mathbb{P}(V)]\,2\phi(D_V) \;=\; -\,d\left(p_*(\tfrac{t}{|t|})\,\phi(D_V)\right)$$

which is zero only in the flat case. It is interesting to observe that the nonorientability of $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ implies that the L^1_{loc} part of $\phi((D_\infty))$ is not d–closed, while the fact that $\mathbb{P}(V)$ is codimension 1 implies that the residue is the odd degree form $-2T_{\infty,0}(u)$, which also fails to be d–closed.

Theorem 4.10. *Let V be an oriented bundle over a manifold X. Let $\nu : X \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ be an atomic section and suppose that $\nu^*\mathcal{O} \cong \mathcal{O}(TX)$. Set $\phi = \chi\,\psi(p_1, ...p_n)$ and let $T_{r,s}$ and $\phi(\overrightarrow{D}_r)$ be the corresponding smooth $\mathcal{O}(TX)$–twisted transgresion and characteristic forms* *on X (c.f. Corollary 2.14). Then the current limit as $r \to \infty$ and $s \to 0$ of the equation*

$$dT_{r,s} \;=\; \phi(\overrightarrow{D}_r) \;-\; \phi(\overrightarrow{D}_s)$$

between smooth $\mathcal{O}(TX)$–twisted forms on X is the current equation

(4.11)
$$dT_{\infty,0} \;=\; \nu^*\left(p_*(\tfrac{t}{|t|})\right)\,\phi(D_V) \;+\; \mathrm{Div}_\infty(\nu)\,(-2\nu^*T_{\infty,0}(u)) \;-\; \psi(D_V)\,\mathrm{Div}_0(\nu) \qquad \textit{on } X.$$

Furthermore the characteristic currents exist and equal

$$\phi((D_0)) \;=\; \psi(D_V)\,\mathrm{Div}_0(\nu)$$

and

$$\phi((D_\infty)) \;=\; \nu^*\left(p_*(\tfrac{t}{|t|})\right)\,\phi(D_V) \;+\; \mathrm{Div}_\infty(\nu)\,(-2\nu^*T_{\infty,0}(u)).$$

Remark 4.12. Suppose that $V = \underline{\mathbb{R}}^{2n}$ is trivial and $\phi = \chi$. Then (4.11) reduces to

$$dT_{\infty,0} \;=\; \tfrac{2}{\gamma_{2n-1}}\,\mathrm{Div}_\infty(\nu)\,\nu^*\Theta \;-\; \mathrm{Div}_0(\nu) \qquad \text{on } X,$$

where $\gamma_{2n-1} = \mathrm{Vol}(S^{2n-1})$ and Θ is the volume form on $\mathbb{P}(\mathbb{R}^{2n})$ which pulls back to the solid angle kernel on $\mathbb{R}^{2n} \sim \{0\}$. Suppose further that $\dim(X) = 2n$, that the zero set consists of isolated points $\{z_j\}$ and that $\mathrm{Sing}\,P = \varnothing$. Let $\{P_i\}$ denote the connected components of P which are orientable. Then, by Proposition 3.5 and Remark 3.17, there exist integers n_j so that

$$dT_{\infty,0} \;=\; \tfrac{2}{\gamma_{2n-1}}\sum_i [P_i]\,\nu^*\Theta \;-\; \sum_j n_j[z_j] \qquad \text{on } X. \tag{4.13}$$

Also, by Remark 3.16, $X \sim P$ is orientable. In (4.13) we have chosen the orientation on $X \sim P$ which is compatible with the standard one on the chart $\mathbb{R}^{2n} \subset \mathbb{P}(\mathbb{R} \oplus \mathbb{R}^{2n})$ defined by (2.1). This induces an orientation on $\partial X \subset X \sim P$ and on each of the the spheres $\partial B(z_i) \subset X \sim P$ described in Corollary 4.14 below.

Letting each current in (4.13) act on the test function $[X]$, we have the following result.

Corollary 4.14. *Let X be a compact $2n$–dimensional manifold with boundary ∂X. Let $f : X \to \mathbb{P}(\mathbb{R}\oplus\mathbb{R}^{2n})$ be a smooth map. Suppose that $f^*\mathcal{O} \cong \mathcal{O}(TX)$. Suppose that f vanishes algebraically and that its zero set consists of a finite number of points, $Z = \{z_1, ...z_N\}$. Suppose (for the sake of simplicity) that the pole set $P = f^{-1}(\mathbb{P}(\mathbb{R}^{2n}))$ is a smooth oriented connected $(2n-1)$–dimensional submanifold of X and that $(Z \cup P) \cap \partial X = \varnothing$. Then*

$$\sum_{i=1}^{N} \mathrm{Mult}(f, z_i) \;-\; \mathrm{Deg}(f\,\big|_P) \;=\; \mathrm{Wind}(f, \partial X).$$

Here $\mathrm{Deg}(f\,\big|_P)$ *is the degree of the map $f : P \to \mathbb{P}(\mathbb{R}^{2n})$. Let $\pi : \mathbb{R}^{2n} \sim \{0\} \to S^{2n-1}$ be the radial projection onto the unit sphere. Then* $\mathrm{Mult}(f, z_i) := \mathrm{Deg}(\pi \circ f : \partial B(z_i) \to S^{2n-1})$ *where $B(z_i)$ is a ball about z_i not containing any other points of $Z \cup P$. Finally the winding number,* $\mathrm{Wind}(f, \partial X)$, *of f around ∂X is defined to be the sum of the degrees of the induced maps from the various connected components of ∂X to the unit sphere in $\mathbb{R}^{2n} \cong \mathbb{P}(\mathbb{R} \oplus \mathbb{R}^{2n}) \sim \mathbb{P}(\mathbb{R}^{2n})$.*

Remark 4.15. In a similar fashion we can use (4.13) to prove the theorem stated in the introduction concerning maps into the sphere S^{2n}. The case $n = 1$ is particularly interesting since $S^2 \cong \mathbb{CP}^1$ and the argument principle for maps $f : X \to \mathbb{CP}^1$ states that

$$\#S \;-\; \#N \;=\; W, \tag{4.16}$$

where the winding number W is the degree of the induced map from ∂X to the unit circle in $\mathbb{C} \cong \mathbb{CP}^1 \sim \{N\}$. In particular, for a compact oriented surface without boundary,

$$\#N \;=\; \#S \;=\; \mathrm{Deg}(f\,\big|_E).$$

Actually (4.16) holds for maps from an n–manifold X to S^n. This is because the solid angle kernel is invariant under the transition function $\varphi(x) = \frac{x}{|x|^2}$ for the two charts on S^n given by stereographic projection from the north and south poles to the equatorial plane.

Proof of Theorem 4.3.

(1) We explicitly compute $dT^{\mathrm{odd}}_{\infty,0}$ in a system of positively oriented charts for $S(\underline{\mathbb{R}} \oplus V)$ using the expression for $dT^{\mathrm{odd}}_{\infty,0}$ given in Proposition 4.1. Orient $S(\underline{\mathbb{R}} \oplus V) \to X$ using the standard orientation inherited from $V \to X$. Let $m = \operatorname{rank} V$. Then the following system of charts is a positively oriented system of charts for $S(\underline{\mathbb{R}} \oplus V)$ whose restriction to $S(V)$ forms a positively oriented system of charts for $S(V)$. For each positively oriented orthonormal local frame e for V the charts are defined by $\psi_j^{\pm} := \frac{\varphi_j^{\pm}}{|\varphi_j^{\pm}|} : \mathbb{R}^m \to S(\mathbb{R} \oplus \mathbb{R}^m)$, where $\varphi_j^{\pm} : \mathbb{R}^m \to \mathbb{R} \oplus \mathbb{R}^m$ are given by

$$\begin{aligned}
\varphi_0^{\pm}(u_1, ..., u_m) &= (\pm 1, \pm u_1, u_2, ..., u_m), \\
\varphi_1^{\pm}(t, u_1, ..., u_{m-1}) &= (-t, \pm 1, \pm u_1, u_2, ..., u_{m-1}), \\
\varphi_{2j}^{\pm}(t, u_1, ..., u_{m-1}) &= (-t, \mp u_1, u_2, ..., u_{2j-1}, \pm 1, u_{2j}, ..., u_{m-1}), \\
\varphi_{2j+1}^{\pm}(t, u_1, ..., u_{m-1}) &= (-t, \pm u_1, u_2, ..., u_{2j}, \pm 1, u_{2j+1}, ..., u_{m-1}).
\end{aligned}$$

Let $\widetilde{T}$ denote the local expression on $\underline{\mathbb{R}} \oplus \underline{\mathbb{R}}^m$ of the pullback of $T^{\mathrm{odd}}_{\infty,0}$ to $\underline{\mathbb{R}} \oplus V$. Let $T_{\infty,0}(u)$ be given by (4.2) and note that by Theorem 1.9

$$dT_{\infty,0}(u) = \phi(D_V) - \psi(D_V)\,[X] \qquad \text{on } \underline{\mathbb{R}}^{2n}. \tag{4.17}$$

Then in the chart ψ_0^+, $(\varphi_0^+)^* \widetilde{T} = T_{\infty,0}(u)$ and so on the image of ψ_0^+,

$$dT^{\mathrm{odd}}_{\infty,0} = d(\psi_0^+)_* (\varphi_0^+)^* \widetilde{T} = (\psi_0^+)_* \, dT_{\infty,0}(u) = \phi(D_V) - \psi(D_V)\,[X_+]$$

by (4.17) and the definition of $[X_+]$, as required.

As for the chart ψ_0^-, let $\eta(u_1, ..., u_m) = (-u_1, u_2, ..., u_m)$ and note that, since η is orientation reversing, $\eta_* \omega = -\eta^* \omega$. Then $(\varphi_0^-)^* \widetilde{T} = \eta_* T_{\infty,0}(u)$ and so

$$d(\varphi_0^-)^* \widetilde{T} = \eta_* \, dT_{\infty,0}(u) = \eta_* \, \phi(D_V) - \psi(D_V)\, \eta_*[X] = -\phi(D_V) - \psi(D_V)[X].$$

Taking $(\psi_0^-)_*$ of both sides gives

$$dT^{\mathrm{odd}}_{\infty,0} = -\phi(D_V) - \psi(D_V)\,[X_-],$$

as required.

Finally let $\varphi = \varphi_k^{\pm}$ where $k > 0$. Then $\varphi^*\widetilde{T} = -\frac{t}{|t|}\varphi^* T_{\infty,0}(u)$ and so

$$d(\varphi^*\widetilde{T}) \;=\; [t=0]\,\varphi^*(-2T_{\infty,0}(u)) \;-\; \tfrac{t}{|t|}\,\phi(D_V)$$

by (4.17). Taking ψ_* gives

$$dT_{\infty,0}^{\mathrm{odd}} \;=\; [S(V)](-2T_{\infty,0}(u)) \;+\; \tfrac{t}{|t|}\,\phi(D_V)$$

since $\psi_*[t=0] = [S(V)]$, as required.

(2) To prove (4.5) we push (4.4) forward by $p : S(\underline{\mathbb{R}} \oplus V) \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ using Lemma 3.3. The rest of the proof follows along the same lines as the proof of Theorem 3.7 of [Z]. □

Proof of Theorem 4.10. Since $\nu^*\mathcal{O} \cong \mathcal{O}(TX)$, the pullback of the $\mathcal{O}$–twisted transgression form $T_{r,s}^{\mathcal{O}}$ on $\mathbb{P}(\underline{\mathbb{R}}\oplus V)$ to X via ν defines a current $T_{r,s}$ on X. By Proposition 4.1 $T_{r,s} \to T_{\infty,0}$ in $L^1_{\mathrm{loc}}(X)$. Now compute $dT_{\infty,0}$ just as in the previous proof using the pullbacks of the charts $\psi_i^{\pm}$. □

5. The odd rank case.

Let $\pi : V \to X$ be an oriented real Riemannian vector bundle of odd rank $m = 2n-1$ over an oriented manifold. Harvey and Lawson defined a family of Thom forms $\boldsymbol{\tau}_s$ on V associated with a metric compatible connection D_V. These forms have all the properties of the Thom forms defined in the even rank case (see Remark 1.10). They also showed that the forms $\boldsymbol{\tau}_s$ on V extend smoothly to all of the orientable manifold $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$. The main aim of this section is to compute the **Euler current at time infinity** of an atomic section $\nu : X \to \mathbb{P}(\underline{\mathbb{R}}\oplus V)$, which is defined by $\tau_\infty := \lim_{s\to\infty} \nu^*\boldsymbol{\tau}_s$ on X. We begin by recalling Harvey and Lawson's definition of the Thom forms. Note that since the Chern–Euler form of a connection on an odd rank bundle is zero the construction of the Thom forms is somewhat different in this case.

Definition 5.1. [HL IV 2.1] Fix a metric compatible connection D_V on V and equip $\widetilde{V} := \underline{\mathbb{R}} \oplus V$ with the direct sum metric and metric connection. Let $\widetilde{\boldsymbol{\sigma}}$ be the spherical potential for the even rank $2n$ bundle $\widetilde{V}$ (see (Remark (4.8)). Fix $s > 0$. First restrict $-2\widetilde{\boldsymbol{\sigma}}$ as a differential form to the affine subbundle $\{s\} \times V \subset \widetilde{V}$. Second pullback to V using the obvious identification of $\{s\} \times V$ with V. The Thom form $\boldsymbol{\tau}_s$ is defined to be the resulting smooth $(2n-1)$–form on V.

To calculate the Euler currents we seek a smooth family of approximate spherical potentials $\boldsymbol{\sigma}_{r,s}$ satisfying the "transgression" formula $d\boldsymbol{\sigma}_{r,s} = \boldsymbol{\tau}_r - \boldsymbol{\tau}_s$ on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$.

Definition 5.2. [HL IV 2.3] Let e_0 denote the global frame 1 for $\underline{\mathbb{R}}$. Fix $0 < s < r < \infty$. Let $\underline{\mathbb{R}}_{r,s}$ denote the submanifold with boundary $\underline{\mathbb{R}}_{r,s} = \{te_0 \,/\, s \le t \le r\}$, and let $\rho_{r,s} : \underline{\mathbb{R}}_{r,s} \oplus V \to V$ be the natural projection. The **approximate spherical potential** $\boldsymbol{\sigma}_{r,s}$ on V is defined by the fibre integral

$$\boldsymbol{\sigma}_{r,s} := -2 \int_{\rho_{r,s}^{-1}(v)} \widetilde{\boldsymbol{\sigma}} = -2(\rho_{r,s})_* \widetilde{\boldsymbol{\sigma}}. \tag{5.3}$$

Lemma 5.4. *The approximate spherical potentials $\boldsymbol{\sigma}_{r,s}$ have the following properties.*

(1) *$\boldsymbol{\sigma}_{r,s}$ is a smooth $(2n-2)$–form on V.*

(2) *$\boldsymbol{\sigma}_{r,s}$ has a smooth extension to the compactification $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ of V, and*

(3) *$d\boldsymbol{\sigma}_{r,s} = \boldsymbol{\tau}_r - \boldsymbol{\tau}_s$ on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$.*

Proof.

(1) Let $(s,u) = (s, u_1, \ldots, u_{2n-1})$ be local fibre variables on $\underline{\mathbb{R}} \oplus V$ corresponding to a positively oriented orthonormal local frame $(e_0, e) = (e_0, e_1, \ldots, e_{2n-1})$ for $\underline{\mathbb{R}} \oplus V$. Let $\lambda = e_1 \wedge \cdots \wedge e_{2n-1}$ and $\widetilde{\lambda} = e_0 \wedge \lambda$ be unit volume elements for V and $\underline{\mathbb{R}} \oplus V$ respectively. Harvey and Lawson [HL: IV,2.6] have shown that the spherical potential $\widetilde{\boldsymbol{\sigma}}$ on $\underline{\mathbb{R}} \oplus V$ is

given by

$$\widetilde{\boldsymbol{\sigma}}\widetilde{\lambda} = \sum_{p=0}^{n-1} K_{n,p}\, \frac{(se_0 + ue)(dse_0 + Due)^{2p+1}}{(|u|^2 + s^2)^{p+1}}\, (e^t\Omega_V e)^{n-p-1}$$

where

$$K_{n,p} = \frac{(-1)^{n-p}}{\pi^n}\, \frac{p!}{(n-p-1)!(2p+1)!2^{2n-2p-1}}.$$

So by (5.3),

$$\boldsymbol{\sigma}_{r,s}\widetilde{\lambda} = \sum_{p=0}^{n-1} 2K_{n,p}\, \frac{(ue)(Due)^{2p}}{|u|^{2p+1}}\, (e^t\Omega_V e)^{n-p-1} \int_s^r \frac{|u|^{2p+1} dse_0}{(|u|^2 + s^2)^{p+1}} \tag{5.5}$$

$$= \sum_{p=0}^{n-1} 2K_{n,p}\, (ue)(Due)^{2p}\, (e^t\Omega_V e)^{n-p-1} \int_s^r \frac{dse_0}{(|u|^2 + s^2)^{p+1}}, \tag{5.6}$$

which is smooth on V

(2) To show that $\boldsymbol{\sigma}_{r,s}$ extends to a smooth form on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ we just need to show that the expression $(\boldsymbol{\sigma}_{r,s})_{HC}$ for $\boldsymbol{\sigma}_{r,s}$ in the local homogeneous coordinate (t,u) corresponding to the fibre variable u is smooth in t at $t=0$. Replacing u by $\frac{u}{t}$ in (5.6) and noting that $(ue)^2 = 0$ we obtain the following expression for $\boldsymbol{\sigma}_{r,s}$ in homogeneous coordinates.

$$(\boldsymbol{\sigma}_{r,s})_{HC}\widetilde{\lambda} = \sum_{p=0}^{n-1} 2K_{n,p}\, \frac{ue((Du - u\frac{dt}{t})e)^{2p}\, (e^t\Omega_V e)^{n-p-1}}{t^{2p+1}} \int_s^r \frac{dse_0}{(\frac{|u|^2}{t^2} + s^2)^{p+1}}$$

$$= \sum_{p=0}^{n-1} 2K_{n,p}\, \frac{ue(Due)^{2p}}{|u|^{2p+1}}\, (e^t\Omega_V e)^{n-p-1} \int_{\frac{st}{|u|}}^{\frac{rt}{|u|}} \frac{dve_0}{(1+v^2)^{p+1}}. \tag{5.7}$$

To see that $(\boldsymbol{\sigma}_{r,s})_{HC}$ is a smooth function of t note that

$$\frac{\partial}{\partial t} \int_{st}^{rt} \frac{dv}{(v^2 + |u|^2)^{p+1}} = \frac{r}{(r^2t^2 + |u|^2)^{p+1}} - \frac{s}{(s^2t^2 + |u|^2)^{p+1}}$$

is a smooth function of t since $r, s > 0$ and t and u are not simultaneously zero.

(3) Finally, all the terms in the equation $d\boldsymbol{\sigma}_{r,s} = \boldsymbol{\tau}_r - \boldsymbol{\tau}_s$ are smooth and since the equation is valid on the open dense chart $V \subset \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ — apply Stokes theorem to the fibre $\underline{\mathbb{R}}_{r,s}$ of $\rho_{r,s}$ — it is valid on all of $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$. $\square$

Proposition 5.8. *Let* $\sigma_{r,s} := \nu^*(\boldsymbol{\sigma}_{r,s})$ *denote the pullback to* X *of the approximate spherical potential* $\boldsymbol{\sigma}_{r,s}$ *by an atomic section* ν *of* $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$. *Then the approximate potentials* $\sigma_{r,s}$ *converge to the* **spherical kernel** $\sigma_{\infty,0}$ *in* $L^1_{\text{loc}}(X)$ *as* $r \to \infty$ *and* $s \to 0$. *Let* $[t, u]$ *be the local coordinate expression for* ν *defined with respect to a positively oriented orthonormal local frame* e *for* V *by* $\nu = [t, ue]$. *Then the spherical kernel is given by*

$$\sigma_{\infty,0} \;=\; \tfrac{t}{|t|}\,\sigma, \tag{5.9}$$

where

$$\sigma\lambda \;=\; \tfrac{-1/2}{(n-1)!}\left(\tfrac{-1}{4\pi}\right)^{n-1}\frac{ue}{|u|}\left(e^t\Omega_V e - \frac{(Due)^2}{|u|^2}\right)^{n-1}. \tag{5.10}$$

Note. The spherical kernel σ on $V \sim X$ was defined by Chern [C2]. In the universal case the spherical kernel $\sigma_{\infty,0}$ is a bona fide L^1_{loc} form on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$. However its factors $\frac{t}{|t|}$ and σ are $\mathcal{O}$–twisted L^1_{loc} forms where $\mathcal{O} := \mathcal{O}(U)$. This corresponds to the fact that $\mathbb{P}(V)$ is a nonorientable submanifold of the orientable manifold $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$. Note that in the trivial case $\sigma = -\frac{1}{\gamma_{2n-2}}\,\Theta$ where Θ is the pullback to $\mathbb{R}^{2n-1}$ of the volume form on the unit sphere S^{2n-2} and $\gamma_{2n-2} = \text{Vol}(S^{2n-2})$.

Proof. Since ν is atomic, we know that $Z \cup P$ has Lebesgue measure zero in X and so $\sigma_{r,s} \to \sigma_{\infty,0}$ a.e. on X as $r \to \infty$ and $s \to 0$. Also by (5.7) $|\sigma_{r,s}| \le |\sigma_{\infty,0}|$ a.e. on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$. So to prove that $\sigma_{r,s} \to \sigma_{\infty,0}$ in $L^1_{\text{loc}}(X)$ it suffices to prove that $\sigma_{\infty,0} \in L^1_{\text{loc}}(X)$. To do this we replace u by $\frac{u}{t}$ in (5.5) to obtain

(5.11)

$$\begin{aligned}(\sigma_{\infty,0})\widetilde{\lambda} \;&=\; \sum_{p=0}^{n-1} 2K_{n,p}\,\frac{t}{|t|}\frac{ue(Due)^{2p}}{|u|^{2p+1}}\,(e^t\Omega_V e)^{n-p-1}\int_0^\infty \frac{|t|\,|u|^{2p+1}dse_0}{(s^2t^2+|u|^2)^{p+1}} \\ &=\; \frac{t}{|t|}\sum_{p=0}^{n-1} 2K_{n,p}\,\frac{ue(Due)^{2p}}{|u|^{2p+1}}\,(e^t\Omega_V e)^{n-p-1}\int_0^\infty \frac{dv}{(1+v^2)^{p+1}}\end{aligned}$$

which defines an $L^1_{\text{loc}}(X)$ form since $P = \{t = 0\}$ has measure zero and $\frac{du^I}{|u|^{2p}} \in L^1_{\text{loc}}(X)$ for $p \le n-1$ by the atomicity of ν. Finally (5.10) is obtained from (5.11) using [HL: IV,2.8–9]. $\square$

When its rank is one V, being oriented, is the trivial bundle $\underline{\mathbb{R}}$ and we study atomic maps $f : X \to \mathbb{RP}^1$. The zero and pole divisors of f are the extensions to all of X of the induced functions from $X \sim P$ and $X \sim Z$ to $\mathbb{R}$. In particular they are both codimension 1 currents on X.

Theorem 5.12. *Let $f : X \to \mathbb{RP}^1$ be atomic. Let $\{Z_j\}$ and $\{P_i\}$ denote the orientable connected components of* $\operatorname{Reg} Z$ *and* $\operatorname{Reg} P$. *Then we can choose orientations on Z_j and P_i so that*

$$d\sigma_{\infty,0} = \sum_i [P_i] - \sum_j [Z_j] \qquad \text{on } X \sim \operatorname{Sing}(Z \cup P) \tag{5.13}$$

is the current limit of the smooth "transgression" formula $d\sigma_{r,s} = \tau_r - \tau_s$ on X.

Note. Pairing the currents in (5.13) with the test function $[a, b]$ on $X = \mathbb{R}$ yields the signum principle discussed in the introduction.

Proof. We compute $d\sigma_{\infty,0}$ using the system of positively oriented charts $\psi_0(u) = [1, u]$ and $\psi_1(t) = [t, -1]$ for $\mathbb{RP}^1$. Write $f = [t, u]$. Now by Proposition 5.6 $\sigma_{\infty,0} = -\frac{1}{2}\frac{t}{|t|}\frac{u}{|u|}$. On $X \sim P$ we set $t = 1$ to get $\sigma_{\infty,0} = -\frac{1}{2}\frac{u}{|u|}$ so that $d\sigma_{\infty,0} = -\operatorname{Div}(u) = -\operatorname{Div}_0(\nu)$, while on $X \sim Z$ we set $u = -1$ so that $d\sigma_{\infty,0} = \operatorname{Div}(t) = \operatorname{Div}_\infty(\nu)$. Combining these two equations proves the theorem. $\square$

Next we define the residue density and the Euler currents in the universal case.

Definition 5.14. The **residue density**, $\operatorname{Res} \in \Omega^{2n-2}(\mathbb{P}(V), \mathcal{O})$, is defined to be the $\mathcal{O}$–twisted form on $\mathbb{P}(V)$ corresponding to the odd form $-2\,\sigma$ on $S(V)$ (see (3.1)).

Note. Recall that $\mathcal{O}$ is the orientation bundle of the tautological bundle $U \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ and that the restriction of U to $\mathbb{P}(V)$ is the tautological bundle $L \to \mathbb{P}(V)$.

Theorem 5.15.

(1) The limit as $r \to \infty$ and $s \to 0$ of the equation $d\boldsymbol{\sigma}_{r,s} = \boldsymbol{\tau}_r - \boldsymbol{\tau}_s$ between smooth forms

on $S(\underline{\mathbb{R}} \oplus V)$ is the current equation

$$(5.16) \qquad d\boldsymbol{\sigma}_{r,s} \;=\; -2\sigma\,[S(V)] \;-\; [X_+] \;-\; [X_-] \qquad \text{on } S(\underline{\mathbb{R}} \oplus V),$$

where $[X_\pm]$is defined as in Theorem 4.3 and σ is given by (5.10).

(2) The limit as $r \to \infty$ and $s \to 0$ of the equation $d\boldsymbol{\sigma}_{r,s} = \boldsymbol{\tau}_r - \boldsymbol{\tau}_s$ between smooth forms on $\mathbb{P}(\underline{\mathbb{R}} \oplus V)$ is the current equation

$$(5.17) \qquad d\sigma_{\infty,0} \;=\; \mathrm{Res}\,[\mathbb{P}(V)] \;-\; [X] \qquad \text{on } \mathbb{P}(\underline{\mathbb{R}} \oplus V),$$

where $[\mathbb{P}(V)]$ is the distributional section of $\Omega^*(\mathbb{P}(\underline{\mathbb{R}} \oplus V), \mathcal{O})$ which integrates $\mathcal{O}$–twisted forms over the nonorientable manifold $\mathbb{P}(V)$. In particular the Euler currents are given by

$$\boldsymbol{\tau}_0 \;=\; [X] \qquad \text{and} \qquad \boldsymbol{\tau}_\infty \;=\; \mathrm{Res}\,[\mathbb{P}(V)].$$

Proof.

(1) The proof is exactly the same as that of Theorem 4.3(1) except that (4.17) is replaced by $d\sigma_{\infty,0}(u) = -[X]$.

(2) To prove (5.17) we push (5.16) forward by $p : S(\underline{\mathbb{R}} \oplus V) \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$. Note that $p_*(-2\sigma[S(V)]) = \mathrm{Res}\ \widetilde{p}_*([S(V)]) =: \mathrm{Res}\,[\mathbb{P}(V)]$, by Lemma 3.3 (5).

Remark 5.18. Just as in the even rank case we can pullback (5.16) using an atomic function $f : X \to S^{2n-1}$ and pair the resulting current equation on X with the test function $[X]$ to obtain the theorem stated in the introduction.

Definition 5.19. Let $\nu : X \to \mathbb{P}(\underline{\mathbb{R}} \oplus V)$ be atomic. Let $\mathbf{L}^*$ denote the pullback to $X \sim Z$ of the dual of the tautological bundle $L \to \mathbb{P}(V)$ via the induced map from $X \sim Z$ to $\mathbb{P}(V)$. Note that $\mathbf{L}^*$ may be nonorientable. The **pole divisor**, $\mathrm{Div}_\infty(\nu)$, is defined to be the extension to all of X of the divisor of the induced section $\nu : X \sim Z \to \mathbf{L}^*$. In particular $\mathrm{Div}_\infty(\nu) \in \Omega^1(X, \nu^*\mathcal{O})'$, where $\mathcal{O} := \mathcal{O}(U)$.

Note. See Propositions 3.5 and 3.8 for some results concerning the geometric structure of the pole divisor. In particular, if $\nu^*\mathcal{O} \not\cong \underline{\mathbb{R}}$, the pole divisor can have support on the nonorientable components of the pole set.

Let $f : X \to Y$ be a smooth map. We can use the local description of $\mathcal{O}$–twisted forms given in Remark 2.7 to pullback $\mathcal{O}$–twisted forms on Y to $f^*\mathcal{O}$–twisted forms on X.

Definition 5.20. Let ν be a smooth section of $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$, and let $p : \mathbb{P}(\underline{\mathbb{R}} \oplus V) \sim X \to \mathbb{P}(V)$ be the natural projection. The **residue density**, $\nu^* \operatorname{Res}$, is defined to be the $\nu^*\mathcal{O}$–twisted form

$$\nu^* \operatorname{Res} := (p \circ \nu)^* \operatorname{Res} \qquad \text{on } X$$

where $\operatorname{Res} \in \Omega^{2n-2}(\mathbb{P}(V), \mathcal{O})$ is given by (5.14).

Finally we are in a position to state and prove the main theorem of this section.

Theorem 5.21. *Let V be an odd rank oriented real vector bundle over an oriented manifold X, and let ν be an atomic section of $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \to X$. Then the Euler currents at time zero and time infinity exist on X and are given by*

$$\tau_0 = \operatorname{Div}_0(\nu) \qquad \textit{and} \qquad \tau_\infty = \nu^* \operatorname{Res} \operatorname{Div}_\infty(\nu).$$

Furthermore the current equation

$$d\sigma_{\infty,0} = \nu^* \operatorname{Res} \operatorname{Div}_\infty(\nu) \;-\; \operatorname{Div}_0(\nu) \qquad \textit{on } X \tag{5.22}$$

is the current limit of the smooth "transgression" formula $d\sigma_{r,s} = \tau_r - \tau_s$ on X.

Corollary 5.23. *Let X be a compact oriented $(2n-1)$–dimensional manifold with boundary ∂X and let $f : X \to \mathbb{P}(\mathbb{R} \oplus \mathbb{R}^{2n-1})$ be a smooth map. Suppose that f vanishes algebraically and that its zero set consists of a finite number of points, $Z = \{z_1, ...z_N\}$. Suppose (for the* *sake of simplicity) that the pole set $P = f^{-1}(\mathbb{P}(\mathbb{R}^{2n-1}))$ is a smooth connected $(2n-2)$–dimensional submanifold of X and that $(Z \cup P) \cap \partial X = \varnothing$. Then*

$$\sum_{i=1}^{N} \mathrm{Mult}(f, z_i) \;-\; \mathrm{Deg}(f\,\big|_P) \;=\; \mathrm{Wind}(f, \partial X).$$

Here $\mathrm{Mult}(f, z_i)$ and $\mathrm{Wind}(f, \partial X)$ are defined just as in Theorem 4.14. However the definition of $\mathrm{Deg}\, f\,\big|_P$ needs to be modified to account for the fact that $\mathbb{P}(\mathbb{R}^{2n-1})$ is not orientable (see below).

Definition 5.24. Suppose that $\nu^*\mathcal{O} \cong \mathcal{O}(TP)$. Let $p : \widetilde{P} \to P$ be the oriented double cover of P. Then the mapping $f : P \to \mathbb{P}(\mathbb{R}^{2n-1})$ lifts to a unique orientation preserving map $\widetilde{f} : \widetilde{P} \to S(\mathbb{R}^{2n-1})$ and we define $\mathrm{Deg}(f\,\big|_P) := \mathrm{Deg}(\widetilde{f}\,\big|_{\widetilde{P}})$. If $\nu^*\mathcal{O} \not\cong \mathcal{O}(TP)$ set $\mathrm{Deg}(f\,\big|_P) := 0$.

Proof of Corollary 5.23. As usual we pair the current equation

$$df^*(\sigma_{\infty,0}) \;=\; \mathrm{Div}_\infty(f)\, f^*\,\mathrm{Res} \;-\; \mathrm{Div}_0(f) \qquad \text{on } X$$

with the test fuction $[X]$. We just need to check that $\langle \mathrm{Div}_\infty(f)\, f^*\,\mathrm{Res}\,,\, 1\rangle = \mathrm{Deg}(f\,\big|_P)$.

Suppose that P is orientable. If $\nu^*\mathcal{O} \not\cong \underline{\mathbb{R}}$ in a neighbourhood of P then, by Proposition 3.8 (1), $\mathrm{Div}_\infty(f) = 0$ as required. If $\nu^*\mathcal{O} \cong \underline{\mathbb{R}}$ near P let $\widetilde{f} : \widetilde{P} \to S^{2n-2}$ be the orientation preserving map covering $f : P \to \mathbb{RP}^{2n-2}$. Here $p : \widetilde{P} \to P$ is the trivial 2–1 cover. Then $\mathrm{Div}_\infty(f) = p_*(\mathrm{Div}_\infty(\widetilde{f})) = p_*[\widetilde{P}] = [P]$ and $(f^*\,\mathrm{Res})^{\mathrm{odd}} = \widetilde{f}^*(-2\sigma)$. So

$$\begin{aligned}
\langle \mathrm{Div}_\infty(f)\, f^*\,\mathrm{Res}\,,\, 1\rangle &= \left\langle p_*\left(\mathrm{Div}_\infty(\widetilde{f})\,(f^*\,\mathrm{Res})^{\mathrm{odd}}\right)\,,\, 1\right\rangle \\
&= \tfrac{1}{2}\left\langle \mathrm{Div}_\infty(\widetilde{f})\,,\, \widetilde{f}^*(-2\sigma)\right\rangle \\
&= \left\langle \widetilde{f}_*[\widetilde{P}]\,,\, -\sigma\right\rangle \\
&= \mathrm{Deg}(\widetilde{f}\,\big|_{\widetilde{P}}) = \mathrm{Deg}(f\,\big|_P),
\end{aligned}$$

as required.

Next suppose that P is nonorientable and let $\widetilde{P} \to P$ be the oriented double cover. If $\nu^*\mathcal{O} \not\cong \mathcal{O}(TP)$ then, by Proposition 3.8 (2), $\mathrm{Div}_\infty(f) = 0$. If $\nu^*\mathcal{O} \not\cong \mathcal{O}(TP)$ let $\widetilde{f} : \widetilde{P} \to S^{2n-2}$ be the orientation preserving map covering f. Now proceed as above. $\square$

Proof of Theorem 5.21. First we prove (5.22). Consider the induced section $\nu : X \sim P \to V$ and note that $\sigma_{\infty,0} = \sigma$ on $X \sim P$. Now σ is a homogeneous form of degree $\leq m-1$ in the fibre differentials which is d–closed on $X \sim Z$ and has residue -1 (see [HL: IV, 2.5(5)]). So by the general Residue Theorem 1.10 of [HL: III]

$$d\sigma_{\infty,0} = -\,\mathrm{Div}_0(\nu) \qquad \text{on } X \sim P.$$

To check (5.22) on $Y := X \sim Z$ we proceed as follows. On $\mathbb{P}(\underline{\mathbb{R}} \oplus V) \sim X$, $\frac{1}{2}\frac{t}{|t|}$ and Res are $\mathcal{O}$–twisted forms and the product $\boldsymbol{\sigma}_{\infty,0} = \frac{1}{2}\frac{t}{|t|}$ Res is a bona fide L^1_{loc} form. Pulling this back to Y by ν gives

$$\sigma_{\infty,0} = \tfrac{1}{2}\tfrac{t}{|t|}\ \mathrm{Res} \qquad \text{on } Y$$

where $\nu = [t, ue]$. Let $p : \widetilde{Y} \to Y$ be the bundle of orientations of $\mathbf{L}^*$ and let $\widetilde{\nu}$ be the induced section of $\widetilde{\mathbf{L}}^* \to \widetilde{Y}$. Now by Lemma 3.3 (4)

$$p^*\sigma_{\infty,0} = \tfrac{1}{2}\tfrac{\widetilde{t}}{|\widetilde{t}|}\,\widetilde{\nu^*\,\mathrm{Res}} \qquad \text{on } \widetilde{Y}.$$

Since $\mathrm{Div}_\infty(\widetilde{\nu}) = d\left(\frac{1}{2}\frac{\widetilde{t}}{|\widetilde{t}|}\right)$ on $\widetilde{Y}$ and $d\sigma = 0$ on Y we have the current equation

$$dp^*\sigma_{\infty,0} = \widetilde{\nu^*\,\mathrm{Res}}\ \mathrm{Div}_\infty(\widetilde{\nu}) \qquad \text{on } \widetilde{Y}. \tag{5.25}$$

Note that $p^*\sigma_{\infty,0}$ is an even form, $\widetilde{\nu^*\,\mathrm{Res}}$ is an odd form and $\mathrm{Div}_\infty(\widetilde{\nu})$ is an odd current on $\widetilde{Y}$. To prove (5.22) we push (5.25) down to Y via the mapping

$$p_* : \Omega^*_{\mathrm{even}}(\widetilde{Y})' \to \Omega^*(Y)'$$

defined by $\langle p_* S, \omega\rangle := \frac{1}{2}\langle S, p^*\omega\rangle$ (see (3.2)). So by (2a) and (5) of Lemma 3.3 we have

$$
\begin{aligned}
d\sigma_{\infty,0} &= d\, p_*(p^*\sigma_{\infty,0}) \\
&= p_*\, d(p^*\sigma_{\infty,0}) \\
&= p_*(\widetilde{\nu^* \operatorname{Res}} \operatorname{Div}_\infty(\widetilde{\nu})) \\
&= \nu^* \operatorname{Res} \widetilde{p}_* \left(\operatorname{Div}_\infty(\widetilde{\nu})\right) \\
&= \nu^* \operatorname{Res} \operatorname{Div}_\infty(\nu)
\end{aligned}
$$

on $X \sim Z$ as required.

Finally we compute the Euler currents. Firstly $\tau_0 = \operatorname{Div}_0(\nu)$ on X since we already know that it is true on $X \sim P$ and $\tau_0 = \lim_{s\to 0} \tau_s = 0$ converges in the C^∞ topology on $X \sim Z$ as can be seen by inspecting the following formula for τ_s (c.f. [HL: IV,2.7]).

$$
\tau_s \lambda = s \sum_{p=0}^{n-1} K_{n,p} \; \frac{(t(Due)^{2p+1} - (Due)^{2p}\, udte)\, (e^t \Omega_V e)^{n-p-1}}{(|u|^2 + s^2 |t|^2)^{p+1}}.
$$

So

$$
\tau_\infty = \lim_{r\to\infty} \tau_r = \lim_{r\to\infty} d\sigma_{r,0} + \operatorname{Div}_0(\nu) = \nu^* \operatorname{Res} \operatorname{Div}_\infty(\nu). \quad \square
$$

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