Exercise 1. Consider the following recursively defined function \( f : \mathbb{Z} \rightarrow \mathbb{Z} \).
\[
f(x) = (x = 0 \rightarrow 0 \mid x > 0 \rightarrow 2 - f(1 - x) \mid x < 0 \rightarrow f(-x))
\]
Find a closed-form definition of \( f \) and prove your answer.

To find a closed-form definition (i.e., one that is non-recursive and does not use \texttt{fix})

it is often useful to define functional \( F \) and then construct the graph of the least fixed point of \( F \). Recall that functional \( F \) is defined by
\[
F(g) = \lambda x . (x = 0 \rightarrow 0 \mid x > 0 \rightarrow 2 - g(1 - x) \mid x < 0 \rightarrow g(-x))
\]

The graph of the least fixed point of \( F \) is the set of input-output pairs that comprises \texttt{fix}(\( F \)). We can construct it incrementally by applying \( F \) to itself starting with \( \perp \):
\[
\begin{align*}
F^0(\perp) &= \{\} \\
F^1(\perp) &= \{(0, 0)\} \\
F^2(\perp) &= \{(0, 0), (1, 2)\} \\
F^3(\perp) &= \{(-1, 2), (0, 0), (1, 2)\} \\
F^4(\perp) &= \{(-1, 2), (0, 0), (1, 2), (2, 0)\} \\
F^5(\perp) &= \{(-2, 0), (-1, 2), (0, 0), (1, 2), (2, 0)\} \\
F^6(\perp) &= \{(-2, 0), (-1, 2), (0, 0), (1, 2), (2, 0), (3, 2)\} \\
F^7(\perp) &= \{(-3, 2), (-2, 0), (-1, 2), (0, 0), (1, 2), (2, 0), (3, 2)\}
\end{align*}
\]

As you can see, eventually a pattern starts to emerge. Function \( f \) appears to return 2 on odd inputs and 0 on even inputs. Thus, we conjecture that \( f = h \) where \( h \) is the following closed-form definition:
\[
h(x) = \begin{cases} 
2 & \text{if } x \text{ is odd} \\
0 & \text{if } x \text{ is even}
\end{cases}
\]

This does not constitute a proof; it is merely a conjecture. We can prove the \( f \subseteq h \) half of the conjecture using fixed point induction.

**Proof.** Define property \( P \) by \( P(g) \equiv \forall x \in g^\rightarrow . g(x)=h(x) \). We wish to prove \( P(f) \). Define functional \( F \) as above, and observe that \texttt{fix}(\( F \)) = \( f \) by the definition of recursion. Thus, to prove \( P(f) \) it suffices to prove \( P(\texttt{fix}(F)) \) by fixed-point induction.
**Base Case:** $P(\bot)$ holds vacuously.

**Inductive Hypothesis:** Assume that $P(g)$ holds for some arbitrary function $g$. That is, assume that $\forall x \in g^- . g(x) = h(x)$.

**Inductive Case:** We will prove that $P(F(g))$ holds. Let $x \in F(g)^-$ be given. Looking at the definition of $F$, there are three cases to consider:

**Case 1:** Suppose $x = 0$. Then by definition of $F$, $F(g)(x) = 0 = h(x)$.

**Case 2:** Suppose $x > 0$. Then by definition of $F$, $F(g)(x) = 2 - g(1 - x)$. By inductive hypothesis, $g(1 - x) = 2$ if $1 - x$ is odd and $0$ if $1 - x$ is even. If $x$ is odd then $1 - x$ is even, so $g(1 - x) = 0$; thus $2 - g(1 - x) = 2 = h(x)$. If $x$ is even then $1 - x$ is odd, so $g(1 - x) = 2$; thus $2 - g(1 - x) = 0 = h(x)$. Either way, $F(g)(x) = 2 - g(1 - x) = h(x)$.

**Case 3:** Suppose $x < 0$. Then by definition of $F$, $F(g)(x) = g(-x)$. By inductive hypothesis, $g(-x) = 2$ if $-x$ is odd and $0$ if $-x$ is even. Since $-x$ has the same parity as $x$, it follows that $F(g)(x) = 2$ if $x$ is odd and $0$ if $x$ is even. Hence, $F(g)(x) = h(x)$.

Functions of multiple arguments can be treated as functions of a single pair argument.

**Exercise 2.** Consider the following recursively defined function $f : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$.

$$f(x, y) = (x=0 \to y \mid y=0 \to x \mid x,y>0 \to f(x-1,y-1)+1)$$

Prove that $f \subseteq \text{max}$.

**Proof.** Define property $P$ by $P(g) \equiv \forall(x,y) \in g^- . g(x,y) = \text{max}(x,y)$. We wish to prove $P(f)$. Define functional $F$ in the usual way:

$$F(g) = \lambda(x,y). (x=0 \to y \mid y=0 \to x \mid x,y>0 \to g(x-1,y-1)+1)$$

To prove $P(f)$ it suffices to prove $P(\text{fix}(F))$ by fixed-point induction.

**Base Case:** $P(\bot)$ holds vacuously.

**Inductive Hypothesis:** Assume that $P(g)$ holds for some arbitrary function $g$. We will prove that $P(F(g))$ holds. Let $(x, y) \in F(g)^-$ be given.

**Case 1:** Suppose $x = 0$. Then by definition of $F$, $F(g)(x, y) = y = \text{max}(x, y)$.

**Case 2:** Suppose $y = 0$. Then by definition of $F$, $F(g)(x, y) = x = \text{max}(x, y)$.

**Case 3:** Suppose $x, y > 0$. Then by definition of $F$, $F(g)(x, y) = g(x - 1, y - 1) + 1$. By inductive hypothesis, $F(g)(x) = \text{max}(x - 1, y - 1) + 1$. If $x \geq y$ then $\text{max}(x - 1, y - 1) = x - 1$, so $F(g)(x, y) = x - 1 + 1 = x$. If $x < y$ then $\text{max}(x - 1, y - 1) = y - 1$, so $F(g)(x, y) = y - 1 + 1 = y$. In either case $F(g)(x, y) = \text{max}(x, y)$.