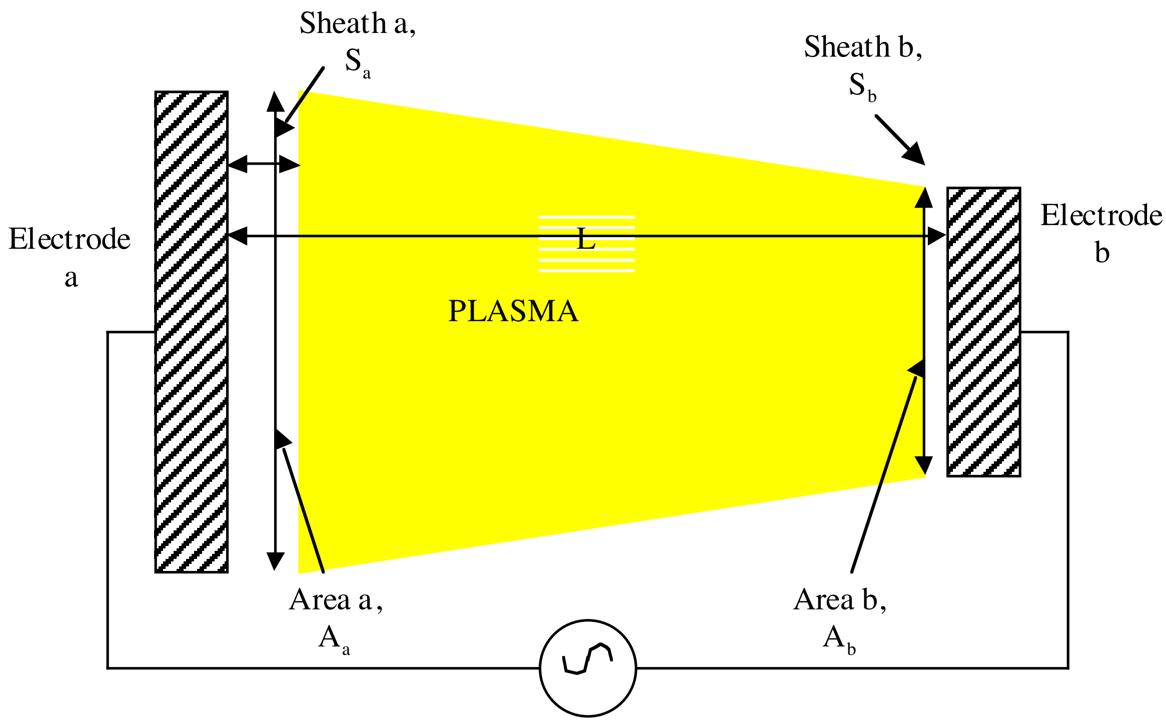
Lecture 10 Capacitively Coupled Plasmas

New homework problems:
Lieberman 11.1 11.2 and 11.3 Due April 16th, 2001. – Note change!



Basic setup.

First, we will start with the simplest model of a CCP system. To do this requires a few simplifying assumptions.

1) The frequency as which the electric field is applied is such that

$$\omega_{pi}^2 << \omega^2 << \omega_{pe}^2 \quad \left(= \frac{ne^2}{\varepsilon_0 m}\right)$$

$$\omega^2 \geq \nu_m^2$$

Effectively this implies that the ions don't respond to the instantaneous electric field but the electrons do respond. Further, the electrons don't collide very often in a cycle of the applied field. Thus the electrons can carry the power put into the plasma from the **E**.

2) The areas of the electrodes are equal, $A_a = A_b$, and large enough so that the **E** and **J** are only directed normal to the electrodes. This implies no transverse fields, which is strictly not correct but in some systems a reasonable approximation. When taken into account with (1) this implies that the total current, free charge current and displacement current is constant across the discharge.

3) The electron density is zero in the sheath – implying that the time-average sheath width, $\langle S \rangle$, is large compared to the Debye length. Further that Child's Law is a good approximation for the sheaths on each side.

4) The sheaths on both sides expand and contract with the applied field but the sum of the sheath widths is a constant. $2\langle S\rangle = S_a(t) + S_b(t) = const$
5) Finally, we will assume that the ion density is uniform across the discharge, including the sheath and the electrons are uniform except for the sheaths.

Now, we can start looking at the total current that flows through the plasma. At sheath a (this also applies to sheath b) we find

$$\partial_x E = \frac{e}{\varepsilon_0} n_i \quad x \le S_a(t)$$
$$\partial_x E = -\frac{e}{\varepsilon_0} n_i \quad x \ge S_b(t)$$

We can integrate and use the boundary condition $E(S,t) \approx 0$ to show that

$$E(x,t) = \frac{e}{\varepsilon_0} n_i (x - S_a) \quad x \le S_a(t)$$
$$E(x,t) = \frac{e}{\varepsilon_0} n_i (S_b - x) \quad x \ge S_b(t)$$

Now, we can get the displacement current in the sheaths.

$$I_{displace}(t) = A_{a/b} j_{displace}(t) = \varepsilon_0 A_{a/b} \partial_t E(t)$$
$$\Downarrow$$
$$I_{displace}(t)\Big|_a = eA_a n_i \partial_t (x - S_a) \quad x \le S_a(t)$$
$$I_{displace}(t)\Big|_b = eA_b n_{ii} \partial_t (S_b - x) \quad x \ge S_b(t)$$

Now at the sheath boundary, the current changes from displacement current to free current in terms of electron motion. We already know that the free current is the form of

$$\begin{aligned} I_{free}(t) &= A_{a/b} j_{free} \\ &= -eA_{a/b} n_e \operatorname{Re}\{\tilde{v} e^{i\omega t}\} \\ &= -eA_{a/b} n_e v_0 \cos(\omega t) \end{aligned}$$

Thus,

$$\begin{aligned} I_{displace}(t)\Big|_a &= I_{free}(t)\Big|_a \\ &= -eA_a n_e v_0 \cos(\omega t) \\ &= eA_a n_i \partial_t (x - S_a) \\ &\Downarrow \qquad\qquad x \le S_a(t) \\ \partial_t S_a &= -\overbrace{\left(\frac{n_e}{n_i}\right)}^{\approx 1} v_0 \cos(\omega t) \\ &\Downarrow \\ S_a &= \langle S_a \rangle - S_{a0} \sin(\omega t) \end{aligned}$$

and

$$I_{displace}(t)\big|_b = I_{free}(t)\big|_b$$
$$= -eA_b n_e v_0 \cos(\omega t)$$
$$= eA_b n_i \partial_t (S_b - x)$$
$$\Downarrow \qquad\qquad x \ge S_b(t)$$
$$\partial_t S_b = -\overbrace{\left(\frac{n_e}{n_i}\right)}^{\approx 1} v_0 \cos(\omega t)$$
$$\Downarrow$$
$$S_b = \langle S_b \rangle + S_{b0}\sin(\omega t)$$

where $S_{a/b0} = \dfrac{v_0}{\omega}$ and the phase difference between sheath edges. Further we note that $S_a + S_b = const$. Now, we can get the sheath potential directly from the electric field.

$$V(t)\big|_a = \int_0^{S_a} E(x,t)dx$$
$$= \frac{e}{\varepsilon_0} n_i \left(\frac{1}{2}x^2 - S_a x + const\right)\Bigg|_0^{S_a}$$
$$= -\frac{e}{2\varepsilon_0} n_i S_a^2$$

where $const = 0$ is determined by making $V(x=0,t)=0$. Likewise,

$$V(t)\big|_b = \int_{S_b}^{l} E(x,t)dx$$
$$= -\frac{e}{\varepsilon_0} n_i \left(\frac{1}{2}x^2 - S_b x + const\right)\Bigg|_{S_b}^{l}$$
$$= -\frac{e}{2\varepsilon_0} n_i S_b^2$$

where $const = -\dfrac{1}{2}l^2 + S_b l$ from $V(x=l,t)=0$. Both sheath voltages are nonlinear in time, as can be easily determined from the above equations.

$$V(t)\big|_a = -\frac{e}{2\varepsilon_0} n_i S_a^2$$
$$= -\frac{e}{2\varepsilon_0} n_i \left(\langle S_a \rangle - S_{a0}\sin(\omega t)\right)^2$$
$$= -\frac{e}{2\varepsilon_0} n_i \left(\langle S_a \rangle^2 + S_{a0}^2 \sin^2(\omega t) - 2\langle S_a \rangle S_{a0}\sin(\omega t)\right)$$

likewise

$$
\begin{aligned}
V(t)\big|_b &= -\frac{e}{2\varepsilon_0} n_i S_b^2 \\
&= -\frac{e}{2\varepsilon_0} n_i \left(\langle S_b\rangle + S_{b0}\sin(\omega t)\right)^2 \\
&= -\frac{e}{2\varepsilon_0} n_i \left(\langle S_b\rangle^2 + S_{b0}^2\sin^2(\omega t) + 2\langle S_b\rangle S_{b0}\sin(\omega t)\right)
\end{aligned}
$$

Like the other electrode, the bias across the sheath is non-linear with the applied frequency. Now the voltage between the electrodes is given by

$$
\begin{aligned}
V(t)\big|_{plasma} &= V(t)\big|_a - V(t)\big|_b{}^2 \\
&= -\frac{e}{2\varepsilon_0} n_i \left(\langle S_b\rangle^2 + S_{b0}^2\sin^2(\omega t) + 2\langle S_b\rangle S_{b0}\sin(\omega t)\right) \\
&\quad + \frac{e}{2\varepsilon_0} n_i \left(\langle S_a\rangle^2 + S_{a0}^2\sin^2(\omega t) - 2\langle S_a\rangle S_{a0}\sin(\omega t)\right) \\
&= -\frac{2e}{\varepsilon_0} n_i \langle S_b\rangle S_{b0}\sin(\omega t)
\end{aligned}
$$

which is linear in the voltage response.

Now let us consider a system in which the areas of the two electrodes are unequal.

1) For the purposes of our example we will assume that $A_a > A_b$.
2) As before, we will assume that the density of the ions is constant across the plasma. (Diffusion of the ions will insure that this is not far form the truth.) Because of this we find that the current densities at each electrode are the same. We will further assume that the current density is constant across the face of each of the electrodes. Thus we find that $j_a \approx j_b$.
3) We will assume that the electrons are excluded from the sheath regions. This implies that we can assume that the sheaths can be adequately modeled with Child's Law. Thus,

$$
\begin{aligned}
(V)^{3/4} &= \frac{3}{2}\left(\frac{j_i}{\varepsilon}\right)^{1/2}\left(\frac{M_i}{2e}\right)^{1/4} S \\
&= \frac{3}{2}\left(\frac{M_i j_i^2}{2e\varepsilon^2}\right)^{1/4} S
\end{aligned}
$$

We can rearrange this to show that

$$
j_i = \frac{4\sqrt{2}}{9}(V)^{3/2}\frac{e^{1/2}\varepsilon}{M_i^{1/2}S^2}
$$

(Note our sign change on the potential.)

4) Each sheath region has a capacitance associated with it that is proportional to the electrode area divided by the sheath width. $C \propto \dfrac{A}{S}$

5) The rf voltage is capacitively divided between the two sheaths. Thus

$$\frac{V_a}{V_b} = \frac{C_b}{C_a}.$$

This make sense as the charge, CV, on the 'sheath/capacitor' is the same on each sheath. Combining the last two assumptions we find that

$$\frac{V_a}{V_b} = \frac{A_b}{S_b}\frac{S_a}{A_a}$$

Further using 2 and 3 we find,

$$\begin{aligned} j_a &= \frac{4\sqrt{2}}{9}(V_a)^{3/2}\frac{e^{1/2}\varepsilon}{M_i^{1/2}S_a^2} \\ &\approx j_b \\ &= \frac{4\sqrt{2}}{9}(V_b)^{3/2}\frac{e^{1/2}\varepsilon}{M_i^{1/2}S_b^2} \\ &\Downarrow \end{aligned}$$

$$(V_a)^{3/2}\frac{1}{S_a^2} = (V_b)^{3/2}\frac{1}{S_b^2}$$

Now we can combine all of the pieces to get

$$\frac{V_b}{V_a} = \left(\frac{A_a}{A_b}\right)^4$$

This means that the smaller electrodes result in larger voltages. This effect is very strong – 4th power! Further a substantial voltage can be generated on even grounded surfaces! (This potential is of course relative to the plasma potential.)

Plasma Heating

Now we want to know how power is deposited into the plasma. Clearly, we are using the electric field to accelerate and decelerate the electrons. However, to 'permanently' deposit the energy, we need some mechanism with which to extract the energy from the electrons, so that they do not give the energy back to the electric field. The simplest mechanism is via ionizing collisions with the neutral gas. Such collisions are reminiscent of resistance to current flow in a wire. This resistance leads naturally to Ohmic heating of the wire. There are two known method from which the electrons gain energy, the first is the general sloshing of the electrons back and forth between the electrodes. This is in effect a bulk motion of the electrons in response to the push and pull of the electric fields. The second method of heating the electrons is known as 'stochastic heating.' Stochastic heating is not unlike hitting a ping pong ball with a paddle. Initially, the paddle is moving at a much higher velocity than the ball but after the collision occurs the ball's velocity greatly exceeds that of the paddle. We will examine Ohmic heating initially.

Ohmic heating

It can be shown that the time-averaged power obtained via Ohmic heating can be described as

$$p_{ave} = \frac{1}{T}\int_0^T \mathbf{J}_{total}(t)\,\Sigma\,\mathbf{E}(t)\,dt$$

$$= \frac{1}{2}\mathrm{Re}\left[\mathbf{J}_{total}(t)\,\Sigma\,\mathbf{E}^*(t)\right]$$

To get at this, we need to know the electric field and the current density.

First, we know that the plasma is driven with an rf electric field. We can model this field as a sinusoidal variation,

$$E = \mathrm{Re}\,\tilde{E}e^{i\omega t}.$$

This field will accelerate the electrons

$$m\frac{dv}{dt} = qE - m\nu_m v$$

where ν_m is the electron-neutral collision frequency. Hence the last term is simply the resistive drag term that we need to have to transfer the power from the electrons to the neutrals and the ions. Now assuming that the electron velocity is also sinusoidal, e.g. they follow the electric field.

$$v = \mathrm{Re}\,\tilde{v}e^{i\omega t}$$

Then we find that

$$mi\omega\,\mathrm{Re}\,\tilde{v}e^{i\omega t} = q\,\mathrm{Re}\,\tilde{E}e^{i\omega t} - m\nu_m\,\mathrm{Re}\,\tilde{v}e^{i\omega t}$$

$$\Downarrow$$

$$\tilde{v} = \frac{q}{m}\frac{1}{(i\omega + \nu_m)}\tilde{E}$$

This is of course related to the free current density

$$\tilde{J}_{free} = qn\tilde{v} = \frac{nq^2}{m}\frac{1}{(i\omega + \nu_m)}\tilde{E}$$

$$= \varepsilon_0\omega_{ps}^2\frac{1}{(i\omega + \nu_m)}\tilde{E}$$

$$= \sigma_p\tilde{E}$$

Where $\sigma_p = \varepsilon_0\omega_{ps}^2\frac{1}{(i\omega + \nu_m)}$ is the plasma conductivity.

Likewise, we have the displacement current density

$$\tilde{J}_{displace} = \varepsilon_0\partial_t\tilde{E} = i\varepsilon_0\omega\tilde{E}e^{i\omega t}.$$

Thus the total current density is simply

$$
\begin{aligned}
\nabla \wedge \tilde{H} &= \tilde{J}_{total} = \tilde{J}_{free} + \tilde{J}_{displace} \\
&= \left[\frac{nq^2}{m} \frac{1}{(i\omega + \nu_m)} + i\varepsilon_0 \omega \right] \tilde{E} \\
&= i\omega\varepsilon_0 \left[1 - \frac{\omega_{ps}^2}{(\omega^2 - i\omega\nu_m)} \right] \tilde{E} \\
&= \left[i\omega\varepsilon_0 + \frac{\varepsilon_0 \omega_{ps}^2}{(i\omega + \nu_m)} \right] \tilde{E} \\
&= \left[i\omega\varepsilon_0 + \sigma_p \right] \tilde{E} \\
&\quad - \text{ or } - \\
&= i\omega\varepsilon_p \tilde{E} = i\omega\varepsilon_0 \kappa_p \tilde{E}
\end{aligned}
$$

where ε_p is know as the plasma dielectric constant.

Now, we can determine the power deposited into the plasma.

$$
\begin{aligned}
p_{ave} &= \frac{1}{T} \int_0^T \mathbf{J}_{total}(t) \Sigma \mathbf{E}(t) dt \\
&= \frac{1}{2} \operatorname{Re}\left[\mathbf{J}_{total}(t) \Sigma \mathbf{E}^*(t) \right] \\
&= \frac{1}{2} \operatorname{Re}\left[i\omega\varepsilon_0 \left[1 - \frac{\omega_{ps}^2}{(\omega^2 - i\omega\nu_m)} \right] \tilde{E}^2 \right] \\
&= \frac{1}{2} \operatorname{Re}\left[\left[i\omega\varepsilon_0 + \frac{\varepsilon_0 \omega_{ps}^2}{(i\omega + \nu_m)} \right] \tilde{E}^2 \right] \\
&= \frac{1}{2} \operatorname{Re}\left[\left[i\omega\varepsilon_0 + \frac{\varepsilon_0 \omega_{ps}^2 (\nu_m - i\omega)}{(\omega^2 + \nu_m^2)} \right] \tilde{E}^2 \right] \\
&= \frac{1}{2} \operatorname{Re}\left[\left[\frac{\varepsilon_0 \omega_{ps}^2 \nu_m}{(\omega^2 + \nu_m^2)} + i\omega \frac{\left(\varepsilon_0 (\omega^2 + \nu_m^2) - 1 \right)}{(\omega^2 + \nu_m^2)} \right] \tilde{E}^2 \right] \\
&= \frac{1}{2} \frac{\varepsilon_0 \omega_{ps}^2 \nu_m}{(\omega^2 + \nu_m^2)} E^2
\end{aligned}
$$

Or we can calculate this in terms of the current density rather than in terms of the electric field to get

$$
p_{ave} = \frac{1}{2} \frac{\nu_m}{\varepsilon_0 \omega_{ps}^2} J^2 .
$$

Now, we know the current density field from our model. Thus we find that the average power deposited into the plasma is

$$
\begin{aligned}
P_{ave} &= \int_0^l p_{ave}dx \\
&= \int_0^l \frac{1}{2}\frac{\nu_m}{\varepsilon_0\omega_{ps}^2}J^2dx \\
&= \frac{1}{2}\frac{\nu_m}{\varepsilon_0\omega_{ps}^2}J^2\int_0^l dx \\
&= \frac{1}{2}\frac{\nu_m}{\varepsilon_0\omega_{ps}^2}J^2 l
\end{aligned}
$$

This of course assumes that the current density is constant across the plasma. Plugging in for the plasma frequency gives

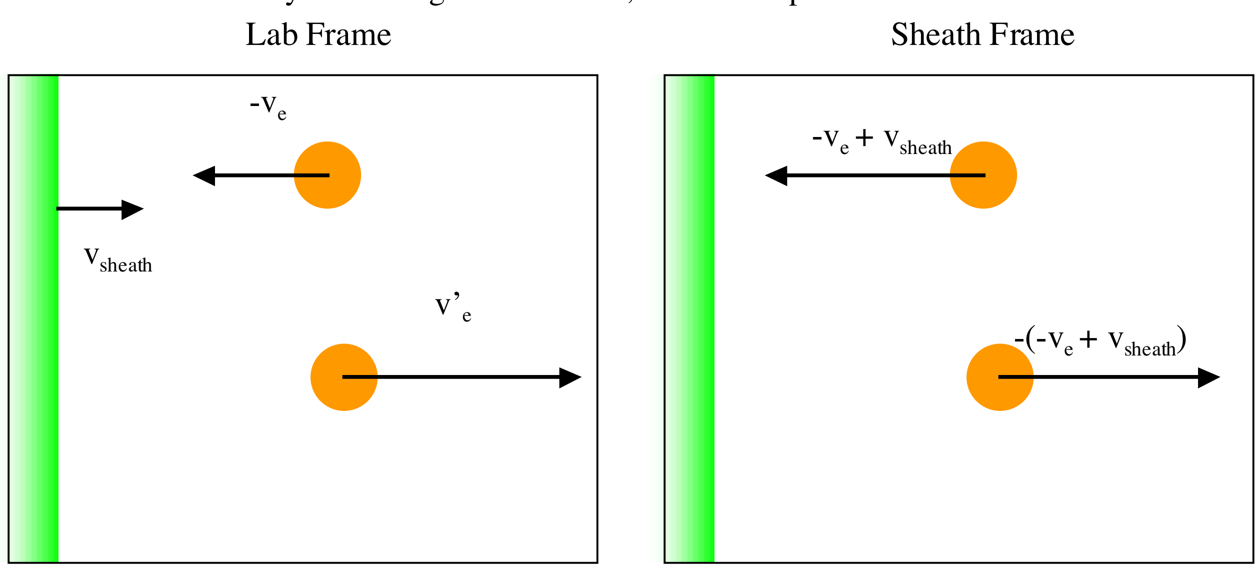
$$
\begin{aligned}
P_{ave} &= \int_0^l p_{ave}dx \\
&= \int_0^l \frac{1}{2}\frac{\nu_m}{\varepsilon_0\omega_{ps}^2}J^2dx \\
&= \frac{1}{2}\frac{\nu_m}{\varepsilon_0\omega_{ps}^2}J^2\int_0^l dx \\
&= \frac{1}{2}\frac{m\nu_m}{n_0e^2}J^2 l
\end{aligned}
$$

(Note that the way we have done this could also apply to ion motion.)

The final thing that we need to look at is stochastic heating.

<u>Stochastic Heating</u>

To understand stochastic heating, we have to examine two different reference frames, the lab frame and the stationary sheath edge frame. First, let us draw pictures of these frames.



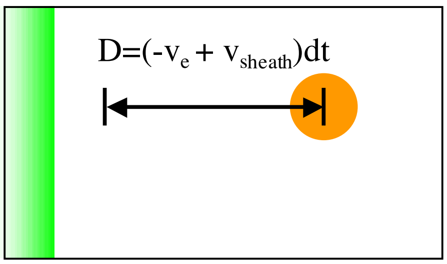
Assuming that the sheath has infinite 'mass,' than in the sheath frame, the electron must retain all of its energy. Thus in the sheath frame the final velocity of the electron must be $-(-v_e + v_{sheath})$.

Transforming back to the lab frame we find that the final velocity of the electron must be
$v_e' = -v_e + 2v_{sheath}$.
This shows us that in the lab frame, the electron has more energy after colliding with the sheath. (The lab frame is what matters as the neutral gas is assumed to be at rest in the lab frame.)

Now we need to find out how this affects the electron energy on average – if one electron is energized while a second losses an equal amount of energy than there is no net heating of the plasma.

To start, we first need to determine how many electrons of a velocity between v_e and $v_e + dv_e$ will collide with the sheath in time dt. This is best understood in the sheath frame.

Sheath Frame



Then the number per unit area is simply

$$\frac{\#}{A} = \frac{D\#}{Vol}$$
$$= Df(v_e,t)dv_e$$
$$= (-v_e + v_{sheath})f(v_e,t)dv_e dt$$

The change in energy of each of these electrons with a velocity between v_e and $v_e + dv_e$ is

$$\Delta Energy = \frac{1}{2}m(v_e'^2 - v_e^2)$$

Integrating over all velocities we find the energy deposited

$$\Delta S_{Stoch} = \int_{-\infty}^{\infty} \frac{1}{2}m(v_e'^2 - v_e^2)(-v_e + v_{sheath})f(v_e,t)dv_e dt$$

BUT THIS IS NOT QUITE CORRECT AS SOME ELECTRONS OUTRUN THE SHEATH SO…

$$\Delta S_{Stoch} = \frac{\Delta energy}{A\ \ dt}$$
$$= \int_{-\infty}^{v_{sheath}} \frac{1}{2}m(v_e'^2 - v_e^2)(-v_e + v_{sheath})f(v_e,t)dv_e$$
$$= \int_{-\infty}^{v_{sheath}} \frac{1}{2}m(v_e^2 - 4v_e v_{sheath} + 4v_{sheath}^2 - v_e^2)(-v_e + v_{sheath})f(v_e,t)dv_e$$
$$= \int_{-\infty}^{v_{sheath}} m(-v_e v_{sheath} + v_{sheath}^2)(-v_e + v_{sheath})f(v_e,t)dv_e$$
$$= \int_{-\infty}^{v_{sheath}} 2mv_{sheath}(-v_e + v_{sheath})^2 f(v_e,t)dv_e$$

While this is 'true' at any point in time in general, again it does not matter if we put energy in to the electrons just to take it out in the next half cycle – so we now average over a cycle of the rf. We do this by noting that the sheath velocity is simply

$$v_{sheath} = v_{sh0}\cos(\omega t)$$

plugging this into the above equation and averaging over a cycle gives

$$\begin{aligned}\langle S_{Stoch}\rangle &= \frac{1}{T}\int_0^T \Delta S_{Stoch}\,dt \\ &= \frac{1}{T}\int_0^T\int_{-\infty}^{v_{sheath}} 2mv_{sh0}\cos(\omega t)\left(-v_e + v_{sh0}\cos(\omega t)\right)^2 f(v_e,t)\,dv_e\,dt \\ &= \frac{1}{T}\int_0^T\int_{-\infty}^{v_{sheath}} 2mv_{sh0}\left(v_e^2\cos(\omega t) - 2v_e v_{sh0}\cos^2(\omega t) + v_{sh0}^2\cos^3(\omega t)\right) f(v_e,t)\,dv_e\,dt \\ &= -\int_{-\infty}^{v_{sheath}} 2mv_e v_{sh0}^2 f(v_e,t)\,dv_e \\ &= \int_{-v_{sheath}}^{\infty} 2mv_e v_{sh0}^2 f(v_e,t)\,dv_e\end{aligned}$$

Now in general the sheath velocity is slow compared so we can let the sheath velocity go to zero. Further we will assume that the electrons have a Maxwellian distribution. Thus

$$\begin{aligned}\langle S_{Stoch}\rangle &\approx \int_0^{\infty} 2mv_e v_{sh0}^2 f(v_e,t)\,dv_e \\ &= 2mv_{sh0}^2\int_0^{\infty} v_e f(v_e,t)\,dv_e \\ &= mv_{sh0}^2\int_{-\infty}^{\infty} |v_e| f(v_e,t)\,dv_e \\ &= mv_{sh0}^2\left[\frac{1}{4}n_e\left(\frac{8kT_e}{\pi m_e}\right)^{1/2}\right]\end{aligned}$$