"Online Supplement to 'Weak $\sigma-$ Convergence: Theory and Applications' "

Jianning Kong[*] Peter C. B. Phillips[†] Donggyu Sul[‡]

January 5, 2019

This document contains supplementary proofs, additional discussion on power trend regression, some further numerical calculations, and additional simulations to those reported in Kong, Phillips and Sul (2019, hereafter KPS). We begin with the proofs of Lemmas 1-5.

Proofs of Lemmas

Proof of Lemma 1: By Euler-Maclaurin summation, the stated representations and large T approximations of $\tau_T(\alpha)$ and $H_T(\alpha,\ell)$ for $\alpha \leq 1$ are well known. When $\alpha > 1$, the exact expressions for the infinite sums are given by the Riemann and Hurwitz zeta functions

$$\begin{aligned} \zeta(\alpha) &= \textstyle\sum_{t=1}^{\infty} t^{-\alpha} = \frac{1}{\alpha-1} + \frac{1}{2} + \Delta_\alpha, \\ \zeta(\alpha,\ell) &= \frac{1}{\ell^\alpha} + \frac{1}{(1+\ell)^\alpha}\left(\frac{1}{2} + \frac{1+\ell}{\alpha-1}\right) + \Delta_{\alpha,\ell}, \end{aligned} \tag{1}$$

where

$$\Delta_\alpha = \sum_{j=1}^{J} \binom{\alpha+2j-2}{2j-1}\left(\frac{B_{2j}}{2j}\right) - (2J+1)!\binom{\alpha+2J}{2J+1}\int_1^\infty \frac{P_{2J+1}(t)}{t^{\alpha+2J+1}}dt, \tag{2}$$

$$\begin{aligned} \Delta_{\alpha,\ell} &= \sum_{j=1}^{J} \binom{\alpha+2j-2}{2j-1}\left(\frac{B_{2j}}{2j}\right)\frac{1}{(1+\ell)^{\alpha+2j-1}} \\ &\quad - (2J+1)!\binom{\alpha+2J}{2J+1}\int_1^\infty \frac{P_{2J+1}(t)}{(t+\ell)^{\alpha+2J+1}}dt, \end{aligned} \tag{3}$$

the B_{2j} are Bernoulli numbers, and $P_{2J+1}(t) = (-1)^{J-1}\sum_{k=1}^{\infty} 2\sin(2k\pi t)/(2k\pi)^{2J+1}$. Thus, the expressions given in (1) provide upper bounds for $Z_T(\alpha)$ and $\zeta_T(\alpha,\ell)$. Simpler bounds are readily constructed (e.g., Kac and Cheung, 2002; KC). Indeed, since $t^{-\alpha}$ is positive, continuous, and tends to zero, the Euler-Maclaurin-Cauchy constant $\gamma_\alpha = \lim_{T\to\infty}\left\{\sum_{t=1}^{T} t^{-\alpha} - \int_1^T t^{-\alpha}dt\right\}$ exists and is finite for all $\alpha > 1$. Note the

[*]Shandong University, China

[†]Yale University, USA; University of Auckland, New Zealand; Singapore Management University, Singapore; University of Southampton, UK.

[‡]University of Texas at Dallas, USA

explicit form

$$\sum_{t=1}^{\infty} t^{-\alpha} = \frac{1}{\alpha-1} + \frac{1}{2} + \alpha \int_1^{\infty} \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{\alpha+1}} dx, \tag{4}$$

where the floor function $\lfloor x \rfloor$ denotes the integer part of x. Since $-\frac{1}{2} < \lfloor x \rfloor - x + \frac{1}{2} \leq \frac{1}{2}$ we have the bound

$$\left| \int_1^{\infty} \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{\alpha+1}} dx \right| \leq \int_1^{\infty} \frac{\frac{1}{2}}{x^{\alpha+1}} dx = \frac{1}{2\alpha},$$

from which we deduce that $|\Delta_\alpha| < \frac{1}{2}$ for all $\alpha > 1$. The first element of (4) is, of course, unbounded as $\alpha \to 1$. Finally, the inequality $\zeta(\alpha, \ell) \leq \zeta(\alpha)$ holds trivially for all $\ell \geq 1$ when $\alpha > 1$.
$\square$

Proof of Lemma 2 The proof of lemma 2 follows in a straightforward way by direct calculation using lemma 1. In particular, we have as $T \to \infty$

$$\mathcal{T}_T(1,\alpha) = \sum_{t=1}^{T} \tilde{t}\widetilde{t^{-\alpha}} = \sum_{t=1}^{T} t^{-(\alpha-1)} - \left(\frac{T+1}{2}\right) \sum_{t=1}^{T} t^{-\alpha} \tag{5}$$

$$= \begin{cases} -\dfrac{\alpha}{2(\alpha-2)(\alpha-1)} T^{2-\alpha} + O\left(T^{1-\alpha}\right) & \text{if } \alpha < 1, \\ -\dfrac{1}{2} T \ln T + T + O(1) & \text{if } \alpha = 1, \\ -\dfrac{1}{2}\zeta(\alpha) T + O(1) & \text{if } \alpha > 1, \end{cases} \tag{6}$$

$$\begin{aligned} \mathcal{S}_T(\alpha) &= \sum_{t=1}^{T} \widetilde{t^{-\alpha}}\widetilde{t^{-\alpha}} = \sum_{t=1}^{T} \left(t^{-\alpha} - T^{-1} \sum_{t=1}^{T} t^{-\alpha} \right) t^{-\alpha} \\ &= \begin{cases} \dfrac{\alpha^2}{(\alpha-1)^2 (1-2\alpha)} T^{1-2\alpha} + O(1) & \text{if } \alpha < 1/2, \\ \ln T + O(1) & \text{if } \alpha = 1/2, \\ \zeta(2\alpha) + o(1) & \text{if } \alpha > 1/2. \end{cases} \end{aligned} \tag{7}$$

For $\mathcal{B}_T(\alpha)$,we first simplify as follows

$$\begin{aligned} \mathcal{B}_T(\alpha) &= \frac{1}{T} \sum_{t=1}^{T} \left[\widetilde{t^{-\alpha}} - \tilde{t} \left(\sum_{t=1}^{T} \tilde{t}^2 \right)^{-1} \sum_{t=1}^{T} \tilde{t}\widetilde{t^{-\alpha}} \right]^2 \\ &= \frac{1}{T} \sum_{t=1}^{T} \left[\widetilde{t^{-\alpha}} - \tilde{t} \frac{\mathcal{T}_T(1,\alpha)}{\sum_{t=1}^{T} \tilde{t}^2} \right]^2 \\ &= \frac{1}{T} \sum_{t=1}^{T} \left(\widetilde{t^{-\alpha}} \right)^2 + \frac{1}{T} \sum_{t=1}^{T} \tilde{t}^2 \left[\frac{\mathcal{T}_T(1,\alpha)}{\sum_{t=1}^{T} \tilde{t}^2} \right]^2 - 2\frac{1}{T} \sum_{t=1}^{T} \tilde{t}^2 \left[\frac{\mathcal{T}_T(1,\alpha)}{\sum_{t=1}^{T} \tilde{t}^2} \right]^2 \\ &= \frac{1}{T} \sum_{t=1}^{T} \left(\widetilde{t^{-\alpha}} \right)^2 - \frac{1}{T} \frac{\left[\mathcal{T}_T(1,\alpha)\right]^2}{\sum_{t=1}^{T} \tilde{t}^2} \\ &= T^{-1} \mathcal{S}_T(\alpha) - 12 T^{-4} \left[\mathcal{T}_T(1,\alpha)\right]^2 . \end{aligned} \tag{8}$$

Then, using the results just established we have

$$\begin{aligned}
\mathcal{B}_T(\alpha) &= T^{-1}\mathcal{S}_T(\alpha) - 12T^{-4}\left[\mathcal{T}_T(1,\alpha)\right]^2 = T^{-1}\mathcal{S}_T(\alpha)\left\{1+O\left(T^{-1}\right)\right\} \qquad (9)\\
&= \begin{cases} \dfrac{\alpha^2}{(\alpha-1)^2(1-2\alpha)}T^{-2\alpha}+O\left(T^{-1}\right) & \text{if } \alpha<1/2,\\ T^{-1}\ln T+O\left(T^{-1}\right) & \text{if } \alpha=1/2,\\ T^{-1}\zeta(2\alpha)+o\left(T^{-1}\right) & \text{if } \alpha>1/2,\end{cases}\\
&= \begin{cases} O\left(T^{-2\alpha}\right) & \text{if } \alpha<1/2,\\ O\left(T^{-1}\ln T\right) & \text{if } \alpha=1/2,\\ O\left(T^{-1}\right) & \text{if } \alpha>1/2,\end{cases}
\end{aligned}$$

as required. □

Proof of Lemma 3 To derive the required asymptotic orders, it is sufficient to compute the order of the variances since all quantities have zero mean. By direct calculation

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=1}^{T} v_{it}t^{-\alpha}\right]^2 = \mathbb{E}\left[\sum_{t=1}^{T}\sum_{s=1}^{T} v_{it}v_{is}t^{-\alpha}s^{-\alpha}\right] = \sum_{t=1}^{T}\sum_{s=1}^{T}\gamma_{s-t,v,i}t^{-\alpha}s^{-\alpha}\\
=&\ \sum_{h=-T+1}^{T-1}\gamma_{h,v,i}\sum_{t=1}^{T}\sum_{s=1}^{T}t^{-\alpha}s^{-\alpha}\mathbf{1}\{s-t=h\}\\
=&\ \sum_{h=-T+1}^{T-1}\gamma_{h,v,i}\sum_{\substack{t=1\\1\le t+h\le T}}^{T}t^{-\alpha}(t+h)^{-\alpha},
\end{aligned}$$

and we deduce by summability and Cauchy-Schwarz that

$$\mathbb{E}\left[\sum_{t=1}^{T} v_{it}t^{-\alpha}\right]^2 \le \sum_{h=-\infty}^{\infty}\left|\gamma_{h,v,i}\right|\left\{\left(\sum_{t=1}^{T}t^{-2\alpha}\right)^2\right\}^{1/2} = O\left(\tau_T(2\alpha)\right).$$

Hence, $\sum_{t=1}^{T} v_{it}t^{-\alpha} = O_p\left([\tau_T(2\alpha)]^{1/2}\right)$ since $\mathbb{E}\left(\sum_{t=1}^{T} v_{it}t^{-\alpha}\right)=0$. Next,

$$\begin{aligned}
\sum_{t=1}^{T} v_{it}\widetilde{t^{-\alpha}} &= \sum_{t=1}^{T} v_{it}\left(t^{-\alpha}-\frac{1}{T}\sum_{t=1}^{T}t^{-\alpha}\right)\\
&= \sum_{t=1}^{T} v_{it}t^{-\alpha}-\frac{1}{T}\sum_{t=1}^{T}t^{-\alpha}\sum_{t=1}^{T}v_{it}\\
&= O_p\left([\tau_T(2\alpha)]^{1/2}\right)+O_p\left(T^{-1/2}\tau_T(\alpha)\right) = O_p\left([\tau_T(2\alpha)]^{1/2}\right).
\end{aligned}$$

and

$$\begin{aligned}
\sum_{t=1}^{T}\tilde{t}t^{-\alpha}v_{it} &= \sum_{t=1}^{T} v_{it}t^{1-\alpha}-\frac{T+1}{2}\sum_{t=1}^{T}v_{it}t^{-\alpha}\\
&= O_p\left(\left(\sum_{t=1}^{T}t^{2-2\alpha}\right)^{1/2}\right)+O_p\left(T\left(\sum_{t=1}^{T}t^{-2\alpha}\right)^{1/2}\right)\\
&= O_p\left(T[\tau_T(2\alpha)]^{1/2}\right).
\end{aligned}$$

For the final result, note that

$$\mathbb{E}\left[\sum_{t=1}^{T} b_i\tilde{t}t^{-\alpha}\right]^2 = \mathbb{E}\left[b_i\mathcal{T}_T(1,\alpha)\right]^2 = \sigma_b^2\mathcal{T}_T^2(1,\alpha),$$

from which we deduce that $\sum_{t=1}^{T} b_i\tilde{t}t^{-\alpha} = O_p\left(\mathcal{T}_T(1,\alpha)\right)$. □

Proof of Lemma 4 Denote $t_\ell = t + \ell$ for any given integer $\ell \geq 1$ and observe that

$$
\begin{aligned}
& G(T,\lambda) := \frac{1}{T}\sum\nolimits_{\ell=1}^{L}\sum\nolimits_{t=1}^{T-\ell}\left(1-\frac{\ell}{L+1}\right)\tilde{m}_t\tilde{m}_{t+\ell} \qquad (10)\\
= \; & \frac{1}{T}\sum\nolimits_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\sum\nolimits_{t=1}^{T-\ell}\tilde{m}_t\left(m_{t+\ell}-\frac{1}{T-\ell}\sum\nolimits_{s=1}^{T-\ell}m_{s+\ell}\right)\\
= \; & \frac{1}{T}\sum\nolimits_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\sum\nolimits_{t=1}^{T-\ell}\tilde{m}_t m_{t+\ell}\\
= \; & \frac{1}{T}\sum\nolimits_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\sum\nolimits_{t=1}^{T-\ell}\left[\widetilde{t^{-\lambda}}-\tilde{t}\left(\sum\nolimits_{t=1}^{T}\tilde{t}^2\right)^{-1}\sum\nolimits_{t=1}^{T}\widetilde{\tilde{t}t^{-\lambda}}\right]\times\\
& \left[t_\ell^{-\lambda}-t_\ell\left(\sum\nolimits_{t=1}^{T}\tilde{t}^2\right)^{-1}\sum\nolimits_{t=1}^{T}\widetilde{\tilde{t}t^{-\lambda}}\right]\\
=: \; & \frac{1}{T}\sum\nolimits_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\left(\Psi_{\ell,1}+\Psi_{\ell,2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi_{\ell,1} \; &: \; = \sum\nolimits_{t=1}^{T-\ell}\left[\widetilde{t^{-\lambda}}t_\ell^{-\lambda}-\widetilde{t^{-\lambda}}t_\ell\left(\sum\nolimits_{t=1}^{T}\tilde{t}^2\right)^{-1}\left(\sum\nolimits_{t=1}^{T}\widetilde{\tilde{t}t^{-\lambda}}\right)\right],\\
\Psi_{\ell,2} \; &: \; = \sum\nolimits_{t=1}^{T-\ell}\left[-\tilde{t}t_\ell^{-\lambda}\left(\sum\nolimits_{t=1}^{T}\tilde{t}^2\right)^{-1}\left(\sum\nolimits_{t=1}^{T}\widetilde{\tilde{t}t^{-\lambda}}\right)\right.\\
& \qquad \left.+\tilde{t}t_\ell\left(\sum\nolimits_{t=1}^{T}\tilde{t}^2\right)^{-2}\left(\sum\nolimits_{t=1}^{T}\widetilde{\tilde{t}t^{-\lambda}}\right)^2\right].
\end{aligned}
$$

We decompose the term $\Psi_{\ell,1}$ in (10) first, writing

$$
\begin{aligned}
\Psi_{\ell,1} \; &= \; \sum\nolimits_{t=1}^{T-\ell}\left[\widetilde{t^{-\lambda}}t_\ell^{-\lambda}-\widetilde{t^{-\lambda}}t_\ell\left(\sum\nolimits_{t=1}^{T}\tilde{t}^2\right)^{-1}\left(\sum\nolimits_{t=1}^{T}\widetilde{\tilde{t}t^{-\lambda}}\right)\right] \qquad (11)\\
&= \; \sum\nolimits_{t=1}^{T-\ell}\left(t^{-\lambda}-\frac{1}{T-\ell}\sum\nolimits_{t=1}^{T-\ell}t^{-\lambda}\right)t_\ell^{-\lambda}-\sum\nolimits_{t=1}^{T-\ell}\left(\widetilde{t^{-\lambda}}t_\ell\right)\left(\sum\nolimits_{t=1}^{T}\tilde{t}^2\right)^{-1}\left(\sum\nolimits_{t=1}^{T}\widetilde{\tilde{t}t^{-\lambda}}\right)\\
&= \; \sum\nolimits_{t=1}^{T-\ell}\left(t^2+t\ell\right)^{-\lambda}-\frac{1}{T-\ell}\sum\nolimits_{t=1}^{T-\ell}t_\ell^{-\lambda}\sum\nolimits_{t=1}^{T-\ell}t^{-\lambda}\\
& \quad -\sum\nolimits_{t=1}^{T-\ell}\left(\widetilde{t^{-\lambda}}t_\ell\right)\left(\sum\nolimits_{t=1}^{T}\tilde{t}^2\right)^{-1}\left(\sum\nolimits_{t=1}^{T}\widetilde{\tilde{t}t^{-\lambda}}\right)\\
&=: \; \Psi_{\ell,11}+\Psi_{\ell,12}+\Psi_{\ell,13}.
\end{aligned}
$$

When $\lambda < 1$, as $T \to \infty$ with a finite $\ell \geq 1$, we have

$$
\sum\nolimits_{t=1}^{T-\ell}(t+\ell)^{-2\lambda} < \sum\nolimits_{t=1}^{T-\ell}\left(t^2+t\ell\right)^{-\lambda} < \min\left(\sum\nolimits_{t=1}^{T-\ell}t^{-2\lambda},\sum\nolimits_{t=1}^{T-\ell}(t\ell)^{-\lambda}\right).
$$

We calculate the upper bound first. Note that

$$
\sum\nolimits_{t=1}^{T-\ell}t^{-2\lambda} = \begin{cases} \dfrac{1}{1-2\lambda}(T-\ell)^{1-2\lambda}+O(1) & \text{if } \lambda < 1/2,\\ \ln(T-\ell)+O(1) & \text{if } \lambda = 1/2,\\ \zeta(2\lambda) = O(1) & \text{if } \lambda > 1/2, \end{cases}
$$

so that

$$\frac{1}{T}\sum_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\sum_{t=1}^{T-\ell}t^{-2\lambda}$$

$$\sim \begin{cases} \dfrac{1}{1-2\lambda}T^{-2\lambda+\kappa} & \text{if } \lambda<1/2, \\ T^{-1+\kappa}\ln T & \text{if } \lambda=1/2, \\ \zeta(2\lambda)\,T^{-1+\kappa} & \text{if } 1>\lambda>1/2, \end{cases}$$

$$= \begin{cases} O\left(T^{-2\lambda+\kappa}\right) & \text{if } \lambda<1/2, \\ O\left(T^{-1+\kappa}\ln T\right) & \text{if } \lambda=1/2, \\ O\left(T^{-1+\kappa}\right) & \text{if } 1>\lambda>1/2. \end{cases}$$

Meanwhile for $\lambda<1$,

$$\frac{1}{T}\sum_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\sum_{t=1}^{T-\ell}(t\ell)^{-\lambda} < \frac{1}{T}\sum_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\ell^{-\lambda}\sum_{t=1}^{T}t^{-\lambda} \tag{12}$$

$$\begin{aligned} &= \frac{1}{1-\lambda}\frac{1}{T}\sum_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\left[T^{1-\lambda}\ell^{-\lambda}+O(1)\right] \\ &\sim \frac{1}{1-\lambda}T^{-\lambda}\sum_{\ell=1}^{T^{\kappa}}\ell^{-\lambda}-\frac{1}{1-\lambda}T^{-\lambda-\kappa}\sum_{\ell=1}^{T^{\kappa}}\ell^{1-\lambda} \\ &\sim \frac{1}{(1-\lambda)^2}T^{-\lambda}T^{\kappa(1-\lambda)}-\frac{1}{(1-\lambda)(2-\lambda)}T^{-\lambda-\kappa}T^{\kappa(2-\lambda)} \end{aligned}$$

$$= \frac{1}{(1-\lambda)^2(2-\lambda)}T^{-\lambda+\kappa-\lambda\kappa} = O\left(T^{-\lambda+\kappa-\lambda\kappa}\right) \tag{13}$$

uniformly in $\ell \le L = \lfloor T^{\kappa}\rfloor$ with $\kappa<1$. Hence

$$\frac{1}{T}\sum_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\sum_{t=1}^{T-\ell}\left(t^2+t\ell\right)^{-\lambda} \tag{14}$$

$$< \begin{cases} \min\left[O\left(T^{-2\lambda+\kappa}\right),O\left(T^{-\lambda+\kappa-\lambda\kappa}\right)\right]=O\left(T^{-2\lambda+\kappa}\right) & \text{if } \lambda<1/2, \\ \min\left[O\left(T^{\kappa-1}\ln T\right),O\left(T^{\kappa-1/2-\kappa/2}\right)\right]=O\left(T^{\kappa-1}\ln T\right) & \text{if } \lambda=1/2, \\ \min\left[O\left(T^{\kappa-1}\right),O\left(T^{-\lambda+\kappa-\lambda\kappa}\right)\right]=O\left(T^{\kappa-1}\right) & \text{if } 1/2<\lambda<1/(1+\kappa), \\ \min\left[O\left(T^{\kappa-1}\right),O\left(T^{-\lambda+\kappa-\lambda\kappa}\right)\right]=O\left(T^{-\lambda+\kappa-\lambda\kappa}\right) & \text{if } 1/(1+\kappa)\le\lambda<1. \end{cases}$$

Next, we consider the lower bound. We have

$$\begin{aligned} \sum_{t=1}^{T-\ell}(t+\ell)^{-2\lambda} &= \sum_{t=1}^{T}t^{-2\lambda}-\sum_{t=1}^{\ell}t^{-2\lambda} \\ &= \begin{cases} \dfrac{T^{1-2\lambda}}{1-2\lambda}-\dfrac{\ell^{1-2\lambda}}{1-2\lambda} & \text{if } \lambda<1/2, \\ \ln T-\ln\ell & \text{if } \lambda=1/2, \\ \zeta(2\lambda)-\sum_{t=1}^{\ell}t^{-2\lambda} & \text{if } \lambda>1/2. \end{cases} \end{aligned}$$

Then

$$\frac{1}{T}\sum_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\sum_{t=1}^{T-\ell}\left(t^2+t\ell\right)^{-\lambda} \tag{15}$$

$$> \begin{cases} \dfrac{T^{-2\lambda}L}{1-2\lambda} & \text{if } \lambda<1/2 \\ \frac{L}{T}\ln T & \text{if } \lambda=1/2 \\ T^{-1+\kappa}\zeta(2\lambda) & \text{if } \lambda>1/2 \end{cases} = \begin{cases} O\left(T^{-2\lambda+\kappa}\right) & \text{if } \lambda<1/2, \\ O\left(T^{-1+\kappa}\ln T\right) & \text{if } \lambda=1/2, \\ O\left(T^{-1+\kappa}\right) & \text{if } 1>\lambda>1/2. \end{cases}$$

Combining (15) with (14) yields

$$\frac{1}{T}\sum\nolimits_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\sum\nolimits_{t=1}^{T-\ell}\left(t^2+t\ell\right)^{-\lambda} \tag{16}$$
$$\sim \begin{cases} \dfrac{T^{-2\lambda+\kappa}}{1-2\lambda} & \text{if } \lambda<1/2, \\ T^{-1+\kappa}\ln T & \text{if } \lambda=1/2, \\ T^{-1+\kappa}\zeta\left(2\lambda\right) & \text{if } 1/2<\lambda<1/\left(1+\kappa\right), \\ \dfrac{1}{\left(1-\lambda\right)^2\left(2-\lambda\right)}T^{-\lambda+\kappa-\lambda\kappa} & \text{if } 1/\left(1+\kappa\right)\le\lambda<1, \end{cases}$$
$$= \begin{cases} O\left(T^{-2\lambda+\kappa}\right) & \text{if } \lambda<1/2, \\ O\left(T^{\kappa-1}\ln T\right) & \text{if } \lambda=1/2, \\ O\left(T^{\kappa-1}\right) & \text{if } 1/2<\lambda<1/\left(1+\kappa\right), \\ O\left(T^{-\lambda+\kappa-\lambda\kappa}\right) & \text{if } 1/\left(1+\kappa\right)\le\lambda<1. \end{cases}$$

In the first three cases, the asymptotic order of the sum is the same as the asymptotic order of the upper and lower bounds, as confirmed in the above derivation. In the last case when $1/\left(1+\kappa\right)\le\lambda<1$, we use the upper bound. Note that when $\lambda<\kappa/2$, this term increases as T increases, but when $\lambda\ge\kappa/2$, this term decreases as T increases.[1]

When $\lambda=1$ and for $\ell\to\infty$ we have

$$\begin{aligned} \Psi_{\ell,11} &= \sum\nolimits_{t=1}^{T-\ell}\frac{1}{t^2+t\ell}=\frac{1}{\ell}\sum\nolimits_{t=1}^{T-\ell}\left(\frac{1}{t}-\frac{1}{t+\ell}\right)=\frac{1}{\ell}\left\{\left(\sum\nolimits_{t=1}^{T-\ell}\frac{1}{t}\right)-\left(\sum\nolimits_{t=1}^{T}\frac{1}{t}-\sum\nolimits_{s=1}^{\ell}\frac{1}{s}\right)\right\} \\ &= \frac{1}{\ell}\left\{\left(\ln\left(T-\ell\right)+\gamma_e\right)-\left(\ln T+\gamma_e-\left[\ln\ell+\gamma_e\right]\right)+O\left(\frac{1}{\ell}+\frac{\ell}{T}\right)\right\} \\ &= \frac{1}{\ell}\left(\ln\ell+\gamma_e\right)+O\left(\frac{1}{\ell^2}+\frac{1}{T}\right), \end{aligned} \tag{17}$$

and when ℓ is fixed we have

$$\begin{aligned} \Psi_{\ell,11} &= \frac{1}{\ell}\sum\nolimits_{s=1}^{\ell}\frac{1}{s}-\frac{1}{\ell}\sum\nolimits_{t=T-\ell+1}^{T}\frac{1}{t}=\frac{1}{\ell}\sum\nolimits_{s=1}^{\ell}\frac{1}{s}+\frac{1}{\ell}\left\{\ln\frac{T-\ell}{T}+O\left(T^{-1}\right)\right\} \\ &= \frac{1}{\ell}\sum\nolimits_{s=1}^{\ell}\frac{1}{s}+O\left(\frac{1}{T}\right). \end{aligned} \tag{18}$$

Then, using (17), (18), and with $L=T^\kappa\to\infty$ as $T\to\infty$, we obtain

$$\begin{aligned} &\frac{1}{T}\sum\nolimits_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\Psi_{\ell,1}\sim\frac{1}{T}\sum\nolimits_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\left(\frac{1}{\ell}\sum\nolimits_{s=1}^{\ell}\frac{1}{s}\right) \\ =\ &\frac{1}{T}\sum\nolimits_{\ell=1}^{L}\ell^{-1}\sum\nolimits_{s=1}^{\ell}s^{-1}-\frac{1}{T\left(L+1\right)}\sum\nolimits_{\ell=1}^{L}\sum\nolimits_{s=1}^{\ell}s^{-1} \\ \sim\ &\frac{1}{T}\int_{\ell=1}^{L}\ell^{-1}\int_{s=1}^{\ell}s^{-1}ds d\ell-\frac{1}{T\left(L+1\right)}\int_{\ell=1}^{L}\int_{s=1}^{\ell}s^{-1}ds d\ell \\ =\ &\frac{1}{T}\int_{\ell=1}^{L}\ell^{-1}\left(\ln\ell\right)d\ell-\frac{1}{T\left(L+1\right)}\int_{\ell=1}^{L}\ln\ell d\ell \\ =\ &\frac{1}{2T}\ln^2 L+O\left(T^{-1}\ln T\right)=\frac{\kappa^2}{2T}\ln^2 T+O\left(T^{-1}\ln T\right). \end{aligned}$$

[1] A graphical demonstration of the relevance of the decay rate (13) in determining the behavior of $G\left(T,\lambda\right)$ as $T\to\infty$ is given in Fig. S5 at the end of this Appendix.

When $\lambda > 1$, we find from Lemma 1 that

$$\begin{aligned}
\sum_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\Psi_{\ell,11} &= \lim_{T\to\infty}\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\left(1-\frac{\ell}{L+1}\right)\left(t^2+t\ell\right)^{-\lambda} \\
&= \lim_{L\to\infty}\sum_{\ell=1}^{L}\sum_{t=1}^{L}\left(1-\frac{\ell}{L+1}\right)\frac{1}{t^{\lambda}\left(t+\ell\right)^{\lambda}} \\
&= \lim_{L\to\infty}\sum_{t=1}^{L}\frac{1}{t^{\lambda}}\sum_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\frac{1}{\left(t+\ell\right)^{\lambda}} \\
&= \lim_{L\to\infty}\sum_{t=1}^{L}\frac{1}{t^{\lambda}}\sum_{\ell=1}^{L}\frac{1}{\left(t+\ell\right)^{\lambda}}+O\left(L^{1-\lambda}\right) \\
&= \sum_{t=1}^{\infty}\frac{1}{t^{\lambda}}\left\{\zeta\left(\lambda,t\right)-\frac{1}{t^{\lambda}}\right\}+O\left(L^{1-\lambda}\right) \\
&= \sum_{t=1}^{\infty}\frac{\zeta\left(\lambda,t\right)}{t^{\lambda}}-\zeta\left(2\lambda\right)
\end{aligned}$$

where $\zeta\left(\lambda,\ell\right)$ is the Hurwitz zeta function which is well defined for all $\lambda > 1$ and $t > 0$. Note, in particular, that

$$\zeta\left(\lambda,t\right)=\sum_{\ell=0}^{\infty}\frac{1}{\left(t+\ell\right)^{\lambda}}=\sum_{\ell=1}^{\infty}\frac{1}{\left(t+\ell\right)^{\lambda}}+\frac{1}{t^{\lambda}}<\zeta\left(\lambda\right)+\frac{1}{t^{\lambda}},$$

with strict inequality showing that[2]

$$\sum_{t=1}^{\infty}\frac{\zeta\left(\lambda,t\right)}{t^{\lambda}}<\sum_{t=1}^{\infty}\frac{\zeta\left(\lambda\right)}{t^{\lambda}}+\sum_{t=1}^{\infty}\frac{1}{t^{2\lambda}}$$

$$\sum_{t=1}^{\infty}\frac{\zeta\left(\lambda,t\right)}{t^{\lambda}}-\zeta\left(2\lambda\right)<\zeta\left(\lambda\right)^2=O\left(1\right).$$

Our next step is to calculate $\sum_{\ell=1}^{L}\left(1-\frac{\ell}{L+1}\right)\left(\Psi_{\ell,12}+\Psi_{\ell,13}+\Psi_{\ell,2}\right)$ but before doing so we provide summation results for $\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\tilde{t}t_{\ell}^{-\lambda}$ and $\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\widetilde{t^{-\lambda}}t_{\ell}$. These results and their orders of magnitude are given as follows:

$$\begin{aligned}
\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\tilde{t}t_{\ell}^{-\lambda} &= \sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\left(t+\ell-\frac{1}{T-\ell}\sum_{t=1}^{T-\ell}\left(t+\ell\right)\right)t_{\ell}^{-\lambda} \\
&= \sum_{\ell=1}^{L}\sum_{s=\ell+1}^{T}\tilde{s}s^{-\lambda}=\sum_{\ell=1}^{L}\left(\sum_{s=1}^{T}\tilde{s}s^{-\lambda}-\sum_{s=1}^{\ell}\tilde{s}s^{-\lambda}\right) \\
&= \sum_{\ell=1}^{L}\left(\mathcal{T}_T\left(1,\lambda\right)-\mathcal{T}_{\ell}\left(1,\lambda\right)\right) \\
&\sim \begin{cases} -\dfrac{\lambda}{2\left(\lambda-2\right)\left(\lambda-1\right)}T^{2-\lambda+\kappa} & \text{if } \lambda<1, \\ -\dfrac{1}{2}T^{1+\kappa}\ln T & \text{if } \lambda=1, \\ -\dfrac{1}{2}\zeta\left(\lambda\right)T^{1+\kappa} & \text{if } \lambda>1, \end{cases}
\end{aligned} \tag{19}$$

[2] It is not needed in the results for Lemma 4 but is interesting to note (and will be used later) that as $\lambda\to\infty$, we get $\lim_{\lambda\to\infty}\left\{\sum_{t=1}^{\infty}t^{-\lambda}\zeta\left(\lambda,t\right)-\zeta\left(2\lambda\right)\right\}=1-1=0$, which corresponds to

$$\lim_{\lambda\to\infty}\lim_{M\to\infty}\sum_{\ell=1}^{M}\sum_{t=1}^{M}\frac{1}{t^{\lambda}\left(t+\ell\right)^{\lambda}}=\lim_{M\to\infty}\sum_{\ell=1}^{M}\sum_{t=1}^{M}\lim_{\lambda\to\infty}\frac{1}{t^{\lambda}\left(t+\ell\right)^{\lambda}}=0,$$

whereas $\lim_{\lambda\to\infty}\zeta\left(\lambda\right)^2=1$. Therefore, as $\lambda\to\infty$ we find that the serial correlation terms

$$\frac{1}{T}\sum_{\ell=1}^{M}\sum_{t=1}^{T-\ell}\left(1-\frac{\ell}{M+1}\right)\tilde{m}_t\tilde{m}_{t+\ell}\to 0.$$

and

$$
\begin{aligned}
&\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\widetilde{t^{-\lambda}}t_{\ell} \qquad (20)\\
=\;& \sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\left(\left(t^{-\lambda}-\frac{1}{T-\ell}\sum_{t=1}^{T-\ell}t^{-\lambda}\right)(t+\ell)\right)\\
=\;& \sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\left(t^{-\lambda}-\frac{1}{T-\ell}\sum_{t=1}^{T-\ell}t^{-\lambda}\right)t\\
=\;& \sum_{\ell=1}^{L}\mathcal{T}_{T-\ell}(1,\lambda)
\end{aligned}
$$

$$
\sim \begin{cases} -\dfrac{\lambda}{2(\lambda-2)(\lambda-1)}T^{2-\lambda+\kappa} & \text{if } \lambda<1,\\ -\dfrac{1}{2}T^{1+\kappa}\ln T & \text{if } \lambda=1,\\ -\dfrac{1}{2}\zeta(\lambda)T^{1+\kappa} & \text{if } \lambda>1, \end{cases}
$$

$$
= \begin{cases} O\left(T^{\kappa+2-\lambda}\right) & \text{if } \lambda<1,\\ O\left(T^{\kappa+1}\ln T\right) & \text{if } \lambda=1,\\ O\left(T^{\kappa+1}\right) & \text{if } \lambda>1. \end{cases}
$$

Then, using results (19) - (20) and Lemmas 1 and 2, we have

$$
\begin{aligned}
\sum_{\ell=1}^{L}\Psi_{\ell,12} &= -\sum_{\ell=1}^{L}\frac{1}{T-\ell}\left(\sum_{t=1}^{T-\ell}t^{-\lambda}\right)\sum_{t=1}^{T-\ell}t_{\ell}^{-\lambda}\\
&= -\sum_{\ell=1}^{L}\frac{1}{T-\ell}\left(\sum_{t=1}^{T-\ell}t^{-\lambda}\right)\sum_{\ell+1}^{T}t^{-\lambda}\\
&= -\sum_{\ell=1}^{L}\frac{1}{T-\ell}\left(\sum_{t=1}^{T-\ell}t^{-\lambda}\right)\left(\sum_{t=1}^{T}t^{-\lambda}-\sum_{t=1}^{\ell}t^{-\lambda}\right)\\
&= \begin{cases} O\left(T^{\kappa+1-2\lambda}\right) & \text{if } \lambda<1,\\ O\left(T^{\kappa-1}\ln^2 T\right) & \text{if } \lambda=1,\\ O\left(T^{\kappa-1}\right) & \text{if } \lambda>1, \end{cases}
\end{aligned}
$$

Note, when $\lambda<1, \sum_{\ell=1}^{L}\Psi_{\ell,12}\sim -\frac{1}{(1-\lambda)^2}T^{\kappa+1-2\lambda}$.

$$
\sum_{\ell=1}^{L}\Psi_{\ell,13} = -\left(\sum_{t=1}^{T}\widetilde{t}^{2}\right)^{-1}\left(\sum_{t=1}^{T}\widetilde{t}\widetilde{t^{-\lambda}}\right)\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\left(\widetilde{t^{-\lambda}}t_{\ell}\right)
$$

$$
= \begin{cases} O\left(T^{-3}\right)O\left(T^{2-\lambda}\right)O\left(T^{\kappa+2-\lambda}\right) = O\left(T^{\kappa+1-2\lambda}\right) & \text{if } \lambda<1,\\ O\left(T^{-3}\right)O\left(T\ln T\right)O\left(T^{\kappa+1}\ln T\right) = O\left(T^{\kappa-1}\ln^2 T\right) & \text{if } \lambda=1,\\ O\left(T^{-3}\right)O\left(T\right)O\left(T^{\kappa+1}\right) = O\left(T^{\kappa-1}\right) & \text{if } \lambda>1. \end{cases}
$$

When $\lambda < 1, \sum_{\ell=1}^{L} \Psi_{\ell,13} \sim -\dfrac{3\lambda^2}{(\lambda-2)^2(\lambda-1)^2} T^{1-2\lambda+\kappa}$.

$$
\begin{aligned}
&\sum\nolimits_{\ell=1}^{L} \Psi_{\ell,2} = -\left(\sum\nolimits_{t=1}^{T} \tilde{t}^2\right)^{-1} \left(\sum\nolimits_{t=1}^{T} \tilde{t}\widetilde{t^{-\lambda}}\right) \sum\nolimits_{\ell=1}^{L} \sum\nolimits_{t=1}^{T-\ell} \tilde{t}t_{\ell}^{-\lambda} \\
&+ \left(\sum\nolimits_{t=1}^{T} \tilde{t}^2\right)^{-2} \left(\sum\nolimits_{t=1}^{T} \tilde{t}\widetilde{t^{-\lambda}}\right)^2 \sum\nolimits_{\ell=1}^{L} \sum\nolimits_{t=1}^{T-\ell} \tilde{t}t_{\ell} \\
= &\begin{cases} O\left(T^{-3}\right) O\left(T^{2-\lambda}\right) O\left(T^{\kappa+2-\lambda}\right) + O\left(T^{-6}\right) O\left(T^{4-2\lambda}\right) O\left(T^{\kappa+3}\right) & \text{if } \lambda < 1, \\ O\left(T^{-3}\right) O\left(T \ln T\right) O\left(T^{\kappa+1} \ln T\right) + O\left(T^{-6}\right) O\left(T^2 \ln^2 T\right) O\left(T^{\kappa+3}\right) & \text{if } \lambda = 1, \\ O\left(T^{-3}\right) O\left(T\right) O\left(T^{\kappa+1}\right) + O\left(T^{-6}\right) O\left(T^2\right) O\left(T^{\kappa+3}\right) & \text{if } \lambda > 1, \end{cases} \\
= &\begin{cases} O\left(T^{\kappa+1-2\lambda}\right) & \text{if } \lambda < 1, \\ O\left(T^{\kappa-1} \ln^2 T\right) & \text{if } \lambda = 1, \\ O\left(T^{\kappa-1}\right) & \text{if } \lambda > 1. \end{cases}
\end{aligned}
$$

When $\lambda < 1$, $\sum_{\ell=1}^{L} \Psi_{\ell,2} \sim -3\left(\dfrac{\lambda}{(\lambda-2)(\lambda-1)}\right)^2 T^{1-2\lambda+\kappa} + 3T^{-6}\left(\dfrac{\lambda}{(\lambda-2)(\lambda-1)} T^{2-\lambda}\right)^2 T^{3+\kappa} = 0$.

Combining these three terms we find that

$$
\sum\nolimits_{\ell=1}^{L} \left(1 - \frac{\ell}{L+1}\right) \left(\Psi_{\ell,12} + \Psi_{\ell,13} + \Psi_{\ell,2}\right) = \begin{cases} O\left(T^{\kappa+1-2\lambda}\right) & \text{if } \lambda < 1, \\ O\left(T^{\kappa-1} \ln^2 T\right) & \text{if } \lambda = 1, \\ O\left(T^{\kappa-1}\right) & \text{if } \lambda > 1. \end{cases} \tag{21}
$$

When $\lambda < 1$, $\sum_{\ell=1}^{L} \left(1 - \frac{\ell}{L+1}\right) \left(\Psi_{\ell,12} + \Psi_{\ell,13} + \Psi_{\ell,2}\right) \sim -4\frac{\lambda^2-\lambda+1}{(\lambda^2-3\lambda+2)^2} T^{1-2\lambda+\kappa}$.

Hence, we have

$$
\begin{aligned}
&\frac{1}{T} \sum\nolimits_{\ell=1}^{L} \sum\nolimits_{t=1}^{T-\ell} \left(1 - \frac{\ell}{L+1}\right) \tilde{m}_t \tilde{m}_{t+\ell} \\
= &\sum\nolimits_{\ell=1}^{L} \left(1 - \frac{\ell}{L+1}\right) \left(\Psi_{\ell,11} + \Psi_{\ell,12} + \Psi_{\ell,13} + \Psi_{\ell,2}\right) \\
= &\begin{cases} \dfrac{\lambda^2 (\lambda+1)^2}{(1-2\lambda)\left(\lambda^2 - 3\lambda + 2\right)^2} T^{-2\lambda+\kappa} & \text{if } \lambda < 1/2, \\ T^{-1+\kappa} \ln T & \text{if } \lambda = 1/2, \\ T^{-1+\kappa} \zeta(2\lambda) & \text{if } 1/2 < \lambda < 1/(1+\kappa), \\ \dfrac{1}{(1-\lambda)^2 (2-\lambda)} T^{-\lambda+\kappa-\lambda\kappa} & \text{if } 1/(1+\kappa) \leq \lambda < 1, \\ \frac{\kappa^2}{2} T^{-1} \ln^2 T + O\left(T^{-2} \ln T\right) & \text{if } \lambda = 1, \\ T^{-1} \left\{\sum_{t=1}^{\infty} t^{-\lambda} \zeta(\lambda, \ell) - \zeta(2\lambda)\right\} & \text{if } \lambda > 1. \end{cases}
\end{aligned}
$$

□

Proof of Lemma 5 Since $\xi_{b,n} = O_p\left(n^{-1/2}\right)$ and $n/T \to \infty$, we have

$$
\xi_{b,n} \widetilde{t^{-\alpha}} = O_p\left(n^{-1/2} \widetilde{t^{-\alpha}}\right) = o_p\left(T^{-1/2} \widetilde{t^{-\alpha}}\right).
$$

Note that

$$\widetilde{t^{-\alpha}}T^{-1/2} = \begin{cases} t^{-\alpha}T^{-1/2} - \dfrac{1}{1-\alpha}T^{-1/2}T^{-\alpha} & \text{if } \alpha < 1, \\ t^{-\alpha}T^{-1/2} - T^{-1}T^{-1/2}\ln T & \text{if } \alpha = 1, \\ t^{-\alpha}T^{-1/2} - \mathcal{Z}_T(\alpha)\, T^{-1}T^{-1/2} & \text{if } \alpha > 1, \end{cases}$$

and

$$\widetilde{t^{-2\alpha}} = \begin{cases} t^{-2\alpha} - \dfrac{1}{1-2\alpha}T^{-2\alpha} & \text{if } \alpha < 1/2, \\ t^{-2\alpha} - T^{-1}\ln T & \text{if } \alpha = 1/2, \\ t^{-2\alpha} - \mathcal{Z}_T(2\alpha)\, T^{-1} & \text{if } \alpha > 1/2. \end{cases}$$

Hence

$$\begin{aligned} & \widetilde{t^{-2\alpha}} - \widetilde{t^{-\alpha}}T^{-1/2} \\ = & \begin{cases} t^{-2\alpha} - \dfrac{1}{1-2\alpha}T^{-2\alpha} - t^{-\alpha}T^{-1/2} + \dfrac{1}{1-\alpha}T^{-1/2}T^{-\alpha} & \text{if } \alpha < 1/2, \\ t^{-2\alpha} - T^{-1}\ln T - t^{-\alpha}T^{-1/2} + \dfrac{1}{1-\alpha}T^{-1/2}T^{-\alpha} & \text{if } \alpha = 1/2, \\ t^{-2\alpha} - \mathcal{Z}_T(2\alpha)\, T^{-1} - t^{-\alpha}T^{-1/2} + \dfrac{1}{1-\alpha}T^{-1/2}T^{-\alpha} & \text{if } 1/2 < \alpha < 1, \\ t^{-2\alpha} - \mathcal{Z}_T(2\alpha)\, T^{-1} - t^{-\alpha}T^{-1/2} + T^{-1}T^{-1/2}\ln T & \text{if } \alpha = 1, \\ t^{-2\alpha} - \mathcal{Z}_T(2\alpha)\, T^{-1} - t^{-\alpha}T^{-1/2} + \mathcal{Z}_T(\alpha)\, T^{-1}T^{-1/2} & \text{if } \alpha > 1, \end{cases} \\ = & \widetilde{t^{-2\alpha}} + o\left(\widetilde{t^{-2\alpha}}\right). \end{aligned}$$

For example, when $\alpha = 1/2$, as $T \to \infty$ we have

$$\begin{aligned} \lim_{T\to\infty} \frac{-\widetilde{t^{-\alpha}}T^{-1/2}}{\widetilde{t^{-2\alpha}}} &= \lim_{T\to\infty} \frac{-t^{-\alpha}T^{-1/2} + \dfrac{1}{1-\alpha}T^{-1/2}T^{-\alpha}}{\widetilde{t^{-2\alpha}}} = \lim_{T\to\infty} \frac{-t^{-1/2}T^{-1/2} + 2T^{-1}}{t^{-1} - T^{-1}\ln T} \\ &= \lim_{T\to\infty} \frac{-(t/T)^{1/2} + 2(t/T)}{1 - (t/T)\ln T} \to 0, \end{aligned}$$

since $t/T \in (0,1]$ as $T \to \infty$ for $1 \leq t \leq T$. Hence, for all $t \leq T$, $-\widetilde{t^{-\alpha}}T^{-1/2} = o\left(\widetilde{t^{-2\alpha}}\right)$. The required result (38) in KPS now follows.

□

Proof of Theorem 1: The Asymptotic Limit of $\hat{\phi}_{nT}$

To analyze the asymptotic behavior of the trend regression coefficient $\hat{\phi}_{nT}$ we use the convenient decomposition (21) in KPS, viz.,

$$\hat{\phi}_{nT} = \sum\nolimits_{t=1}^{T} a_{tT}\eta_t + \sum\nolimits_{t=1}^{T} a_{tT}\xi_{n,t} + \sum\nolimits_{t=1}^{T} a_{tT}\varepsilon_{n,t} =: I_A + I_B + I_C,$$

where

$$a_{tT} = \frac{\tilde{t}}{\sum_{s=1}^{T}\tilde{s}^2} = \frac{t - T^{-1}\sum_{s=1}^{T}s}{T^3 \times \frac{1}{T^3}\sum_{s=1}^{T}\tilde{s}^2} = \frac{12}{T^3}\left(t - \frac{T+1}{2}\right)\left\{1 + O\left(T^{-1}\right)\right\},$$

The dominant term in η_t,denoted by $\eta_{t,d}$, can be classified according to the three models as follows

Case	M1	M2	M3
$\alpha, \beta > 0$,and $\sigma_{a\mu} \neq 0$	$2\sigma_{a\mu}t^{-\alpha}$	$\sigma_\epsilon^2 t^{-2\beta}$	$2\sigma_{a\mu}t^{-\alpha} + \sigma_\epsilon^2 t^{-2\beta}$
$\alpha, \beta > 0$,and $\sigma_{a\mu} = 0$	$\sigma_\mu^2 t^{-2\alpha}$	$\sigma_\epsilon^2 t^{-2\beta}$	$\sigma_\mu^2 t^{-2\alpha} + \sigma_\epsilon^2 t^{-2\beta}$
$\alpha < 0$ or $\beta < 0$	$\sigma_\mu^2 t^{-2\alpha}$	$\sigma_\epsilon^2 t^{-2\beta}$	$\sigma_\mu^2 t^{-2\alpha} + \sigma_\epsilon^2 t^{-2\beta}$

Using the general form $\eta_{t,d} = bt^{-\lambda}$,where λ represents the decay parameter and b is the corresponding coefficient in that term, we first obtain the following expression for I_A. We specify b and λ later in the case of each individual model.

$$\begin{aligned} I_A &= \left(\sum_{t=1}^T \tilde{t}^2\right)^{-1} \sum_{t=1}^T \tilde{t}\eta_t \sim b\left(\sum_{t=1}^T \tilde{t}^2\right)^{-1} \sum_{t=1}^T \tilde{t}t^{-\lambda} \\ &= -6b \times \begin{cases} \dfrac{\lambda}{(\lambda-2)(\lambda-1)} T^{-1-\lambda} + O\left(T^{-2-\lambda}\right) & \text{if } \lambda < 1, \\ T^{-2}\ln T + O\left(T^{-2}\right) & \text{if } \lambda = 1, \\ T^{-2} Z_T(\lambda)\left\{1 + O\left(T^{-1}\right)\right\} & \text{if } \lambda > 1. \end{cases} \end{aligned} \tag{22}$$

In particular:

(i) Case $\lambda < 1$

$$\begin{aligned} \sum_{t=1}^T a_{tT}t^{-\lambda} &= \frac{\sum_{t=1}^T t^{1-\lambda} - \frac{T+1}{2}\sum_{t=1}^T t^{-\lambda}}{\sum_{s=1}^T \tilde{s}^2} = \frac{\left\{\frac{T^{-\lambda-1}}{2-\lambda} - \frac{T^{-1-\lambda}}{2(1-\lambda)}\right\}\left\{1+O\left(T^{-1}\right)\right\}}{T^{-3}\sum_{s=1}^T \tilde{s}^2} \\ &= -\left(\frac{6\lambda T^{-1-\lambda}}{(2-\lambda)(1-\lambda)}\right)\left\{1+O\left(T^{-1}\right)\right\} \end{aligned}$$

(ii) Case $\lambda = 1$

$$\begin{aligned} \sum_{t=1}^T a_{tT}t^{-\lambda} &= \frac{T - \frac{T+1}{2}\sum_{t=1}^T t^{-1}}{\sum_{s=1}^T \tilde{s}^2} = \frac{\left\{T^{-2} - \frac{T^{-2}}{2}\ln T\right\}\left\{1+O\left(T^{-1}\right)\right\}}{T^{-3}\sum_{s=1}^T \tilde{s}^2} \\ &= -6T^{-2}\ln T\left\{1+O\left(T^{-1}\right)\right\} \end{aligned}$$

(iii) Case $\lambda > 1$

$$\begin{aligned} \sum_{t=1}^T a_{tT}t^{-\lambda} &= \frac{\sum_{t=1}^T t^{1-\lambda} - \frac{T+1}{2}\sum_{t=1}^T t^{-\lambda}}{\sum_{s=1}^T \tilde{s}^2} = \frac{\frac{T^{2-\lambda}}{2-\lambda}\left\{1+O\left(T^{-1}\right)\right\} - \frac{T+1}{2}Z_T(\lambda)}{T^3 \times \frac{1}{T^3}\sum_{s=1}^T \tilde{s}^2} \\ &= -\left(\frac{\frac{T^{-2}}{2}Z_T(\lambda)\left\{1+O\left(T^{-1}\right)\right\}}{\frac{1}{T^3}\sum_{s=1}^T \tilde{s}^2}\right) = -6T^{-2}Z_T(\lambda)\left\{1+O\left(T^{-1}\right)\right\} \end{aligned}$$

Next consider $I_B = \sum_{t=1}^T a_{tT}\xi_{n,t}$.When $\sigma_{a\mu} \neq 0$, we have

$$I_B = O_p\left(n^{-1/2}\right) \times \left(\sum_{t=1}^T \tilde{t}^2\right)^{-1} \sum_{t=1}^T \tilde{t}t^{-\lambda}$$

Thus, $I_B = O_p\left(n^{-1/2}\right) \times I_A$ and I_A dominates I_B always. When $\sigma_{a\mu} = 0$ (this only influences M1 and M3), term $2\sigma_{a\mu}t^{-\alpha}$ disappears in I_A, but term $2\sigma_{a\mu,n}t^{-\alpha}$ is still present in I_B. Hence, when $\sigma_{a\mu} = 0$,

$$I_B = \begin{cases} O_p\left(n^{-1/2}T^{-3}\right)\mathcal{T}_T(1,\alpha) & \text{for M1,} \\ O_p\left(n^{-1/2}T^{-3}\right)\mathcal{T}_T(1,\lambda^*), \text{ with } \lambda^* = \min[\alpha, 2\beta] & \text{for M3,} \end{cases}$$

since $n^{-1}\sum_{i=1}^n \tilde{a}_i\tilde{\mu}_i = O_p\left(n^{-1/2}\right)$ by (19) in KPS, and

$$\begin{aligned} & \left(\sum_{t=1}^T \tilde{t}^2\right)^{-1} \sum_{t=1}^T \sigma_{a\mu,n}\tilde{t}\widetilde{t^{-\alpha}} \\ = \quad & \left(\sum_{t=1}^T \tilde{t}^2\right)^{-1} \left(n^{-1}\sum_{i=1}^n \tilde{a}_i\tilde{\mu}_i\right)\sum_{t=1}^T \tilde{t}\widetilde{t^{-\alpha}} = \begin{cases} O_p\left(n^{-1/2}T^{-1-\alpha}\right) & \text{if } \alpha < 1, \\ O_p\left(n^{-1/2}T^{-2}\ln T\right) & \text{if } \alpha = 1, \\ O_p\left(n^{-1/2}T^{-2}\right) & \text{if } \alpha > 1. \end{cases} \end{aligned}$$

For term I_C, first recall that

$$\varepsilon_{n,t} = \begin{cases} 2n^{-1}\sum_{i=1}^n(\tilde{a}_i\tilde{\epsilon}_{it} + \tilde{\mu}_i\tilde{\epsilon}_{it}t^{-\alpha}) + \left(\sigma^2_{\epsilon,nt} - \sigma^2_{\epsilon,nT}\right) & \text{for M1,} \\ 2n^{-1}\sum_{i=1}^n \tilde{a}_i\tilde{\epsilon}_{it}t^{-\beta} + \left(\sigma^2_{\epsilon,nt} - \sigma^2_{\epsilon,nT}\right)t^{-2\beta} & \text{for M2,} \\ 2n^{-1}\sum_{i=1}^n(\tilde{a}_i\tilde{\epsilon}_{it}t^{-\beta} + \tilde{\mu}_i\tilde{\epsilon}_{it}t^{-\alpha-\beta}) + \left(\sigma^2_{\epsilon,nt} - \sigma^2_{\epsilon,nT}\right)t^{-2\beta} & \text{for M3.} \end{cases}$$

Let $\tilde{\varsigma}_{it} = 2\tilde{a}_i\tilde{\epsilon}_{it}$. Then irrespective of whether $\sigma_{a\mu} = 0$, or $\sigma_{a\mu} \neq 0$, if $\alpha > 0$ and $\beta > 0$ the dominant term in $\varepsilon_{n,t}$ is as follows:

M1	M2	M3
$n^{-1}\sum_{i=1}^n \tilde{\varsigma}_{it} + \left(\sigma^2_{\epsilon,nt} - \sigma^2_{\epsilon,nT}\right)$	$n^{-1}\sum_{i=1}^n \tilde{\varsigma}_{it}t^{-\beta}$	$n^{-1}\sum_{i=1}^n \tilde{\varsigma}_{it}t^{-\beta}$

Using lemma 3, for M2 and M3, we have

$$I_C = \left(\sum_{t=1}^T \tilde{t}^2\right)^{-1} \sum_{t=1}^T \left(n^{-1}\sum_{i=1}^n \tilde{\varsigma}_{it}\right)\tilde{t}\widetilde{t^{-\beta}} = \begin{cases} O_p\left(n^{-1/2}T^{-3/2-\beta}\right) & \text{if } \beta < 1/2, \\ O_p\left(n^{-1/2}T^{-2}\left[\ln T\right]^{1/2}\right) & \text{if } \beta = 1/2, \\ O_p\left(n^{-1/2}T^{-2}\right) & \text{if } \beta > 1/2, \end{cases}$$

and then, for M1,

$$I_C = \left(\sum_{t=1}^T \tilde{t}^2\right)^{-1} n^{-1}\sum_{i=1}^n\sum_{t=1}^T \tilde{\varsigma}_{it}\tilde{t} + \left(\sum_{t=1}^T \tilde{t}^2\right)^{-1}\sum_{t=1}^T \tilde{t}\left(\sigma^2_{\epsilon,nt} - \sigma^2_{\epsilon,nT}\right) = O_p\left(n^{-1/2}T^{-3/2}\right).$$

If $\alpha < 0$ or $\beta < 0$,the order of term I_C will be discussed under each case.

The Asymptotic Limit of $\hat{\phi}_{nT}$ When $\alpha > 0, \beta > 0$

When $\alpha > 0, \beta > 0$,we separate the proof when $\sigma_{a\mu} = 0$ from that when $\sigma_{a\mu} \neq 0$.

(i) The Asymptotic Limit of $\hat{\phi}_{nT}$ when $\sigma_{a\mu} = 0$

Recall that $\eta_{t,d} = bt^{-\lambda}$. Then, when $\sigma_{a\mu} = 0$, we have: $\lambda = 2\alpha$ and $b = \sigma^2_\mu$ in M1; $\lambda = 2\beta$, $b = \sigma^2_\epsilon$ in M2; and $\lambda = \min[2\alpha, 2\beta]$, with $b = \sigma^2_\mu$ if $\lambda = 2\alpha$, and $b = \sigma^2_\epsilon$ if $\lambda = 2\beta$ in M3. We take each model in turn to obtain the final results.

Under M1: We have

$$\begin{aligned}
\hat{\phi}_{nT} &= I_A + I_B + I_C \\
&= \begin{cases} O\left(T^{-1-2\alpha}\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-3/2}\right) & \text{if } \alpha < 1/2, \\ O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-3/2}\right) & \text{if } \alpha = 1/2, \\ O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-3/2}\right) & \text{if } 1/2 < \alpha < 1, \\ O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-3/2}\right) & \text{if } \alpha = 1, \\ O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-2}\right) + O_p\left(n^{-1/2}T^{-3/2}\right) & \text{if } \alpha > 1, \end{cases} \\
&= \begin{cases} O\left(T^{-1-2\alpha}\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) & \text{if } \alpha < 1/2, \\ O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-3/2}\right) & \text{if } \alpha = 1/2, \\ O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-3/2}\right) & \text{if } \alpha > 1/2. \end{cases} \qquad (23)
\end{aligned}$$

So I_A dominates I_B when $n/T \to \infty$.

Under M2 For M2, I_A always dominates I_B as discussed above. Then

$$\begin{aligned}
\hat{\phi}_{nT} &= \left(I_A + I_C\right)\{1 + o(1)\} \\
&= \begin{cases} O\left(T^{-1-2\beta}\right) + O_p\left(n^{-1/2}T^{-3/2-\beta}\right) & \text{if } \beta < 1/2, \\ O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-2}\left[\ln T\right]^{1/2}\right) & \text{if } \beta = 1/2, \\ O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-2}\right) & \text{if } \beta > 1/2, \end{cases} \qquad (24)
\end{aligned}$$

so that I_A dominates I_C.

Under M3 We have

$$I_A = O\left(T^{-3}\right) \times \mathcal{T}_T(1, \lambda) = \begin{cases} O\left(T^{-1-\lambda}\right) & \text{if } \lambda < 1, \\ O\left(T^{-2}\ln T\right) & \text{if } \lambda = 1, \\ O\left(T^{-2}\right) & \text{if } \lambda > 1, \end{cases} \text{ with } \lambda = \min\left[2\alpha, 2\beta\right],$$

$$\begin{aligned}
I_B &= O_p\left(n^{-1/2}T^{-3}\right)\mathcal{T}_T(1, \lambda^*), \text{ with } \lambda^* = \min\left[\alpha, 2\beta\right], \\
&= \begin{cases} O_p\left(n^{-1/2}T^{-1-\lambda^*}\right) & \text{if } \lambda^* < 1, \\ O_p\left(n^{-1/2}T^{-2}\ln T\right) & \text{if } \lambda^* = 1, \\ O_p\left(n^{-1/2}T^{-2}\right) & \text{if } \lambda^* > 1, \end{cases}
\end{aligned}$$

$$I_C = \begin{cases} O_p\left(n^{-1/2}T^{-3/2-\beta}\right) & \text{if } \beta < 1/2, \\ O_p\left(n^{-1/2}T^{-2}\left[\ln T\right]^{1/2}\right) & \text{if } \beta = 1/2, \\ O_p\left(n^{-1/2}T^{-2}\right) & \text{if } \beta > 1/2. \end{cases}$$

We need to consider the following two subcases.

Case 1: $(\alpha \leq \beta)$ Combining all the three terms, we have the following.

$$\begin{aligned}
\hat{\phi}_{nT} &= I_A + I_B + I_C \\
&= \begin{cases}
O\left(T^{-1-2\alpha}\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-3/2-\beta}\right) & \text{if } \alpha \leq \beta < 1/2, \\
O\left(T^{-1-2\alpha}\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-2}\left[\ln T\right]^{1/2}\right) & \text{if } \alpha < \beta = 1/2, \\
O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-2}\left[\ln T\right]^{1/2}\right) & \text{if } \alpha = \beta = 1/2, \\
O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-2}\right) & \text{if } 1/2 = \alpha < \beta, \\
O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-2}\left[\ln T\right]^{1/2}\right) & \text{if } 1/2 = \alpha = \beta, \\
O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-2}\right) + O_p\left(n^{-1/2}T^{-2}\right) & \text{if } 1 < \alpha \leq \beta, \\
O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-2}\right) & \text{if } 1/2 < \alpha < 1 \leq \beta, \\
O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-2}\right) & \text{if } 1 = \alpha \leq \beta, \\
O\left(T^{-1-2\alpha}\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-2}\right) & \text{if } \alpha < 1/2 < \beta,
\end{cases} \\
&= \begin{cases}
O\left(T^{-1-2\alpha}\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) & \text{if } \alpha < 1/2, \\
O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) & \text{if } \alpha = 1/2, \\
O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) & \text{if } 1/2 < \alpha < 1, \\
O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-2}\ln T\right) & \text{if } \alpha = 1, \\
O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-2}\right) & \text{if } \alpha > 1,
\end{cases} \qquad (25)
\end{aligned}$$

so that I_A dominates I_B and I_C when $n/T \to \infty$.

Case 2: $(\alpha > \beta)$ When $\sigma_{a\mu} = 0$ and $\alpha > \beta$, I_A always dominates I_C. When $2\beta \leq \alpha$,

$$\begin{aligned}
\hat{\phi}_{nT} &= I_A + I_B \\
&= \begin{cases}
O\left(T^{-1-2\beta}\right) + O_p\left(n^{-1/2}T^{-3/2-\beta}\right) & \text{if } \beta < 1/2, \\
O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-2}\left[\ln T\right]^{1/2}\right) & \text{if } \beta = 1/2, \\
O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-2}\right) & \text{if } \beta > 1/2.
\end{cases}
\end{aligned}$$

If $2\beta > \alpha > \beta$, we have

$$\begin{aligned}
\hat{\phi}_{nT} &= I_A + I_B \\
&= \begin{cases}
O\left(T^{-1-2\beta}\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) & \text{if } \alpha < 2\beta < 1, \\
O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) & \text{if } \alpha < 2\beta = 1, \\
O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-1-\alpha}\right) & \text{if } \alpha < 1 < 2\beta, \\
O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-2}\ln T\right) & \text{if } \alpha = 1 < 2\beta, \\
O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-2}\right) & \text{if } 1 < \alpha < 2\beta,
\end{cases} \qquad (26)
\end{aligned}$$

so that I_A dominates I_B when $n/T \to \infty$.

(ii) The Asymptotic Limit of $\hat{\phi}_{nT}$ when $\sigma_{a\mu} \neq 0$

When $\sigma_{a\mu} \neq 0$,and $\alpha, \beta > 0$, we have: $\lambda = \alpha$ and $b = \sigma_{a\mu}$ for M1; $\lambda = 2\beta$ and $b = \sigma_{\epsilon}^2$ for M2; $\lambda = \min\left[\alpha, 2\beta\right]$ for M3, with $b = \sigma_{a\mu}$ if $\lambda = \alpha$, and $b = \sigma_{\epsilon}^2$ if $\lambda = 2\beta$ for M3. When $b = \sigma_{a\mu}$, the sign of I_A is consonant with that of $-b$. Hence, when $\sigma_{a\mu} > 0$, I_A is negative, and when $\sigma_{a\mu} < 0$, I_A is positive.

Under M1 $\hat{\phi}_{nT}$ can be written as

$$\begin{aligned}\hat{\phi}_{nT} &= (I_A + I_C)\{1 + o_p(1)\} \\ &= \begin{cases} O\left(T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-3/2}\right) & \text{if } \alpha < 1, \\ O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-3/2}\right) & \text{if } \alpha = 1, \\ O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-3/2}\right) & \text{if } \alpha > 1, \end{cases}\end{aligned} \tag{27}$$

so that if $\alpha < 0.5$, then I_A dominates I_C for any n. Otherwise, I_A dominates I_C when $n/T \to \infty$.

Under M2 For M2, the behavior of $\hat{\phi}_{nT}$ when $\sigma_{a\mu} \neq 0$ is the same as when $\sigma_{a\mu} = 0$.

Under M3 When $\sigma_{a\mu} \neq 0$, we have

$$I_A = \begin{cases} O\left(T^{-3}\mathcal{T}_T(1,\alpha)\right) & \text{if } \alpha \leq 2\beta, \\ O\left(T^{-3}\mathcal{T}_T(1,2\beta)\right) & \text{if } \alpha > 2\beta, \end{cases}$$

where $\mathcal{T}_T(1,\alpha) = \sum_{t=1}^{T} \tilde{t}\widetilde{t^{-\alpha}} = \sum_{t=1}^{T} t^{-(\alpha-1)} - \left(\frac{T+1}{2}\right)\sum_{t=1}^{T} t^{-\alpha}$ is defined in the proof of Lemma 2.

From the analysis above, the term I_C has the following order

$$I_C = \begin{cases} O_p\left(n^{-1/2}T^{-3/2-\beta}\right) & \text{if } \beta < 1/2, \\ O_p\left(n^{-1/2}T^{-2}\left[\ln T\right]^{1/2}\right) & \text{if } \beta = 1/2, \\ O_p\left(n^{-1/2}T^{-2}\right) & \text{if } \beta > 1/2. \end{cases}$$

Case 1: ($\alpha \leq 2\beta$) When $\sigma_{a\mu} \neq 0$ and $\alpha \leq 2\beta$,

$$\begin{aligned}\hat{\phi}_{nT} &= \{I_A + I_C\}\{1 + o(1)\} \\ &= \begin{cases} O\left(T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-3/2-\beta}\right) & \text{if } \alpha \leq 2\beta < 1, \\ O\left(T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-2}\left[\ln T\right]^{1/2}\right) & \text{if } \alpha < 2\beta = 1, \\ O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-2}\left[\ln T\right]^{1/2}\right) & \text{if } \alpha = 2\beta = 1, \\ O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-2}\right) & \text{if } 1 = \alpha < 2\beta, \\ O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-2}\left[\ln T\right]^{1/2}\right) & \text{if } 1 = \alpha = 2\beta, \\ O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-2}\right) & \text{if } 1 < \alpha \leq 2\beta, \\ O\left(T^{-1-\alpha}\right) + O_p\left(n^{-1/2}T^{-2}\right) & \text{if } \alpha < 1 < 2\beta, \end{cases} \\ &= \begin{cases} O\left(T^{-1-\alpha}\right) & \text{if } \alpha < 1, \\ O\left(T^{-2}\ln T\right) & \text{if } \alpha = 1, \\ O\left(T^{-2}\right) & \text{if } \alpha > 1. \end{cases}\end{aligned} \tag{28}$$

Note that since $O\left(T^{-1-\alpha+3/2+\beta}\right) = O\left(T^{-\alpha+1/2+\beta}\right) > O\left(T^{-2\beta+1/2+\beta}\right)$ when $\alpha \leq 2\beta$, the first term dominates the second term.

Case 2: $(\alpha > 2\beta)$ When $\sigma_{a\mu} \neq 0$ and $\alpha > 2\beta$,

$$\begin{aligned} \hat{\phi}_{nT} &= I_A + I_C \\ &= \begin{cases} O\left(T^{-1-2\beta}\right) + O_p\left(n^{-1/2}T^{-3/2-\beta}\right) & \text{if } \beta < 1/2, \\ O\left(T^{-2}\ln T\right) + O_p\left(n^{-1/2}T^{-2}\left[\ln T\right]^{1/2}\right) & \text{if } \beta = 1/2, \\ O\left(T^{-2}\right) + O_p\left(n^{-1/2}T^{-2}\right) & \text{if } \beta > 1/2. \end{cases} \end{aligned} \tag{29}$$

The first term always dominates the second term.

The Asymptotic Limit of $\hat{\phi}_{nT}$ When $\alpha < 0$ or $\beta < 0$

Recall the OLS estimate is decomposed in the form

$$\hat{\phi}_{nT} = \sum_{t=1}^{T} a_{tT}\tilde{\eta}_t + \sum_{t=1}^{T} a_{tT}\tilde{\xi}_{n,t} + \sum_{t=1}^{T} a_{tT}\tilde{\varepsilon}_{n,t} =: I_A + I_B + I_C.$$

Note that when $\alpha < 0$ or $\beta < 0$, we have: $\lambda = 2\alpha$ and $b = \sigma_\mu^2$ in M1; $\lambda = 2\beta$ and $b = \sigma_\epsilon^2$ in M2; $\lambda = \min(2\alpha, 2\beta)$ in M3. As shown above, I_A always dominates I_B since $I_B = O_p\left(n^{-1/2}\right) I_A$.Using Lemma 2, we have

$$I_A = -b\left(\frac{6\lambda T^{-1-\lambda}}{(2-\lambda)(1-\lambda)}\right)\left\{1 + O\left(T^{-1}\right)\right\}.$$

Note the sign of I_A is positive when $\alpha < 0$ or $\beta < 0$.

Under Model M1 and M2:

First, we consider the term I_C. If $\alpha < 0$ (for M1) or $\beta < 0$ (for M2), the dominating term in ε_{nt} is $2n^{-1}\sum_{i=1}^{n}\tilde{\mu}_i\tilde{\epsilon}_{it}t^{-\alpha}$ in M1 and $\left(\sigma_{\epsilon,nt}^2 - \sigma_{\epsilon,nT}^2\right)t^{-2\beta}$ in M2. By using lemma 3, we have

$$\begin{aligned} I_C &= \left(\sum_{t=1}^{T}\tilde{t}^2\right)^{-1}\sum_{t=1}^{T}\tilde{t}\varepsilon_{nt} \\ &= \begin{cases} O_p\left(n^{-1/2}T^{-2}\left[\tau_T(2\alpha)\right]^{1/2}\right) & \text{for M1,} \\ O_p\left(n^{-1/2}T^{-2}\left[\tau_T(4\beta)\right]^{1/2}\right) & \text{for M2.} \end{cases} \end{aligned}$$

Combining the two parts we have

$$\begin{aligned} \hat{\phi}_{nT} &= I_A + I_C \\ &= \begin{cases} O\left(T^{-1-2\alpha}\right) + O_p\left(n^{-1/2}T^{-3/2-\alpha}\right) & \text{for M1,} \\ O\left(T^{-1-2\beta}\right) + O_p\left(n^{-1/2}T^{-3/2-2\beta}\right) & \text{for M2.} \end{cases} \end{aligned}$$

Thus, I_A dominates I_C for M1 and M2.

Under Model M3

We proceed case by case as follows.

Case 1: ($\alpha < 0$ **and** $\beta > 0$) If $\alpha < 0$ but $\beta > 0$, the dominating term is $\sigma_\mu^2 t^{-2\alpha}$ in η_t and is $2n^{-1}\sum_{i=1}^n \tilde{\mu}_i \tilde{\epsilon}_{it} t^{-\alpha-\beta}$ in ε_{nt}. Then $I_A = \left(\sum_{t=1}^T \tilde{t}^2\right)^{-1} \sum_{t=1}^T \tilde{t}\tilde{\eta}_t = O\left(T^{-1-2\alpha}\right)$ and

$$I_C = \left(\sum_{t=1}^T \tilde{t}^2\right)^{-1} \sum_{t=1}^T \tilde{t}\tilde{\varepsilon}_{nt} = O_p\left(n^{-1/2} T^{-2} \left[\tau_T\left(2\alpha + 2\beta\right)\right]^{1/2}\right).$$

Note from Lemma 1 that $\tau_T(2\alpha+2\beta)$ reaches the largest order of $O\left(T^{1-2\alpha-2\beta}\right)$ when $\alpha + \beta < 1/2$. Even at this case, $I_C = O_p\left(n^{-1/2}T^{-3/2-\alpha-\beta}\right)$,and is still dominated by I_A.Hence, we have

$$\hat{\phi}_{nT} = O\left(T^{-1-2\alpha}\right) + O_p\left(n^{-1/2}T^{-2}\left[\tau_T\left(2\alpha+2\beta\right)\right]^{1/2}\right) = O\left(T^{-1-2\alpha}\right).$$

Case 2: ($\alpha > 0$ **and** $\beta < 0$) If $\alpha > 0$ and $\beta < 0$, the dominating term is $\sigma_\epsilon^2 t^{-2\beta}$ in η_t, and is $\left(\sigma_{\epsilon,nt}^2 - \sigma_{\epsilon,nT}^2\right) t^{-2\beta}$ in ε_{nt}. Then $I_A = O\left(T^{-1-2\beta}\right)$ and $I_C = O_p\left(n^{-1/2}T^{-2}\left[\tau_T\left(4\beta\right)\right]^{1/2}\right)$. Hence we have

$$\begin{aligned}
\hat{\phi}_{nT} &= O\left(T^{-1-2\beta}\right) + O_p\left(n^{-1/2}T^{-2}\left[\tau_T\left(4\beta\right)\right]^{1/2}\right) \\
&= O\left(T^{-1-2\beta}\right) + O_p\left(n^{-1/2}T^{-3/2-2\beta}\right) = O\left(T^{-1-2\beta}\right).
\end{aligned}$$

Case 3: ($\alpha < 0$ **and** $\beta < 0$) If $\alpha < 0$ and $\beta < 0$,the dominating terms are $\sigma_\mu^2 t^{-2\alpha} + \sigma_\epsilon^2 t^{-2\beta}$ in η_t, and $2n^{-1}\sum_{i=1}^n \tilde{\mu}_i\tilde{\epsilon}_{it} t^{-\alpha-\beta} + \left(\sigma_{\epsilon,nt}^2 - \sigma_{\epsilon,nT}^2\right) t^{-2\beta}$ in ε_{nt}. Then we have

$$\begin{aligned}
\hat{\phi}_{nT} &= O\left(T^{-3}\right)\left[\sigma_\mu^2 \mathcal{T}_T\left(1, 2\alpha\right) + \sigma_\epsilon^2 \mathcal{T}_T\left(1, 2\beta\right)\right] + O_p\left(n^{-1/2}T^{-2}\left[\tau_T\left(2\alpha+2\beta\right)\right]^{1/2}\right) \\
&\quad + O_p\left(n^{-1/2}T^{-2}\left[\tau_T\left(4\beta\right)\right]^{1/2}\right) \\
&= O\left(T^{-1-2\delta}\right),
\end{aligned}$$

where $\delta = \min\left(\alpha, \beta\right)$.

The Asymptotic Limit of $\hat{\phi}_{nT}$ When $\alpha = \beta = 0$

This case is covered by standard theory and the proof is omitted.

□

Proof of Theorem 2: (t-ratio of $\hat{\phi}_{nT}$)

The proof under $\alpha = \beta = 0$ is standard and is omitted.

(i) The Asymptotic Limit of $t_{\hat{\phi}_{nT}}$ when $\alpha > 0, \beta > 0$

The asymptotic behavior of the t-ratio is determined by the three component factors of

$$t_{\hat{\phi}_{nT}} = \hat{\phi}_{nT}\left(\sum_{t=1}^T \tilde{t}^2\right)^{1/2} / \sqrt{\Omega_{\mathcal{M}}^2}.$$

The behavior of $\hat{\phi}_{nT}$ is described above in (23) - (29), and $\sum_{t=1}^T \tilde{t}^2 = \frac{1}{12}T^3\left\{1 + O\left(T^{-1}\right)\right\}$. The behavior of the long run variance estimate $\Omega_{\mathcal{M}}^2$ is obtained as follows. As in the proof of Theorem 1, it is convenient to use the following specifications of λ for each model.

Model specifications of λ

- M1: $\lambda = \alpha\,(\text{or } 2\alpha)$ when $\sigma_{a\mu} \neq 0$ (respectively, $\sigma_{a\mu} = 0$);
- M2: $\lambda = 2\beta$;
- M3: $\lambda = \min[\alpha, 2\beta]$ (or $\min[2\alpha, 2\beta]$) when $\sigma_{a\mu} \neq 0$ (respectively, $\sigma_{a\mu} = 0$).

We write $\widetilde{\mathcal{M}}_{nt} = \tilde{m}_t + R_{nt}$, where the deterministic part is $\tilde{m}_t = \widetilde{t^{-\lambda}} - \tilde{t}\left(\sum_{t=1}^T \tilde{t}\widetilde{t^{-\lambda}}\right)\left(\sum_{t=1}^T \tilde{t}^2\right)^{-1}$ and the random part is $R_{nt} = \tilde{\xi}_{n,t} - \tilde{t}\,(I_B + I_C)$. Note that as $n/T \to \infty$, $R_{nt} = o_p(\tilde{m}_t)$ uniformly in $t \leq T$. The regression residual is given by

$$\hat{u}_t = \widetilde{K^x_{nt}} - \hat{\phi}_{nT}\tilde{t} = \left(\tilde{\eta}_{n,t} - \hat{\phi}_{nT}\tilde{t}\right) + \tilde{\varepsilon}_{nt} =: \tilde{\mathcal{M}}_{nt} + \tilde{\varepsilon}_{nt}.$$

We decompose $\hat{\Omega}^2_u$, the long run variance estimate with lag truncation parameter L and Bartlett lag kernel $\vartheta_{\ell L}$, as follows

$$\begin{aligned}
\hat{\Omega}^2_u &= \frac{1}{T}\sum_{t=1}^{T}\hat{u}_t^2 + 2\frac{1}{T}\sum_{\ell=1}^{L}\vartheta_{\ell L}\sum_{t=1}^{T-\ell}\hat{u}_t\hat{u}_{t+\ell} \\
&= \frac{1}{T}\sum_{t=1}^{T}\left(\tilde{\mathcal{M}}_{nt} + \tilde{\varepsilon}_{nt}\right)^2 + 2\frac{1}{T}\sum_{\ell=1}^{L}\vartheta_{\ell L}\sum_{t=1}^{T-\ell}\left(\tilde{\mathcal{M}}_{nt} + \tilde{\varepsilon}_{nt}\right)\left(\tilde{\mathcal{M}}_{nt+\ell} + \tilde{\varepsilon}_{nt+\ell}\right) \\
&= \frac{1}{T}\sum_{t=1}^{T}\tilde{\mathcal{M}}^2_{nt} + 2\frac{1}{T}\sum_{\ell=1}^{L}\vartheta_{\ell L}\sum_{t=1}^{T-\ell}\tilde{\mathcal{M}}_{nt}\tilde{\mathcal{M}}_{nt+\ell} \\
&\quad + \frac{1}{T}\sum_{t=1}^{T}\tilde{\varepsilon}^2_{nt} + 2\frac{1}{T}\sum_{\ell=1}^{L}\vartheta_{\ell L}\sum_{t=1}^{T-\ell}\tilde{\varepsilon}_{nt}\tilde{\varepsilon}_{nt+\ell} \\
&\quad + 2\frac{1}{T}\sum_{t=1}^{T}\tilde{\mathcal{M}}_{nt}\tilde{\varepsilon}_{nt} + 2\frac{1}{T}\sum_{\ell=1}^{L}\vartheta_{\ell L}\sum_{t=1}^{T-\ell}\left(\tilde{\mathcal{M}}_{nt}\tilde{\varepsilon}_{nt+\ell} + \tilde{\varepsilon}_{nt}\tilde{\mathcal{M}}_{nt+\ell}\right) \\
&:= \hat{\Omega}^2_{\mathcal{M}} + \hat{\Omega}^2_{\varepsilon} + 2\hat{\Omega}_{\mathcal{M}\varepsilon},
\end{aligned}$$

where

$$\hat{\Omega}^2_{\mathcal{M}} = \frac{1}{T}\sum_{t=1}^{T}\tilde{\mathcal{M}}^2_{nt} + \frac{2}{T}\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\left(1 - \frac{\ell}{T^\kappa + 1}\right)\tilde{\mathcal{M}}_{nt}\tilde{\mathcal{M}}_{nt+\ell}, \tag{30}$$

$$\Omega^2_{\mathcal{M}} = \frac{1}{T}\sum_{t=1}^{T}\tilde{m}^2_t + \frac{2}{T}\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\left(1 - \frac{\ell}{T^\kappa + 1}\right)\tilde{m}_t\tilde{m}_{t+\ell}. \tag{31}$$

Note that

$$\begin{aligned}
\hat{\Omega}^2_{\mathcal{M}} &= \frac{1}{T}\sum_{t=1}^{T}\tilde{\mathcal{M}}^2_{nt} + \frac{2}{T}\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\left(1 - \frac{\ell}{T^\kappa + 1}\right)\tilde{\mathcal{M}}_{nt}\tilde{\mathcal{M}}_{nt+\ell} \\
&= \frac{1}{T}\sum_{t=1}^{T}(\tilde{m}_t + R_{nt})^2 + \frac{2}{T}\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\left(1 - \frac{\ell}{T^\kappa + 1}\right)(\tilde{m}_t + R_{nt})(\tilde{m}_{t+\ell} + R_{nt+\ell}) \\
&= \frac{1}{T}\sum_{t=1}^{T}\tilde{m}^2_t + \frac{2}{T}\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\left(1 - \frac{\ell}{T^\kappa + 1}\right)\tilde{m}_t\tilde{m}_{t+\ell} \\
&\quad + \frac{1}{T}\sum_{t=1}^{T}R^2_{nt} + \frac{2}{T}\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\left(1 - \frac{\ell}{T^\kappa + 1}\right)R_{nt}R_{nt+\ell} \\
&\quad + 2\frac{1}{T}\sum_{t=1}^{T}\tilde{m}_t R_{nt} + \frac{2}{T}\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\left(1 - \frac{\ell}{T^\kappa + 1}\right)(\tilde{m}_t R_{nt+\ell} + R_{nt}\tilde{m}_{t+\ell})
\end{aligned}$$

We have shown that $R_{nt} = o_p(\tilde{m}_t)$ uniformly in $t \leq T$. It follows that $\hat{\Omega}^2_{\mathcal{M}} = \Omega^2_{\mathcal{M}} + o_p\left(\Omega^2_{\mathcal{M}}\right)$, which we write as $\hat{\Omega}^2_{\mathcal{M}} \sim \Omega^2_{\mathcal{M}}$.

From Lemma 4, we know that

$$\Omega_{\mathcal{M}}^2 \sim \begin{cases} \frac{\lambda^2(\lambda+1)^2}{(1-2\lambda)(\lambda^2-3\lambda+2)^2} b^2 T^{-2\lambda+\kappa} & \text{if } \lambda < 1/2, \\ b^2 T^{-1+\kappa} \ln T & \text{if } \lambda = 1/2, \\ b^2 T^{-1+\kappa} \zeta(2\lambda) & \text{if } 1/2 < \lambda < 1/(1+\kappa), \\ \dfrac{1}{(1-\lambda)^2(2-\lambda)} b^2 T^{-\lambda+\kappa-\lambda\kappa} & \text{if } 1/(1+\kappa) \le \lambda < 1, \\ \frac{\kappa^2}{2} b^2 T^{-1} \ln^2 T + O\left(T^{-2}\ln T\right) & \text{if } \lambda = 1, \\ T^{-1} b^2 \left\{\sum_{t=1}^{\infty} t^{-\lambda} \zeta(\lambda, \ell) - \zeta(2\lambda)\right\} & \text{if } \lambda > 1. \end{cases}$$

Recall that

$$\varepsilon_{n,t} = \begin{cases} 2n^{-1}\sum_{i=1}^n \tilde{a}_i \tilde{\epsilon}_{it} + 2n^{-1}\sum_{i=1}^n \tilde{\mu}_i \tilde{\epsilon}_{it} t^{-\alpha} + \left(\sigma_{\epsilon,nt}^2 - \sigma_{\epsilon,nT}^2\right) & \text{for M1}, \\ 2n^{-1}\sum_{i=1}^n \tilde{a}_i \tilde{\epsilon}_{it} t^{-\beta} + \left(\sigma_{\epsilon,nt}^2 - \sigma_{\epsilon,nT}^2\right) t^{-2\beta} & \text{for M2}, \\ 2n^{-1}\sum_{i=1}^n \tilde{a}_i \tilde{\epsilon}_{it} t^{-\beta} + 2n^{-1}\sum_{i=1}^n \tilde{\mu}_i \tilde{\epsilon}_{it} t^{-\alpha-\beta} + \left(\sigma_{\epsilon,nt}^2 - \sigma_{\epsilon,nT}^2\right) t^{-2\beta} & \text{for M3}. \end{cases}$$

For fixed t, $\varepsilon_{n,t} = O_p\left(n^{-1/2}\right)$. When $\alpha > 0$ and $\beta > 0$, the dominant term in $\varepsilon_{n,t}$ is $2n^{-1}\sum_{i=1}^n \tilde{a}_i \tilde{\epsilon}_{it}$ for M1 and $2n^{-1}\sum_{i=1}^n \tilde{a}_i \tilde{\epsilon}_{it} t^{-\beta}$ for M2 and M3. From Lemma 3, we have

$$\begin{aligned} \hat{\Omega}_{\varepsilon}^2 &= \frac{1}{T}\sum_{t=1}^{T} \tilde{\varepsilon}_{nt}^2 + 2\frac{1}{T}\sum_{\ell=1}^{L} \vartheta_{\ell L} \sum_{t=1}^{T-\ell} \tilde{\varepsilon}_{nt}\tilde{\varepsilon}_{nt+\ell} \\ &= \begin{cases} O_p\left(n^{-1}T^{-1}\mathcal{S}(\beta)\right) & \text{in M2 \& M3}, \\ O_p\left(n^{-1}\right) & \text{in M1}. \end{cases} \end{aligned}$$

Comparing $\hat{\Omega}_{\mathcal{M}}^2$ and $\hat{\Omega}_{\varepsilon}^2$, it is evident that the order of $\hat{\Omega}_{\mathcal{M}}^2$ exceeds that of $\hat{\Omega}_{\varepsilon}^2$ as long as $n/T \to \infty$. Next, consider $\hat{\Omega}_{m\varepsilon}$. By Cauchy-Schwarz, $\hat{\Omega}_{m\varepsilon}$ is bounded above as

$$\frac{1}{T}\sum_{t=1}^{T} \tilde{\mathcal{M}}_{nt}\tilde{\varepsilon}_{nt} \le \left(\frac{1}{T}\sum_{t=1}^{T} \tilde{\mathcal{M}}_{nt}^2\right)^{1/2} \left(\frac{1}{T}\sum_{t=1}^{T} \tilde{\varepsilon}_{nt}^2\right)^{1/2} = o_p\left(\frac{1}{T}\sum_{t=1}^{T} \tilde{\mathcal{M}}_{nt}^2\right),$$

since $\frac{1}{T}\sum_{t=1}^{T} \tilde{\varepsilon}_{nt}^2 = o_p\left(\frac{1}{T}\sum_{t=1}^{T} \tilde{\mathcal{M}}_{nt}^2\right)$.

Combining these results, we find that

$$\hat{\Omega}_u^2 = \hat{\Omega}_{\mathcal{M}}^2 + \hat{\Omega}_{\varepsilon}^2 + 2\hat{\Omega}_{\mathcal{M}\varepsilon} \sim \hat{\Omega}_{\mathcal{M}}^2 \sim \Omega_{\mathcal{M}}^2, \tag{32}$$

and therefore

$$\hat{\Omega}_u^2 = \begin{cases} \frac{\lambda^2(\lambda+1)^2}{(1-2\lambda)(\lambda^2-3\lambda+2)^2} b^2 T^{-2\lambda+\kappa} & \text{if } \lambda < 1/2, \\ b^2 T^{-1+\kappa} \ln T & \text{if } \lambda = 1/2, \\ b^2 T^{-1+\kappa} \zeta(2\lambda) & \text{if } 1/2 < \lambda < 1/(1+\kappa), \\ \dfrac{1}{(1-\lambda)^2(2-\lambda)} b^2 T^{-\lambda+\kappa-\lambda\kappa} & \text{if } 1/(1+\kappa) \le \lambda < 1, \\ \frac{\kappa^2}{2} b^2 T^{-1} \ln^2 T + O\left(T^{-2}\ln T\right) & \text{if } \lambda = 1, \\ T^{-1} b^2 \left\{\sum_{t=1}^{\infty} t^{-\lambda} \zeta(\lambda, \ell) - \zeta(2\lambda)\right\} & \text{if } \lambda > 1. \end{cases} \tag{33}$$

Using (32), the t ratio $t_{\hat{\phi}_{nT}}$ has the same asymptotic form as

$$t_{\hat{\phi}_{nT}} \sim \frac{\hat{\phi}_{nT}\left(\sum \tilde{t}^2\right)^{1/2}}{\sqrt{\hat{\Omega}_{\mathcal{M}}^2 + \hat{\Omega}_{\varepsilon}^2 + 2\hat{\Omega}_{\mathcal{M}\varepsilon}}} \sim \frac{\hat{\phi}_{nT}\left(\sum \tilde{t}^2\right)^{1/2}}{\sqrt{\Omega_{\mathcal{M}}^2}},$$

where $\Omega^2_{\mathcal{M}}$ is deterministic. Therefore, using (33) as $n/T \to \infty$ with $\kappa = 1/3$ and a finite λ, the asymptotic behavior of the t-ratio in all models can be represented as follows:

$$t_{\hat{\phi}_{nT}} = \begin{cases} \dfrac{-\sqrt{3}\,(1-2\lambda)^{1/2}}{(\lambda+1)} T^{1/2-\kappa/2} = (-1) \times O\left(T^{1/2-\kappa/2}\right) \to -\infty & \text{if } \lambda < 1/2, \\ -\frac{2}{3}\sqrt{3} T^{1/2-\kappa/2} (\ln T)^{-1/2} = (-1) \times O\left(T^{1/2-\kappa/2} (\ln T)^{-1/2}\right) \to -\infty & \text{if } \lambda = 1/2, \\ \dfrac{-\sqrt{3}\lambda}{(\lambda-2)(\lambda-1)} (\zeta(2\lambda))^{-1/2} T^{1-\lambda-\kappa/2} = (-1) \times O\left(T^{1-\lambda-\kappa/2}\right) \to -\infty & \text{if } 1/2 < \lambda < 1/(1+\kappa), \\ \dfrac{-\sqrt{3}\lambda}{(2-\lambda)^{1/2}} T^{(1-\lambda)(1-\kappa)/2} = (-1) \times O\left(T^{(1-\lambda)(1-\kappa)/2}\right) = -\infty & \text{if } 1/(1+\kappa) \le \lambda < 1, \\ \dfrac{-6bT^{-2}\ln T\left(\frac{1}{12}T^3\right)^{1/2}}{\left(\dfrac{\kappa^2}{2} b^2 T^{-1} \ln^2 T\right)^{1/2}} = -3\sqrt{6} & \text{if } \lambda = 1, \\ \dfrac{-6bT^{-2}\zeta(\lambda)\left(\frac{1}{12}T^3\right)^{1/2}}{\left(T^{-1}b^2\left\{\sum_{t=1}^{\infty} t^{-\lambda}\zeta(\lambda,t)\right\}\right)^{1/2}} = \dfrac{-\zeta(\lambda)\sqrt{3}}{\left(\sum_{t=1}^{\infty} t^{-\lambda}\zeta(\lambda,t)\right)^{1/2}} > -\sqrt{3} & \text{if } \lambda > 1. \end{cases} \tag{34}$$

Note that with $\kappa = 1/3$, we have $1 - \lambda - \kappa/2 > 0$ as long as $1/2 < \lambda < 1/(1+\kappa)$. The last inequality in (34) is obtained by noting that when $t > 1$, $\zeta(\lambda, t) < \zeta(\lambda)$ and hence

$$\sum_{t=1}^{\infty} t^{-\lambda}\zeta(\lambda,t) < \sum_{t=1}^{\infty} t^{-\lambda} \sum_{s=1}^{\infty} \frac{1}{s^{\lambda}} = \zeta(\lambda)^2,$$

giving

$$\frac{\zeta(\lambda)}{\left(\sum_{t=1}^{\infty} t^{-\lambda}\zeta(\lambda,t)\right)^{1/2}} > 1 \text{ for all } \lambda > 1.$$

Moreover, when $\lambda \to \infty$ we have

$$\begin{aligned} \lim_{\lambda\to\infty} \left\{\sum_{t=1}^{\infty} t^{-\lambda}\zeta(\lambda,t) - \zeta(2\lambda)\right\} &= \lim_{\lambda\to\infty} \left\{\sum_{t=1}^{\infty}\sum_{s=0}^{\infty} \frac{1}{t^{\lambda}(t+s)^{\lambda}} - \sum_{t=1}^{\infty} \frac{1}{t^{2\lambda}}\right\} \\ &= \lim_{\lambda\to\infty} \left\{\sum_{t=1}^{\infty}\sum_{s=1}^{\infty} \frac{1}{t^{\lambda}(t+s)^{\lambda}}\right\} = 0. \end{aligned}$$

Then, as $\lambda \to \infty$ we have

$$\lim_{\lambda\to\infty} t_{\hat{\phi}_{nT}} = -\lim_{\lambda\to\infty} \frac{\zeta(\lambda)\sqrt{3}}{\left(\sum_{t=1}^{\infty} t^{-\lambda}\zeta(\lambda,t)\right)^{1/2}} = -\lim_{\lambda\to\infty} \frac{\zeta(\lambda)\sqrt{3}}{\zeta(2\lambda)^{1/2}} = -\sqrt{3},$$

giving a sharp result for the t-ratio for large λ.

On the other hand, as $\lambda \to 0$, the t-ratio has the following limit

$$\lim_{\lambda\to+0} t_{\hat{\phi}_{nT}} = \lim_{\lambda\to+0} \frac{-\sqrt{3}\,(1-2\lambda)^{1/2}}{(\lambda+1)} T^{1/2-\kappa/2} = -\sqrt{3}T^{1/2-\kappa/2} \to -\infty \text{ as } T \to \infty.$$

(ii) The Asymptotic Limit of $t_{\hat{\phi}_{nT}}$ when $\alpha < 0$ or $\beta < 0$

Recall the representation

$$\tilde{\mathcal{M}}_{nt} = b\left[\widetilde{t^{-\lambda}} - \tilde{t}\left(\sum_{t=1}^{T} \tilde{t}^2\right)^{-1} \sum_{t=1}^{T} \tilde{t}\widetilde{t^{-\lambda}}\right] + R_{nt} = b\tilde{m}_t + R_{nt}, \tag{35}$$

where $\lambda = 2\alpha$ for M1, $\lambda = 2\beta$ for M2, and $\lambda = \min[2\alpha, 2\beta]$ for M3. The factor b in (35) is positive since $b = \sigma_\epsilon^2$ or σ_μ^2. Then we have

$$\frac{1}{T}\sum\nolimits_{t=1}^{T}\tilde{m}_t^2 = b^2\mathcal{B}(\lambda) = O_p\left(T^{-2\lambda}\right),$$

since from Lemma 2, $\mathcal{B}(\lambda) = O\left(T^{-2\lambda}\right)$ when $\lambda < 0$. Note also that when $\lambda < 0$,

$$\Psi_{\ell,11} = \sum\nolimits_{t=1}^{T-\ell}\left(t^2 + t\ell\right)^{-\lambda} = \sum\nolimits_{t=1}^{T-\ell} t^{-\lambda}(t+\ell)^{-\lambda} < \sum\nolimits_{t=1}^{T-\ell}(t+\ell)^{-2\lambda} \sim \frac{T^{1-2\lambda}}{1-2\lambda}$$

Then

$$\frac{1}{T}\sum\nolimits_{\ell=1}^{L}\left(1 - \frac{\ell}{L+1}\right)\Psi_{\ell,11} < \frac{T^{-2\lambda}L}{1-2\lambda} = O\left(T^{-2\lambda+\kappa}\right).$$

This implies from lemma 4 that

$$\frac{1}{T}\sum\nolimits_{\ell=1}^{L}\sum\nolimits_{t=1}^{T-\ell}\left(1 - \frac{\ell}{L+1}\right)\tilde{m}_t\tilde{m}_{t+\ell} = O\left(T^{-2\lambda+\kappa}\right).$$

The order of $\Omega_{\mathcal{M}}^2$ is at least $O_p\left(T^{-2\lambda}\right)$ and is less than $O\left(T^{-2\lambda+\kappa}\right)$,where $\lambda = 2\alpha$ for M1, $\lambda = 2\beta$ for M2, and $\lambda = \min[2\alpha, 2\beta]$ for M3. From Lemma 3, we have

$$\hat{\Omega}_\varepsilon^2 = \begin{cases} O_p\left(n^{-1}T^{-1}\mathcal{S}(\alpha)\right) = O_p\left(n^{-1}T^{-2\alpha}\right) & \text{in M1,} \\ O_p\left(n^{-1}T^{-1}\mathcal{S}(2\beta)\right) = O_p\left(n^{-1}T^{-4\beta}\right) & \text{in M2,} \\ \max\left[O_p\left(n^{-1}T^{-2(\alpha+\beta)}\right), O_p\left(n^{-1}T^{-4\beta}\right)\right] & \text{in M3.} \end{cases}$$

Hence, $\hat{\Omega}_{\mathcal{M}}^2$ dominates $\hat{\Omega}_\varepsilon^2$.And $\hat{\Omega}_{\mathcal{M}}^2$ also dominates $\hat{\Omega}_{m\varepsilon}$ since

$$\frac{1}{T}\sum\nolimits_{t=1}^{T}\tilde{\mathcal{M}}_{nt}\tilde{\varepsilon}_{nt} \leq \left(\frac{1}{T}\sum\nolimits_{t=1}^{T}\tilde{\mathcal{M}}_{nt}^2\right)^{1/2}\left(\frac{1}{T}\sum\nolimits_{t=1}^{T}\tilde{\varepsilon}_{nt}^2\right)^{1/2} = o_p\left(\frac{1}{T}\sum\nolimits_{t=1}^{T}\tilde{\mathcal{M}}_{nt}^2\right),$$

Hence the long run variance has the following order

$$\hat{\Omega}_u^2 = O\left(T^{-2\lambda+\kappa}\right).$$

Then

$$t_{\hat{\phi}_{nT}} = \frac{\hat{\phi}_{nT}\sqrt{\sum_{t=1}^{T}\tilde{t}^2}}{\sqrt{\hat{\Omega}_u^2}} = \frac{O\left(T^{-1-\lambda}\right)}{O\left(T^{\kappa/2-\lambda}\right)}O\left(T^{3/2}\right) = O\left(T^{1/2-\kappa/2}\right).$$

In this event, $t_{\hat{\phi}_{nT}} \to +\infty$ as $n, T \to \infty$, since the sign of $t_{\hat{\phi}_{nT}}$ is consistent with $\hat{\phi}_{nT}$ which is positive when $\lambda < 0$.

Power Trend Regression

We explore the impact on Theorem 1 of using a power trend regression of the form (34) in KPS in place of a linear trend regression. In (34) in KPS the empirical regression involves the power trend regressor t^ψ for some given power parameter $\psi > 0$. Direct calculations extending the results in Theorem 1 show that the asymptotic behavior of the regression coefficient $\hat{\phi}_{nT}$ in this case is as follows:

$$\hat{\phi}_{nT} = \begin{cases} O_p\left(T^{-\psi-\lambda}\right) & \text{if } 0 < \lambda < 1, \\ O_p\left(T^{-1-\psi}\ln T\right) & \text{if } \lambda = 1, \\ O_p\left(T^{-1-\psi}\right) & \text{if } \lambda > 1. \end{cases} \tag{36}$$

rather than $\hat{\phi}_{nT} = O_p\left(\mathcal{O}_{T\lambda}\right)$, where λ in (36) is as given in Theorem 1. Upon calculation, we find that

$$\left(\sum_{t=1}^{T}\left[t^{\psi}-\frac{1}{T}\sum_{t=1}^{T}t^{\psi}\right]^{2}\right)^{-1}\sum_{t=1}^{T}\left[t^{\psi}-\frac{1}{T}\sum_{t=1}^{T}t^{\psi}\right]\varepsilon_{n,t} = O_p\left(n^{-1/2}T^{-1/2-\psi}\right),$$

and then

$$n^{1/2}T^{1/2+\psi}\hat{\phi}_{nT} = \begin{cases} O\left(n^{1/2}T^{1/2-\lambda}\right) & \text{if } 0<\lambda<1, \\ O\left(n^{1/2}T^{-1/2}\ln T\right) & \text{if } \lambda = 1, \\ O\left(n^{1/2}T^{-1/2}\right) & \text{if } \lambda > 1. \end{cases}$$

Hence divergence of the scaled statistic $n^{1/2}T^{1/2+\psi}\hat{\phi}_{nT}$ requires $n/T \to \infty$ regardless of the value of ψ. Thus using a power trend regression with regressor t^{ψ} instead of a simple linear trend does not lead to different requirements regarding (n, T).

Additional Numerical Calculations

We extend the numerical calculations given in Section 5 of the paper for model M1 by conducting related computations for models M2 and M3. As either n or β increases, the variance of $e_{n,t}$ shrinks to zero for given T, so that the t-ratio diverges to a negative infinity under the alternative as $n \to \infty$ with a fixed β, or approaches the limit value of $-\sqrt{3}$ given in Theorem 2 as $\beta \to \infty$. We first investigate the finite sample behavior of the t-ratio for given n and T. Figure S1 plots the empirical density functions of the t-ratio with various β values in M2. We set $n = 100$, $T = 200$, $\sigma_a^2 = 1$, and $\epsilon_{it} \sim iid\mathcal{N}(0,1)$. As β decreases, the variance of the t ratio increases and the mean of the distribution moves to the left. Even for moderately large n and T, the entire empirical distribution of the t-ratio still lies in the left side of the critical value, -1.65, with for $\beta = 2$. As β passes to infinity, the empirical distribution collapses to a mass point at $-\sqrt{3} = -1.73$. For $\beta = 0.5$, x_{it} is convergent, the trend regression test is consistent, and its strong discriminatory power is evident in the density shown in Figure S1.

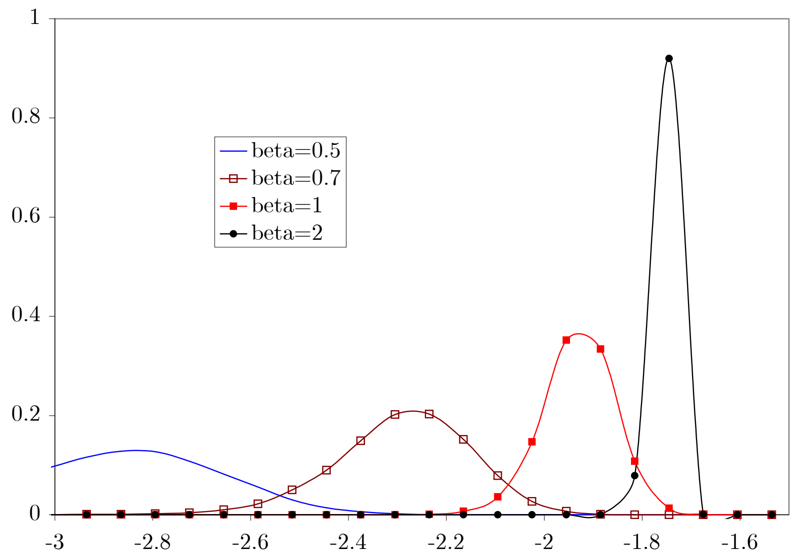

Figure S1: Empirical distribution of $t_{\hat{\phi}_{nT}}$ in M2
$T = 200$, $n = 100$, $\sigma_a^2 = 1$, $\epsilon_{it} \sim iid\mathcal{N}(0,1)$, $\kappa = 1/3$

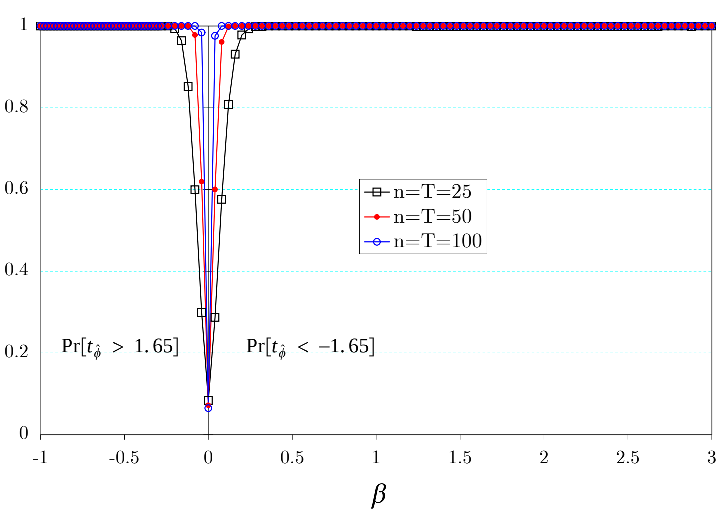

Figure S2: Rejection Frequencies plotted against β in Model M2
for a 5% level test with $\sigma_a^2 = 1$, $\sigma_\epsilon^2 = 4$.

Figure S2 displays the rapid changes in the power function of a 5% level test near $\beta = 0$ as n and T increase with $T = n$. As Theorem 2 indicates, no asymptotic n/T ratio condition is required for test consistency in this case. Evidently, as sample size increases, the rapid movement in the power function near $\beta = 0$ becomes more accentuated. The power function is unity outside a small neighborhood of $\beta = 0$ even for $n = T = 25$ because the empirical distribution of the t-ratio is well separated from the test critical value of -1.65.

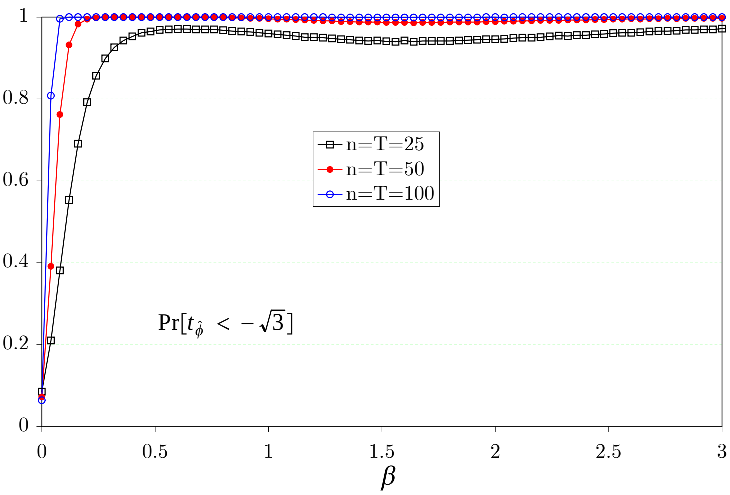

Figure S3: The Rejection Frequencies with
the critical value of $-\sqrt{3}$ over $\lambda = 2\beta$ in Model 2 ($\sigma_a^2 = 1$, $\sigma_\epsilon^2 = 2$)

Since Figure S2 reports the rejection probability for the 5% level test with critical value of -1.65, which exceeds $-\sqrt{3}$, this figure does not reveal the relationship of the density function of the t-ratio to the limit value $-\sqrt{3}$. To explore this issue, we set the critical value of the test to $-\sqrt{3}$ and plot the associated rejection frequencies in Figure S3. When n, T is small, some portion of the t-ratio is in fact larger than $-\sqrt{3}$ so that in that case the rejection frequency is less than unity. However as n increases, the power function reaches unity rapidly, thereby indicating that the asympotic theory holds well in finite samples.

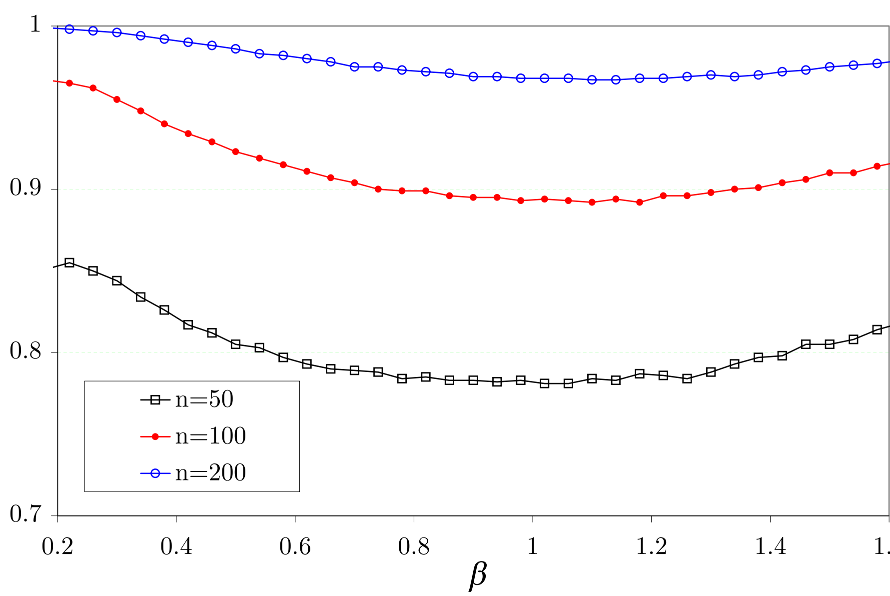

Figure S4: Test Rejection Frequencies in Model M3 ($\alpha = 2\beta - 0.1, \sigma_a^2 = 8, \sigma_{a\mu} = 0, \sigma_\varepsilon^2 = \sigma_\beta^2 = 1$)

Finally, we consider model M3. As shown in Theorems 1 and 2, when either $2\beta > \alpha > \beta$ or $\alpha \leq \beta < 1$ the n/T ratio matters in the limit theory and we explore finite sample performance in this case, setting $\alpha = 2\beta - 0.1$, $\sigma_a^2 = 8$ and fixing $T = 50$ for all values of n. Figure S4 displays the power of the test for various values of β. The power functions are seen in the figure to have a mild $U-$shape and minimum power is found around $\beta = 1$. When $\beta > 1$ power increases as β increases. It is also apparent from Figure S4 that as n increases with T fixed, the n/T ratio increases and test power approaches unity around $\beta = 1$.

Approximation accuracy of (13) for $G(T, \lambda)$

We can assess the adequacy of the approximation (13) in a graphical demonstration by using the deterministic DGP in (35) in KPS to characterize the large T behavior of $G(T,\lambda) = \frac{1}{T}\sum_{\ell=1}^{L}\sum_{t=1}^{T-\ell}\left(1 - \frac{\ell}{L+1}\right)\tilde{m}_t\tilde{m}_{t+\ell}$ when $\lambda < 1$. We let $\kappa = 1/3$ and compute the ratio $\frac{G(10^k,\lambda)}{G(10^6,\lambda)}$ for various values of λ and $k = 3, 4, 5$. The plots are shown in Figure S5. Evidently, the ratios exceed unity for large λ but rapidly decrease as λ decreases. The threshold value of λ for $G(T,\lambda)$ to decay as $T \to \infty$ is $\lambda > \kappa/2 = 1/6 \simeq 0.167$ for $\kappa = 1/3$, which is evidently well-matched in the figure, corroborating the limit behavior $G(T,\lambda) = O\left(T^{-\lambda+\kappa-\lambda\kappa}\right) \searrow 0$ given in (13).

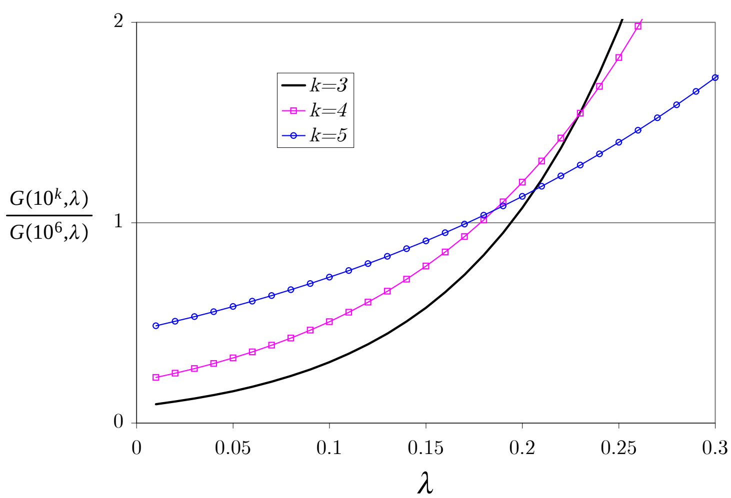

Figure S5: Approximation Accuracy of (13) for $G(T, \lambda)$

References

[1] Kong, J., P.C.B. Phillips and D. Sul (2019). Weak $\sigma-$convergence: Theory and Applications, *Journal of Econometrics*.