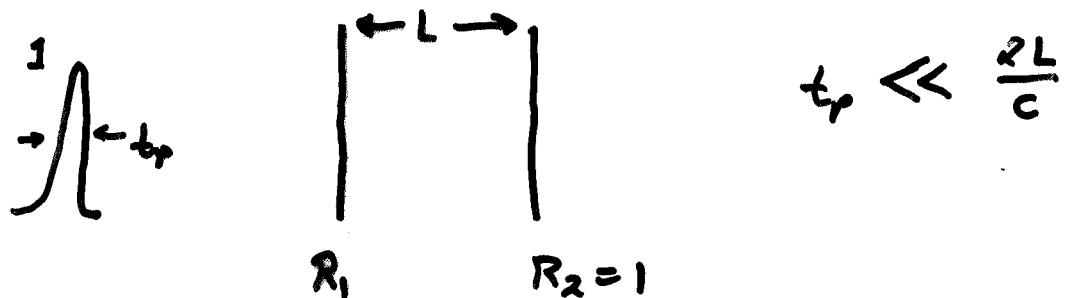


Optical Resonators & Etalons

Consider 2 mirrors facing each other:



WHAT GETS REFLECTED? Multiple Pulses

That is to say ... Echoes.

$$I_1 = R_1 I_0$$

$$I_2 = (1 - R_1)^2 I_0$$

$$I_3 = R_1 (1 - R_1)^2 I_0$$

$$\vdots$$

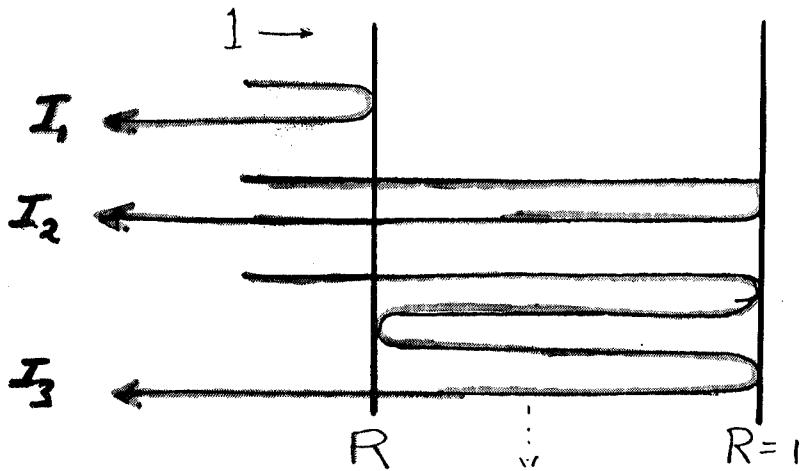
For Fin., Ask: What R_1 gives $I_1 = I_2$.

$$R_1 = (1 - R_1)^2$$

$$0 = R_1^2 - 3R_1 + 1$$

$$R_1 = \frac{3\sqrt{5}}{2} = 0.382..$$

Analysis: Pulse bursts



$$I_1 = R$$

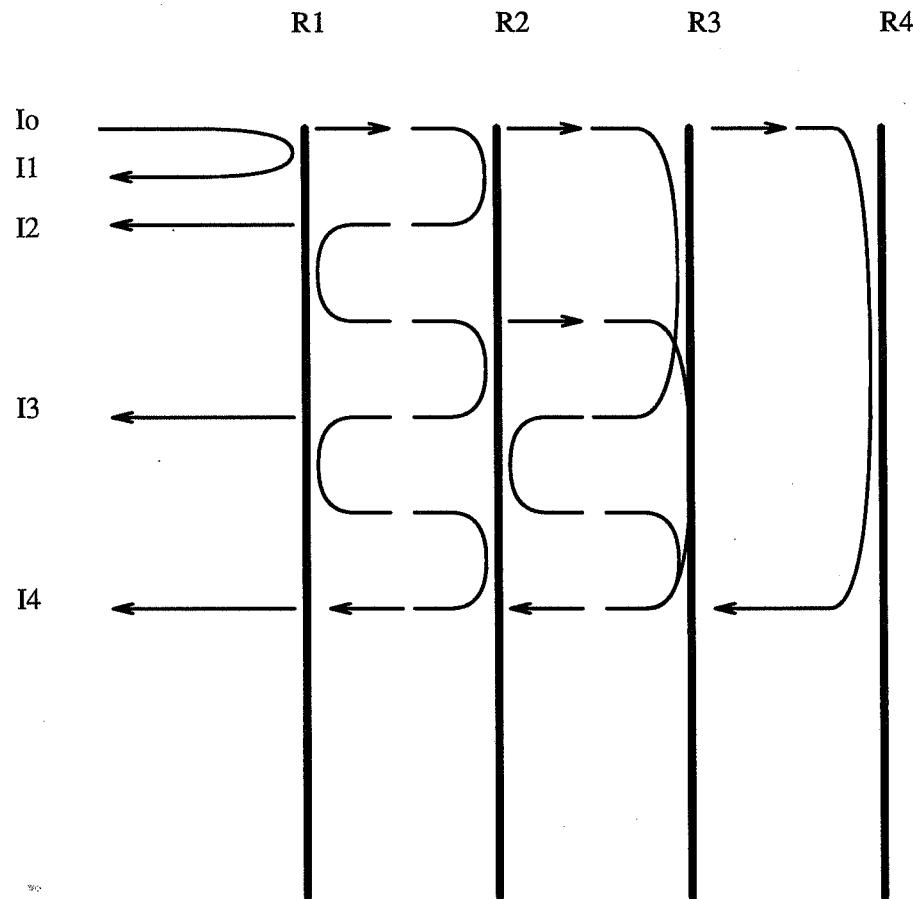
$$I_2 = (1 - R)^2$$

$$I_1 = I_2 \text{ implies } R = \frac{3-\sqrt{5}}{2} = 0.3820$$

$$\eta = I_1 + I_2 = 76.4\%$$

$$I_3 = (1 - R^2)R = 0.144$$

Design and Analysis



$$ct_p \ll d$$

$$R's \Rightarrow I's$$

Decoupled (Random Interference) Analysis

$$\begin{aligned}I_1 &= R_1 \\I_2 &= (1 - R_1)^2 R_2 \\I_3 &= (1 - R_1)^2 (1 - R_2)^2 R_3 \\I_n &= R_n \prod_{i=1}^{n-1} (1 - R_i)^2\end{aligned}$$

If we let $I_1 = I_2 = I_3 = \dots = I_n$ then we may derive the polynomials,

$$\begin{aligned}R_n - 1 &= 0 \\R_{n-1}^2 - 3R_{n-1} + 1 &= 0 \\R_{n-2}^4 - 7R_{n-2}^3 + 13R_{n-2}^2 - 7R_{n-2} + 1 &= 0 \\1, -15, +83, -220, +303, -220, +83, -15, +1 \\1, -31, +413, -3141, +15261, -50187, +115410, -189036, +222621\dots\end{aligned}$$

Alternatively we can build up a solution by,

$$R_{i-1} = \frac{2R_i + 1}{2R_i} - \sqrt{\left(\frac{2R_i + 1}{2R_i}\right)^2 - 1}$$

Analysis Including Interference and Echo Pulses

$$I_1 = R_1$$

$$I_2 = (1 - R_1)^2 R_2$$

$$I_3 = (1 - R_1)^2 \left[(1 - R_2)^2 R_3 \pm R_1 R_2 \right]$$

$$I_4 = (1 - R_1)^2 (1 - R_2)^2 \left\{ \left[(1 - R_3)^2 R_4 \pm 2R_1 R_2 R_3 \pm R_2 R_3^2 \right] \pm R_1^2 R_2^2 \right\}$$

$$I_1 = I_2 = I_3 = \dots = I_n$$

Comparison of designed reflectivities for optical rattlers which produce four bit codes.

| Interference | R1 | R2 | R3 | R4 | Efficiency |
|--------------|--------|--------|--------|-----|------------|
| constructive | 0.1684 | 0.2434 | 0.4078 | 1.0 | 0.673 |
| none | 0.1605 | 0.2278 | 0.3820 | 1.0 | 0.642 |
| destructive | 0.1574 | 0.2216 | 0.3530 | 1.0 | 0.629 |

Comparison of output intensities for optical rattlers which produce four bit codes.

| Interference | I1 | I2 | I3 | I4 | I5 | I6 | I7 |
|--------------|--------|--------|--------|--------|--------|--------|--------|
| constructive | 0.1684 | 0.1684 | 0.1684 | 0.1684 | 0.1056 | 0.0763 | 0.0452 |
| none | 0.1605 | 0.1605 | 0.1605 | 0.1605 | 0.1097 | 0.0760 | 0.0452 |
| destructive | 0.1574 | 0.1574 | 0.1574 | 0.1574 | 0.1117 | 0.0756 | 0.0455 |

Component reflectivities and efficiencies for up to an eight mirror filter

| N | R0 | R1 | R2 | R3 | R4 | R5 | R6 | R7 | η |
|---|-------|-------|-------|-------|-------|-------|-------|-----|--------|
| 1 | 1.0 | | | | | | | | 1 |
| 2 | 0.382 | 1.0 | | | | | | | 0.76 |
| 3 | 0.234 | 0.399 | 1.0 | | | | | | 0.70 |
| 4 | 0.168 | 0.243 | 0.408 | 1.0 | | | | | 0.67 |
| 5 | 0.131 | 0.174 | 0.250 | 0.414 | 1.0 | | | | 0.66 |
| 6 | 0.108 | 0.135 | 0.178 | 0.254 | 0.418 | 1.0 | | | 0.66 |
| 7 | 0.092 | 0.111 | 0.140 | 0.184 | 0.261 | 0.431 | 1.0 | | 0.64 |
| 8 | 0.079 | 0.094 | 0.113 | 0.141 | 0.185 | 0.262 | 0.456 | 1.0 | 0.64 |

Individual components of each intensity equation:

- $I_3 = 5$
- $I_4 = 14$
- $I_5 = 42$
- $I_6 = 126$
- $I_7 = 400$

Drawbacks

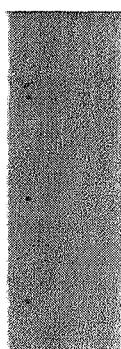
- Cumbbersome
- Cannot analyze filters — only design

Coded pulse trains — component reflectivities and efficiencies

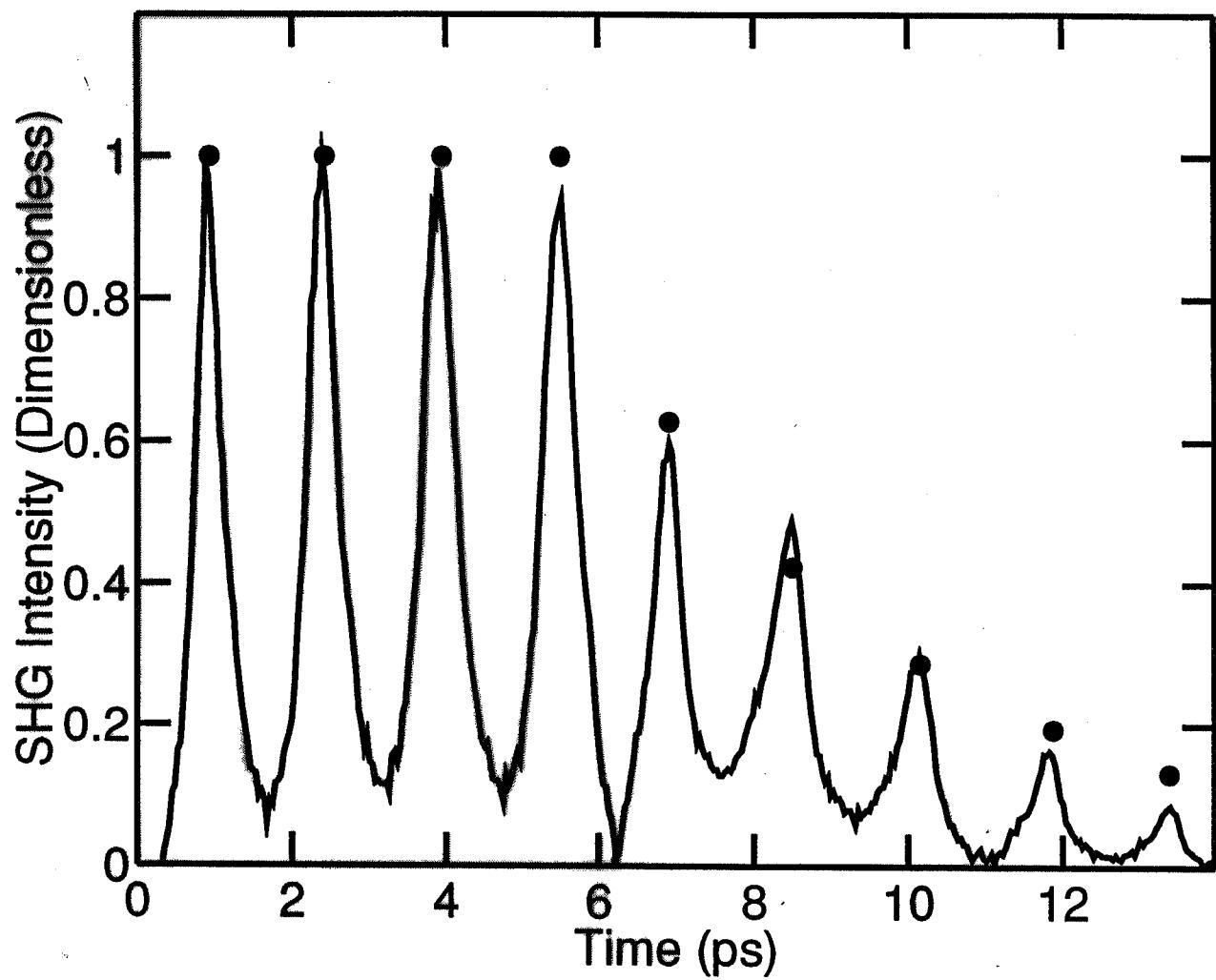
| Specified Waveform | r_0 | r_1 | r_2 | r_3 | Calculated Response |
|-----------------------|--------|--------|--------|-------|--|
| Pulse Burst {a,a,a,a} | 0.1684 | 0.2434 | 0.4079 | 1.000 | {0.1684 0.1684 0.1684 0.1684 0.1056 0.0713 0.0484 0.0327... } |
| Pulse Code {a,0,a,a} | 0.2278 | 0.0000 | 0.3820 | 1.000 | {0.2278 0.0000 0.2278 0.2278 0.1068 0.0729 0.0494 0.0309 ...} |
| Sawtooth {a,2a,3a,4a} | 0.0699 | 0.1615 | 0.3421 | 1.000 | {0.0699 0.1398 0.2096 0.2795 0.1296 0.0738 0.0425 0.0240 ...} |

Multipliers — component reflectivities and efficiencies

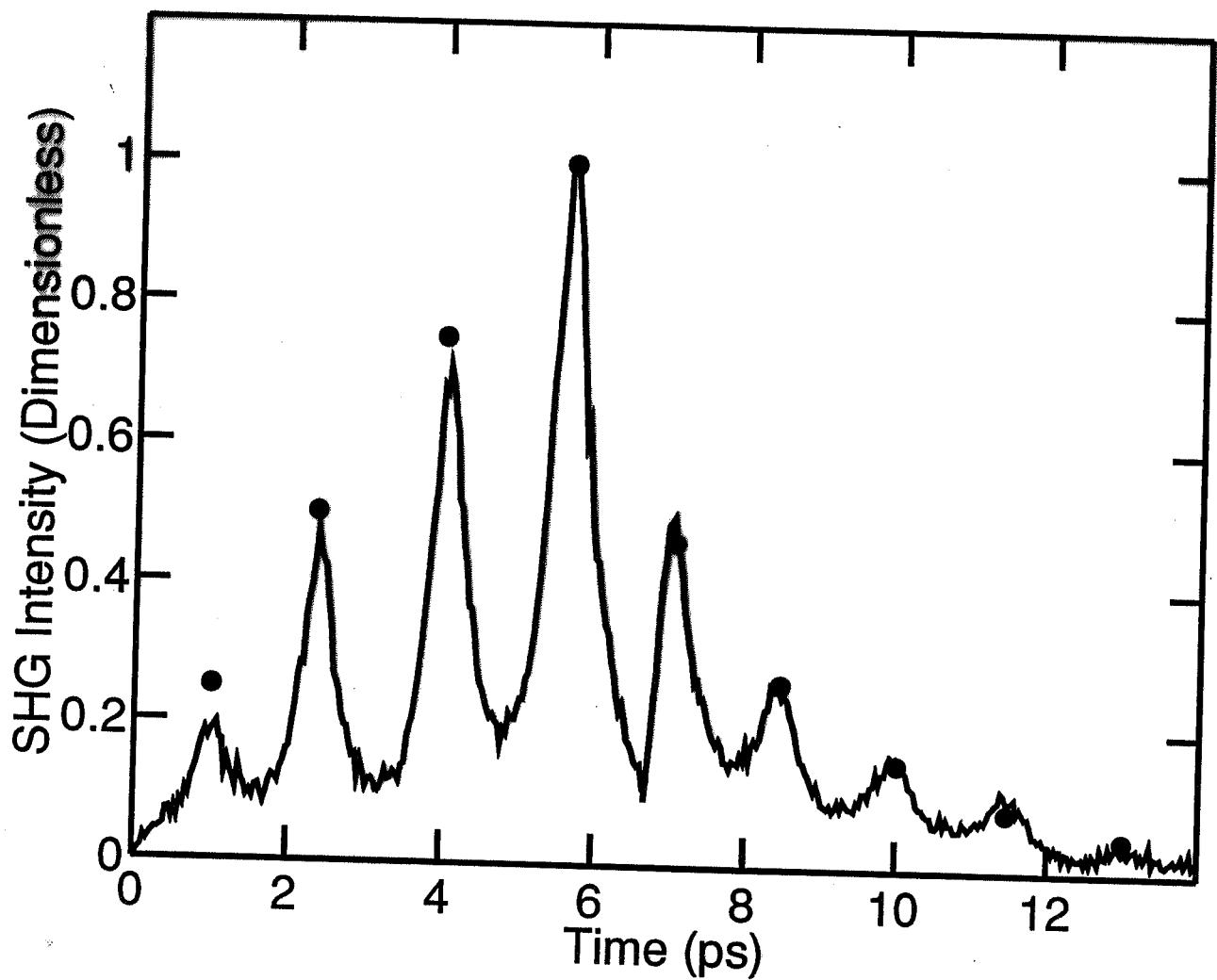
| M | r_0 | r_1 | r_2 | r_3 | r_4 | r_5 | r_6 | r_7 |
|----|--------|--------|--------|--------|--------|--------|--------|-------|
| 2x | 0.3333 | 1.000 | — | — | — | — | — | — |
| 3x | 0.1745 | 0.3656 | 1.000 | — | — | — | — | — |
| 4x | 0.1095 | 0.2046 | 0.3811 | 1.000 | — | — | — | — |
| 5x | 0.0763 | 0.1352 | 0.2192 | 0.3906 | 1.000 | — | — | — |
| 6x | 0.0565 | 0.0975 | 0.1478 | 0.2281 | 0.3972 | 1.000 | — | — |
| 7x | 0.0437 | 0.0744 | 0.1086 | 0.1557 | 0.2344 | 0.4021 | 1.000 | — |
| 8x | 0.0350 | 0.0591 | 0.0841 | 0.1156 | 0.1614 | 0.2392 | 0.4060 | 1.000 |



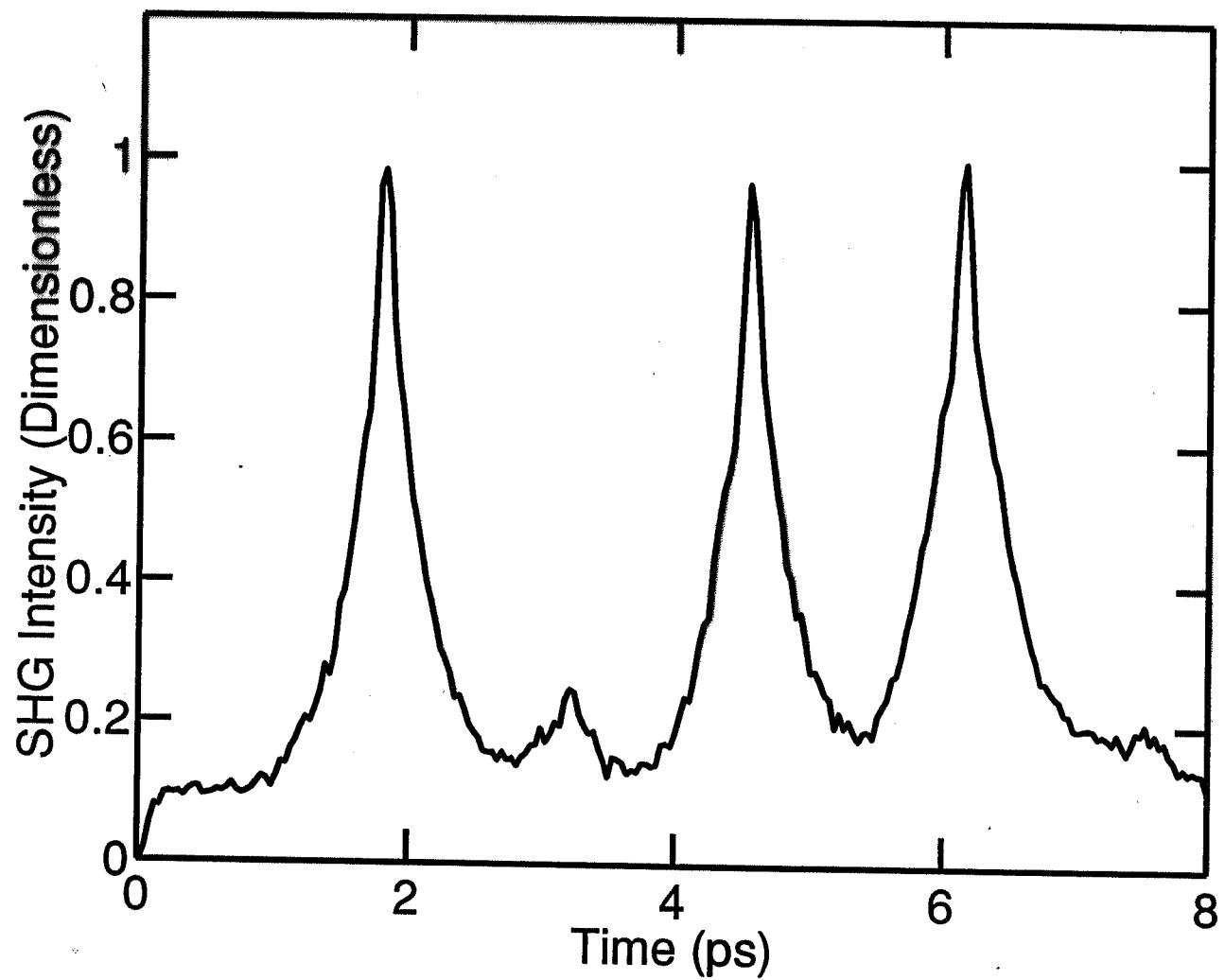
Equal pulse bursts



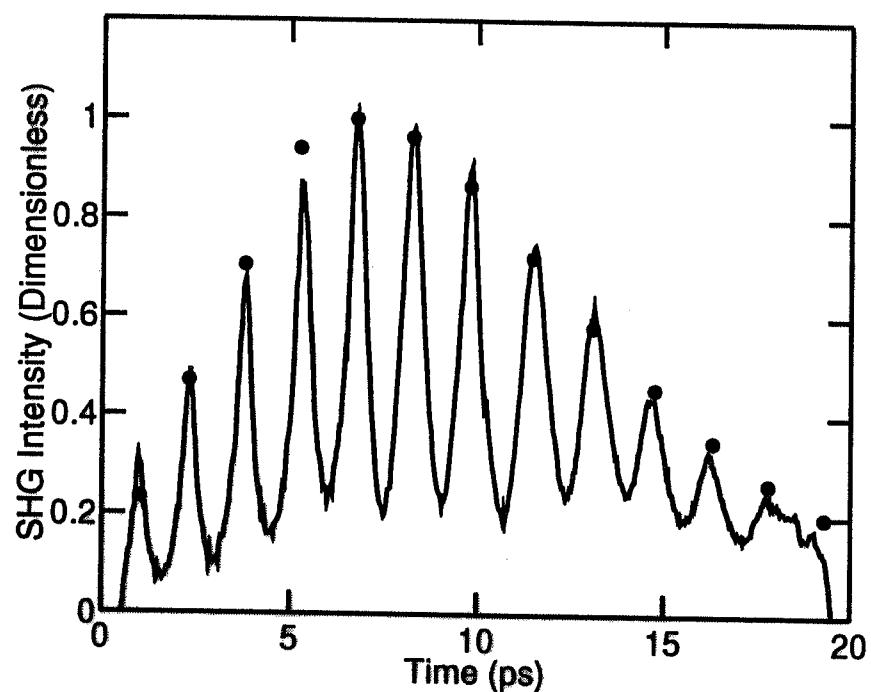
Ramps



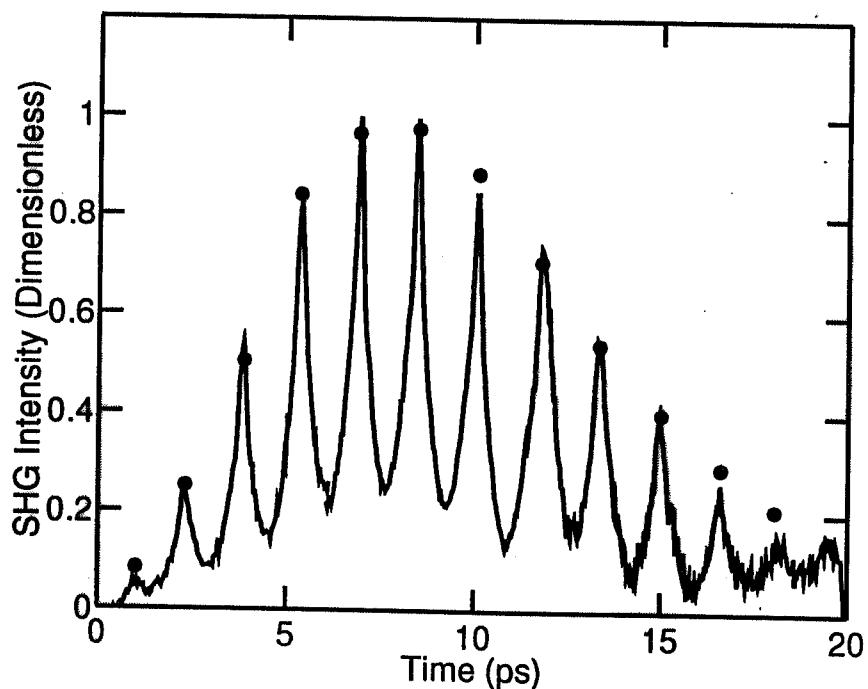
Encoded pulse bursts



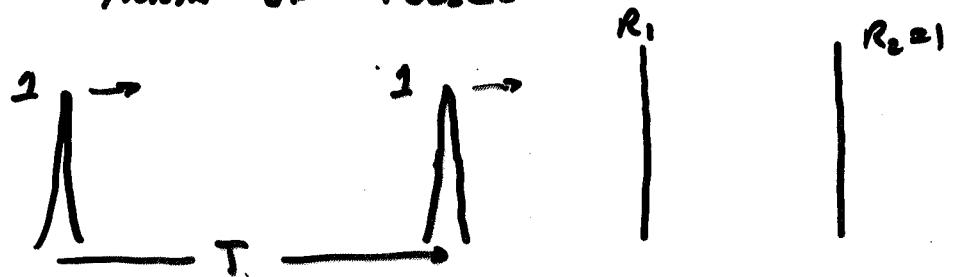
Signal filtering — integration: step to ramp



Signal filtering — integration: ramp to quadratic



Now Excite The Etalon With A Train of Pulses

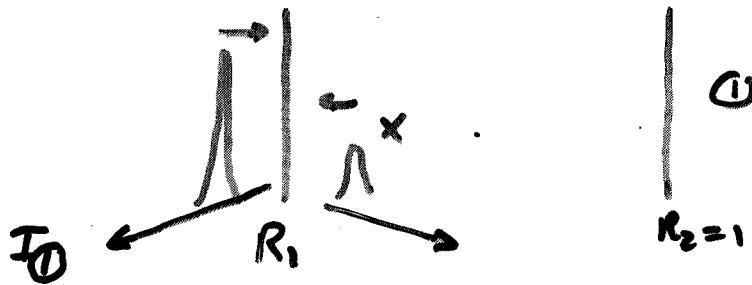


Set the spacing s.t. $\frac{2L}{c} = \frac{T}{2}$

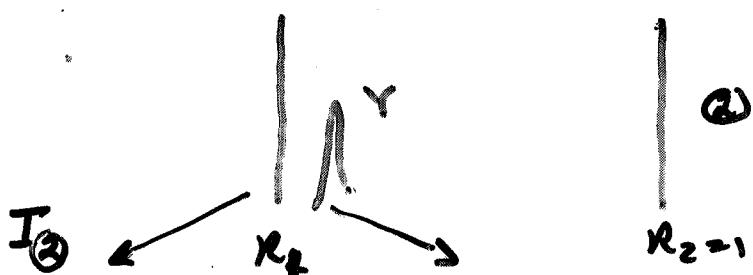
What Gets Reflected?

A PULSE TRAIN WITH A PRF MULTIPLIED BY 2.

How Do I Make This New Pulse Train Uniform?



TWO TYPES
ONE COLLISIONS
 $\textcircled{1} \rightarrow \textcircled{2} \rightarrow \textcircled{1} \rightarrow \textcircled{2} \dots$



The Collision Gas Are:

$$I_0 = R + X(1-R)$$

$$X = YR$$

$$I_0 = Y(1-R)$$

$$Y = XR + (1-R)$$

LEFT HAND PRODUCT

RIGHT HAND PRODUCT

$$\Rightarrow Y = \frac{1}{1+R}$$

$$X = \frac{R}{1+R}$$

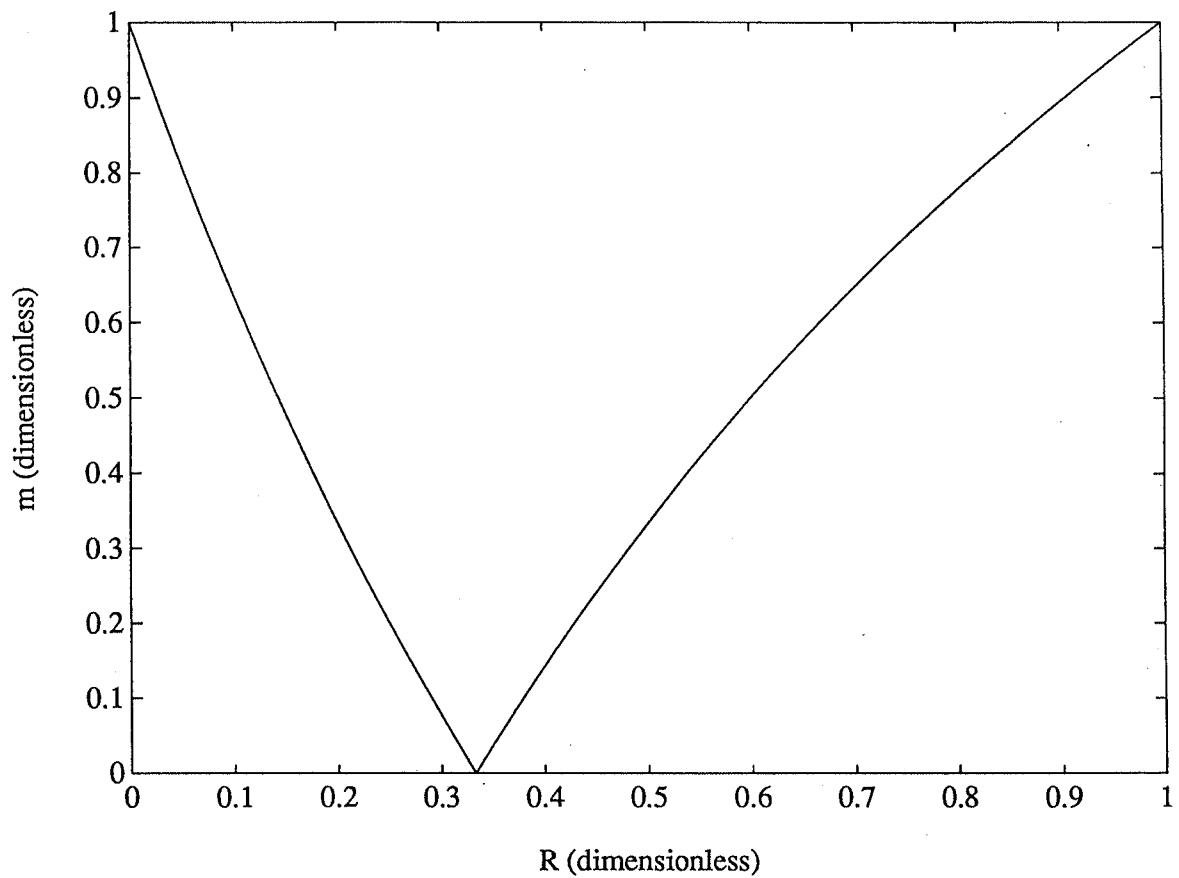
$$\Rightarrow I_0 = \frac{2R}{1+R}$$

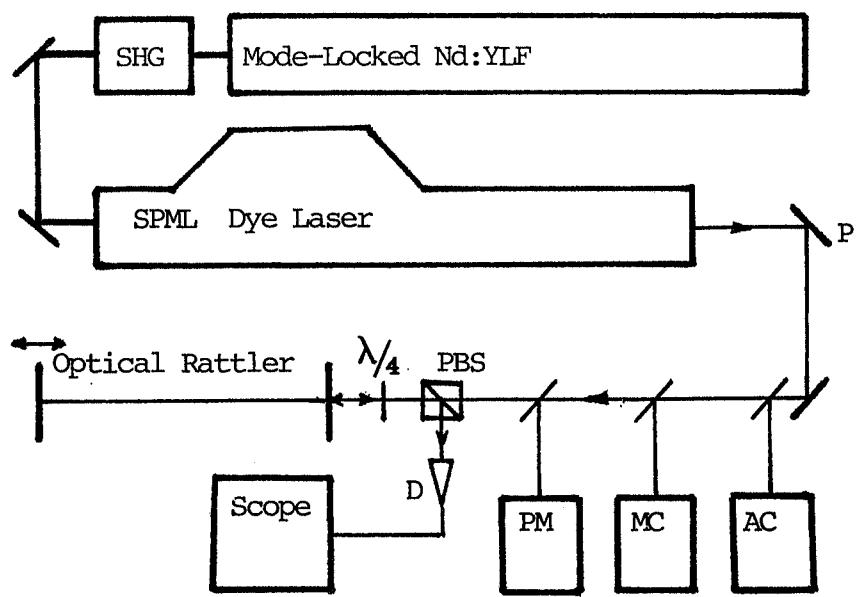
$$I_0 = \frac{1-R}{1+R}$$

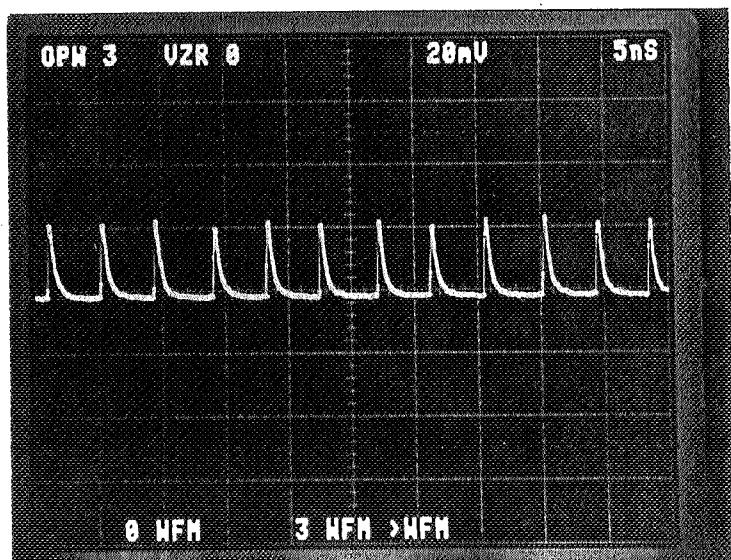
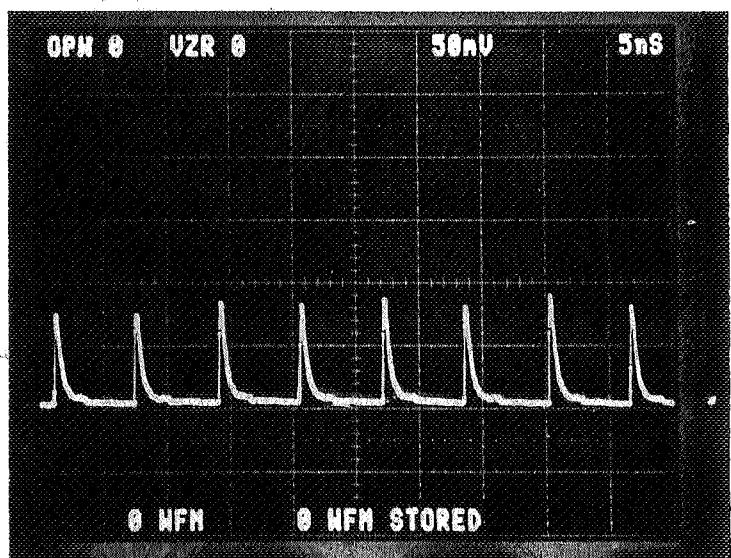
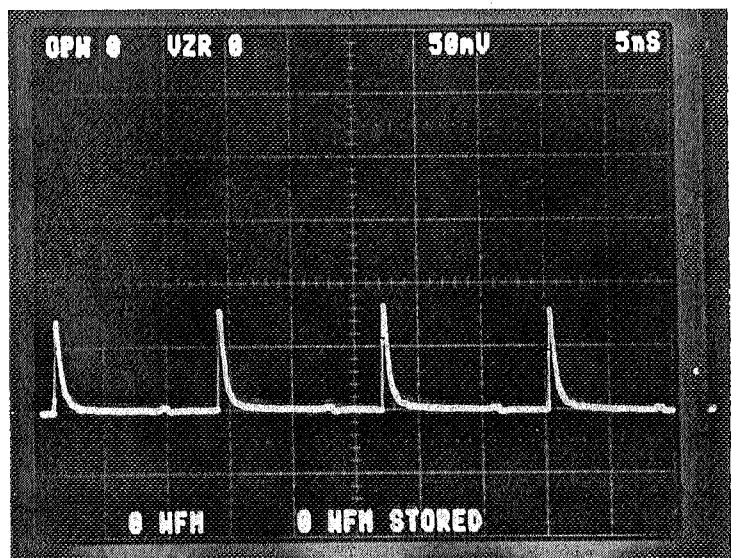
$$I_0 = I_0 \Rightarrow$$

$$R = \frac{1}{3}$$

FOR UNIFORM PULSE
TRANS OUT

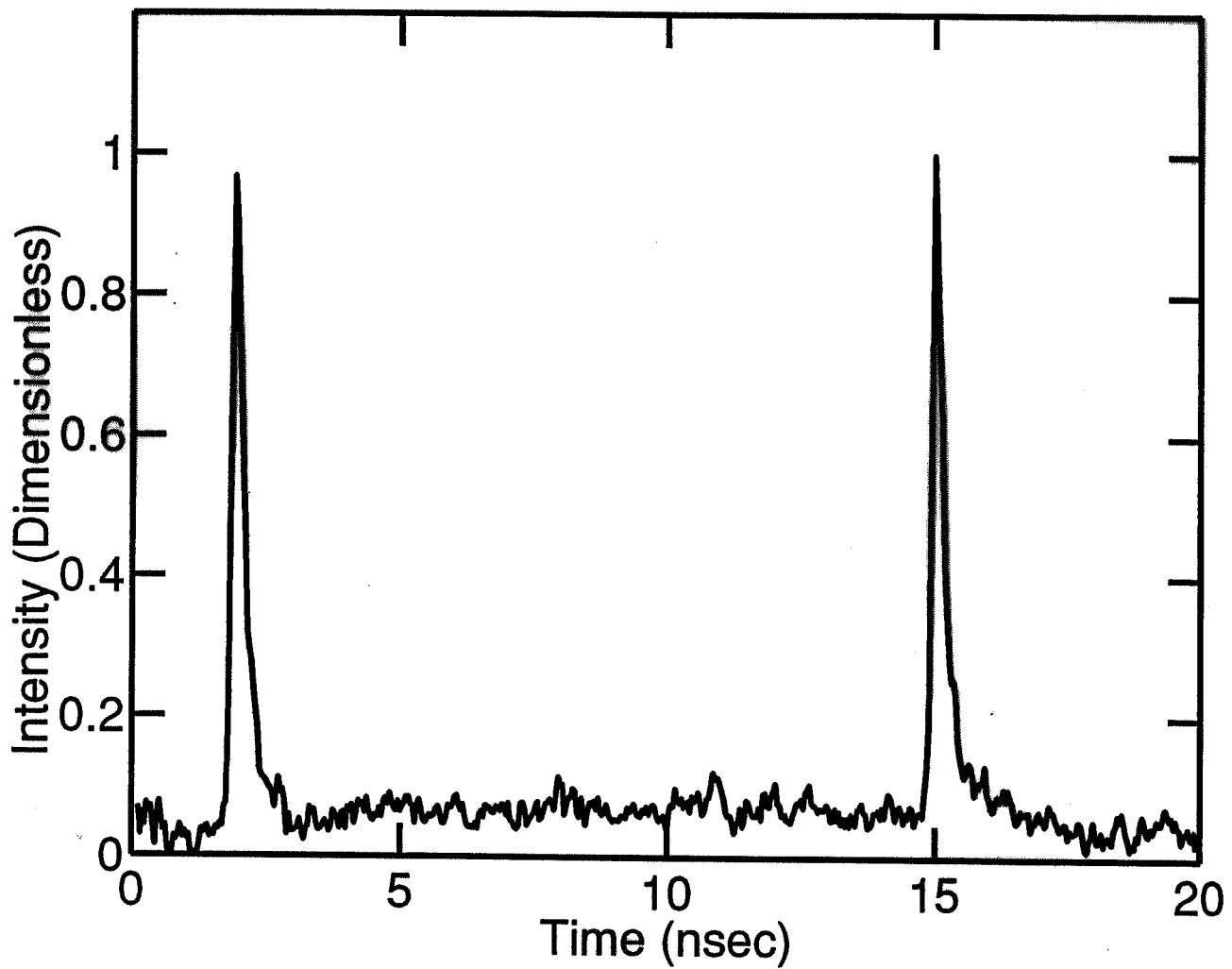




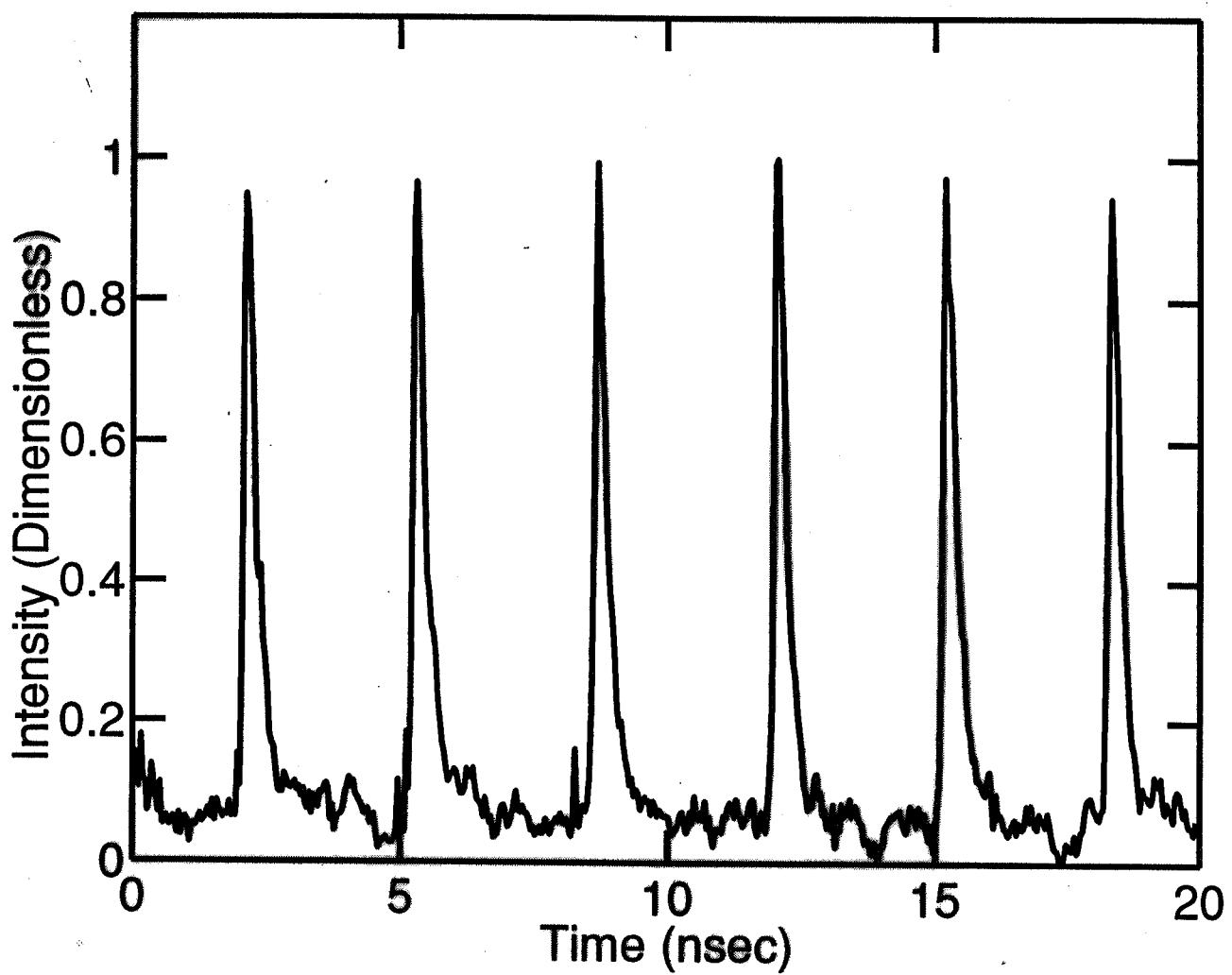


Data — pulse train multipliers

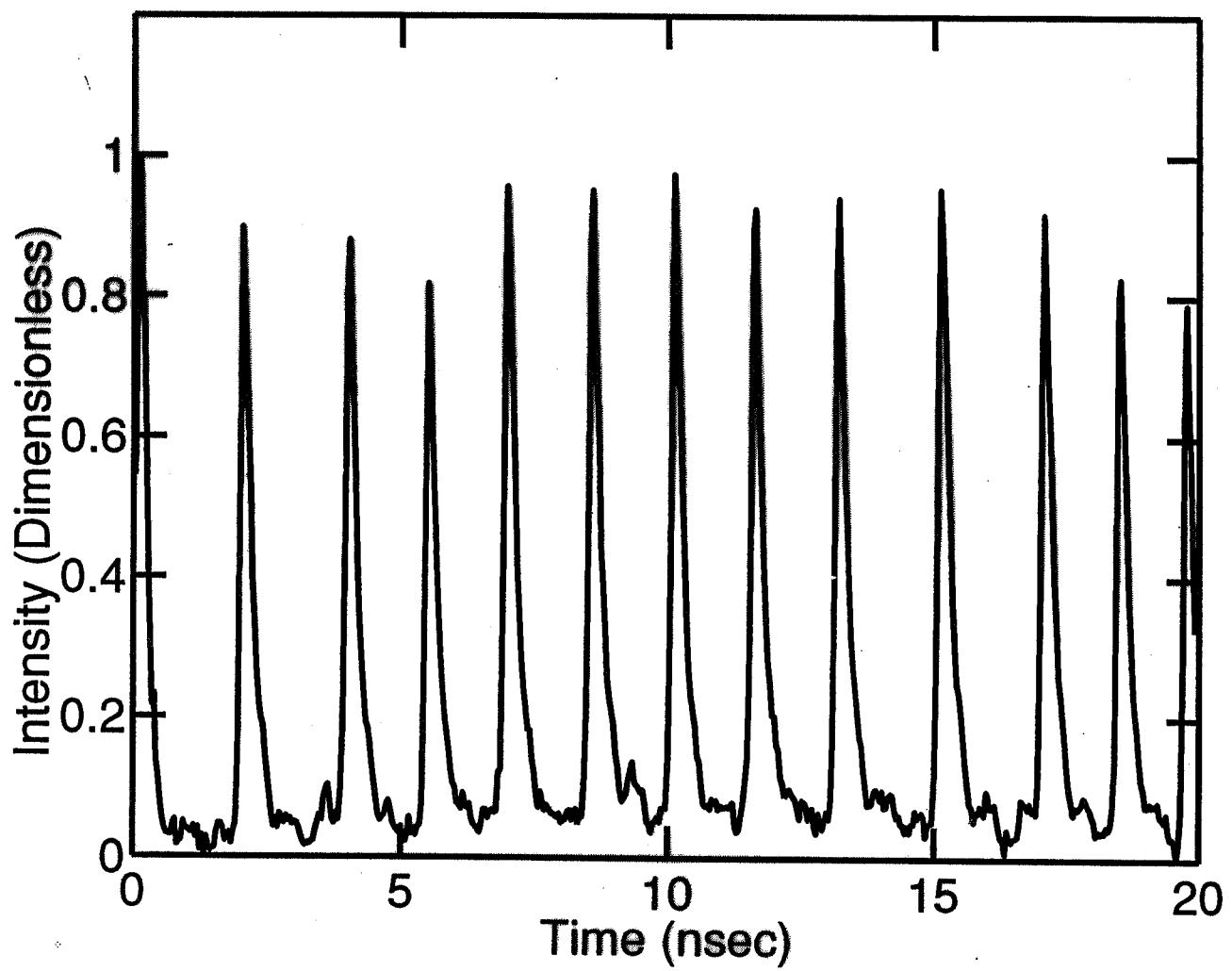
Input pulse train



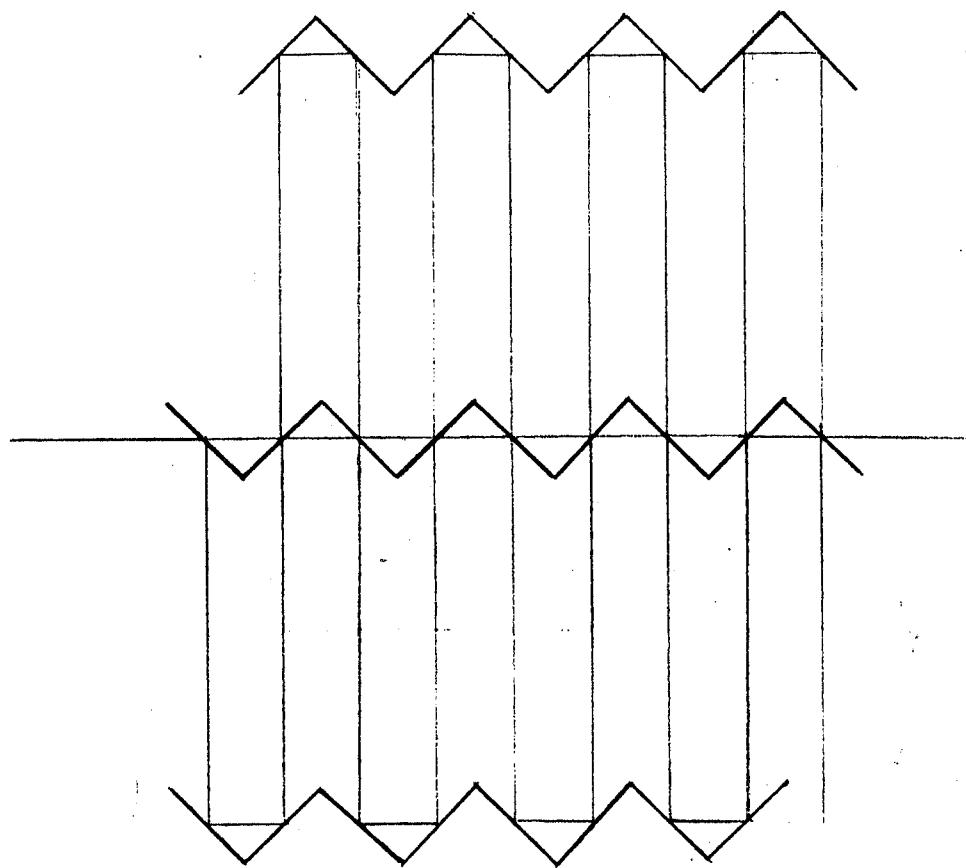
4X multiplication



8X multiplication



Other Configurations:



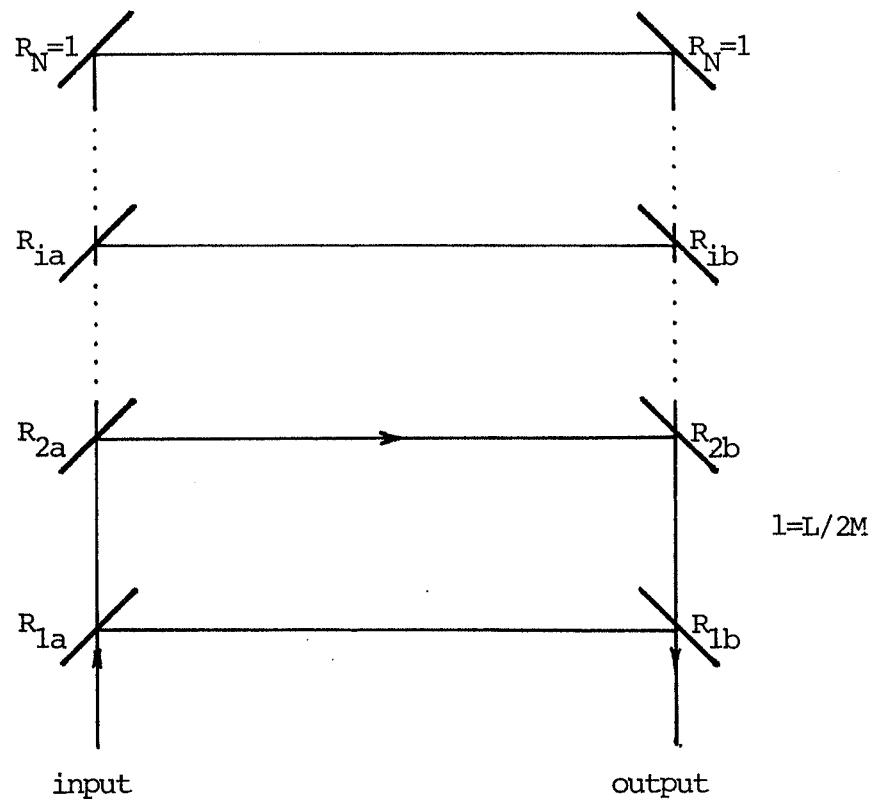
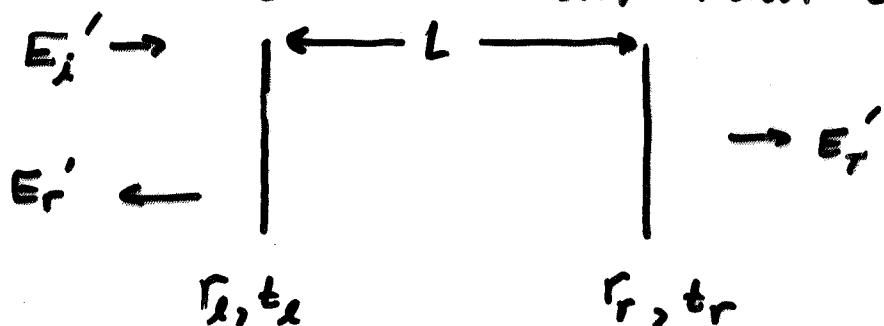


Fig. 1.

LET US ANALYSE THE SAME SOURCE
WITH A SINE WAVE EXCITATION

(MUCH MORE CONVENTIONAL ANALYSIS
OR OUR FABRY-PONOT ETAL)



$$\nabla^2 E' + \kappa^2 E' = 0 \Rightarrow E' = E_0 e^{-i \int k dz} \\ (E(z,t) = E_0 e^{-ikz} e^{-i\omega t})$$

IN STATIONARY STATE E_T' IS SUPERPOSITION OF
(LTII)
ALL INTERNAL CAVITY WAVES BOUNCING AROUND
IN CAVITY.

$$E_T' = t_i t_r E_i' e^{-i\delta} \\ + t_i t_r r_i r_r t_r E_i' e^{-2i\delta} \\ + t_i t_r r_i r_r r_{tr} t_r E_i' e^{-3i\delta} \\ + \dots$$

$$E_T' = t_e t_r E_i' e^{-iS} \underbrace{\left[1 + r_f r_e e^{-iS} + (r_f r_e)^2 e^{-2iS} + \dots \right]}_{1 + x + x^2 + \dots} = \frac{1}{1 - x}$$

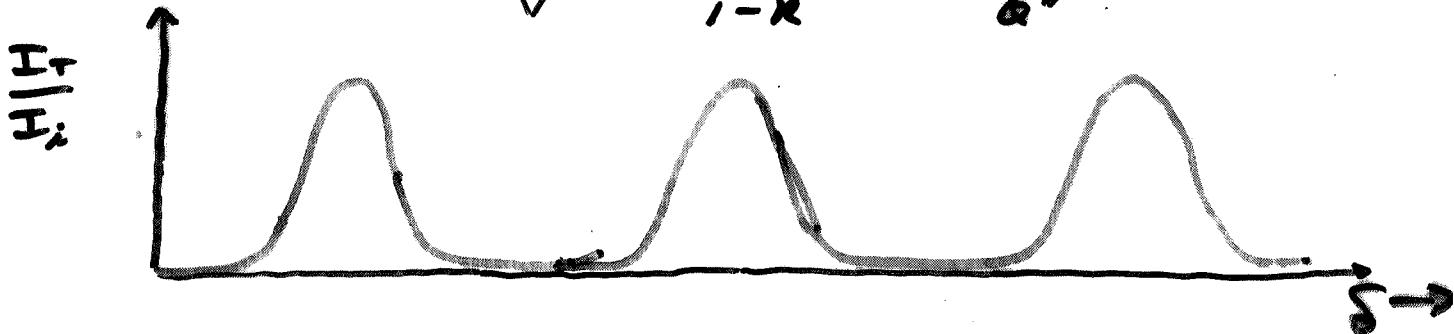
S.t.

$$E_T' = \frac{t_e t_r E_i' e^{-iS}}{1 - r_f r_e e^{-iS}}$$

In terms of Intensities:

$$\frac{I_T}{I_i} = \frac{T^2}{(1-R)^2} \frac{1}{1 + \frac{4R}{(1-R)^2} \sin \delta} \leftarrow \text{Resonance Dampening.}$$

$$\sqrt{\gamma} = \frac{\pi \sqrt{R}}{1-R} \quad \text{CAVITY "Q"}$$



$$\text{Peaks when } \kappa = \frac{1}{L} h \pi \quad h = \text{integer}$$

\Rightarrow Interfer. # wavelength $= L$

$$\Rightarrow FSR = \frac{c}{2n_0 L}$$

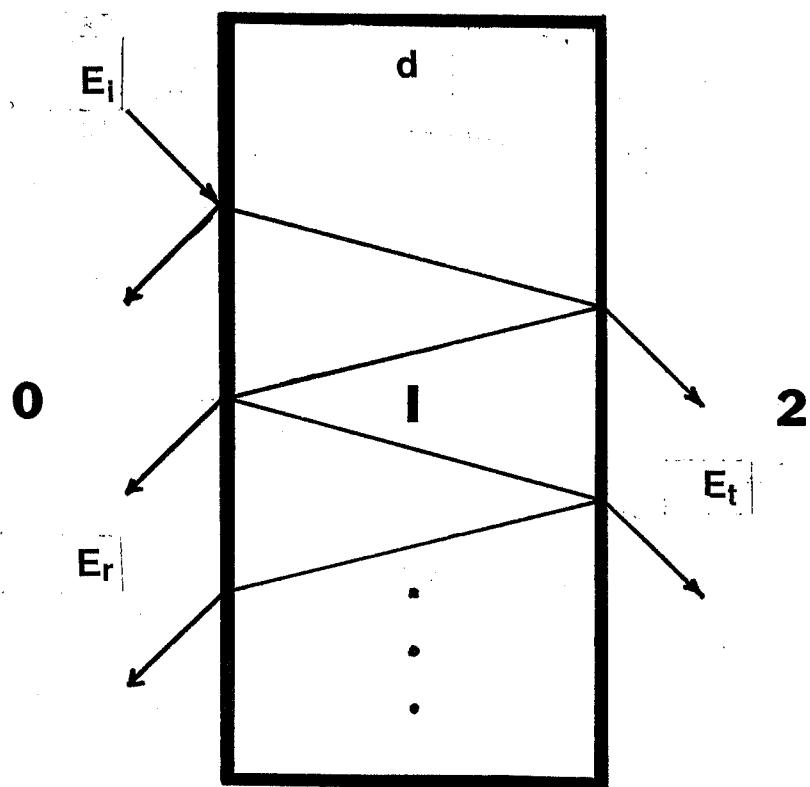


Fig. 1a

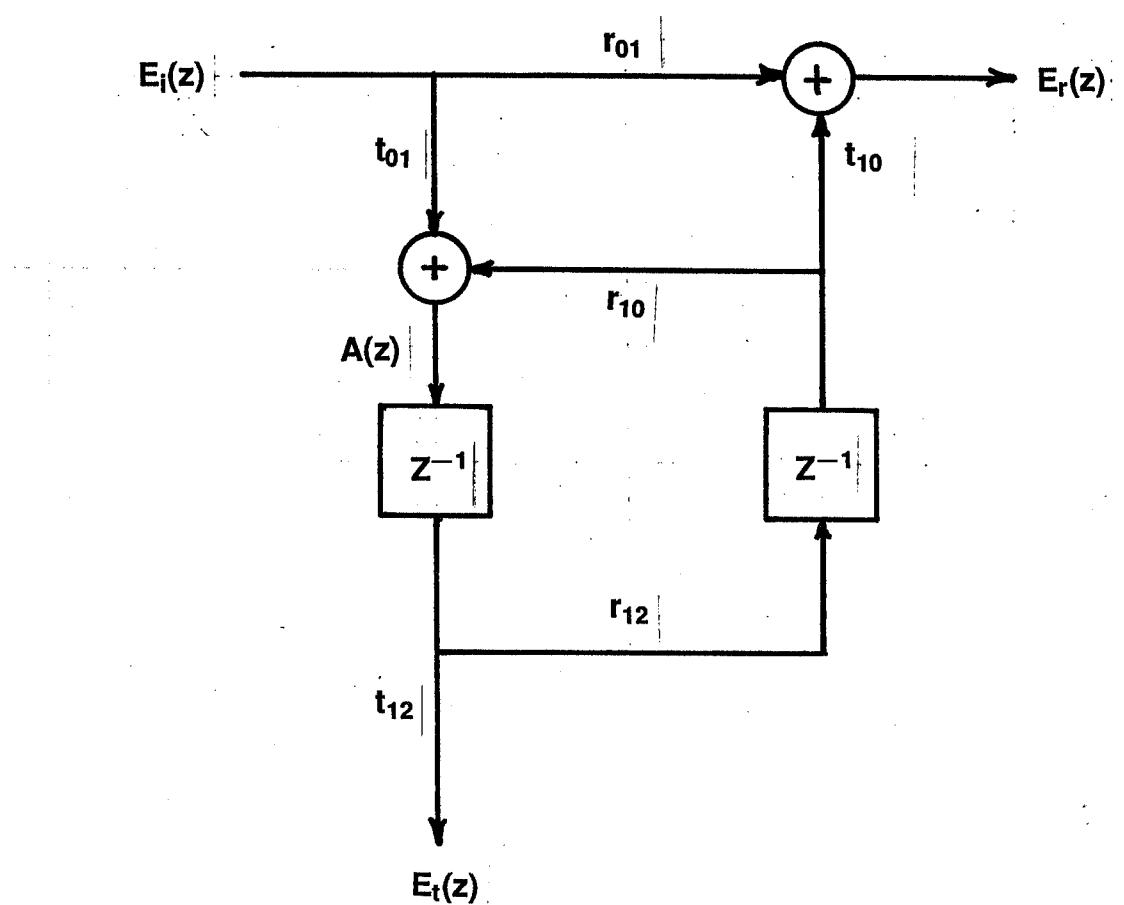


Fig. 1b

Analysis of the Fabry-Perot Etalon, Revisited

Equations of state:

$$E_r(z) = t_{10} r_{12} z^{-2} A(z) + r_{01} E_i(z)$$

$$A(z) = t_{01} E_i(z) + r_{10} r_{12} z^{-2} A(z)$$

$$E_t(z) = t_{12} z^{-1} A(z)$$

Eliminating A(z)

$$E_t(z) = E_i(z) \left[\frac{t_{01} t_{12} z^{-1}}{1 - r_{10} r_{12} z^{-2}} \right]$$

$$E_r(z) = E_i(z) \left[r_{01} + \frac{t_{01} t_{10} r_{12} z^{-2}}{1 - r_{10} r_{12} z^{-2}} \right]$$

or in rational form

$$\frac{E_r(z)}{E_i(z)} = \frac{r_{01} + r_{12} z^{-2}}{1 - r_{10} r_{12} z^{-2}}$$

Region of convergence

$$|z| > \sqrt{r_{10}r_{12}}$$

Difference equations

$$y[n] - r_{10}r_{12} y[n-2] = t_{01}t_{12} x[n-1]$$

$$y[n] - r_{10}r_{12} y[n-2] = r_{01}x[n] - r_{12} x[n-2]$$

Cascade form

$$H(z) = \frac{r_{01} \left(1 - \sqrt{\frac{r_{12}}{r_{10}}} z^{-1}\right) \left(1 + \sqrt{\frac{r_{12}}{r_{10}}} z^{-1}\right)}{\left(1 - \sqrt{r_{10}r_{12}} z^{-1}\right) \left(1 + \sqrt{r_{10}r_{12}} z^{-1}\right)}$$

Parallel form

$$H(z) = \frac{1}{2} \left[\frac{r_{01} + \sqrt{\frac{r_{12}}{r_{10}}} z^{-1}}{1 - \sqrt{r_{10}r_{12}} z^{-1}} + \frac{r_{01} - \sqrt{\frac{r_{12}}{r_{10}}} z^{-1}}{1 + \sqrt{r_{10}r_{12}} z^{-1}} \right]$$

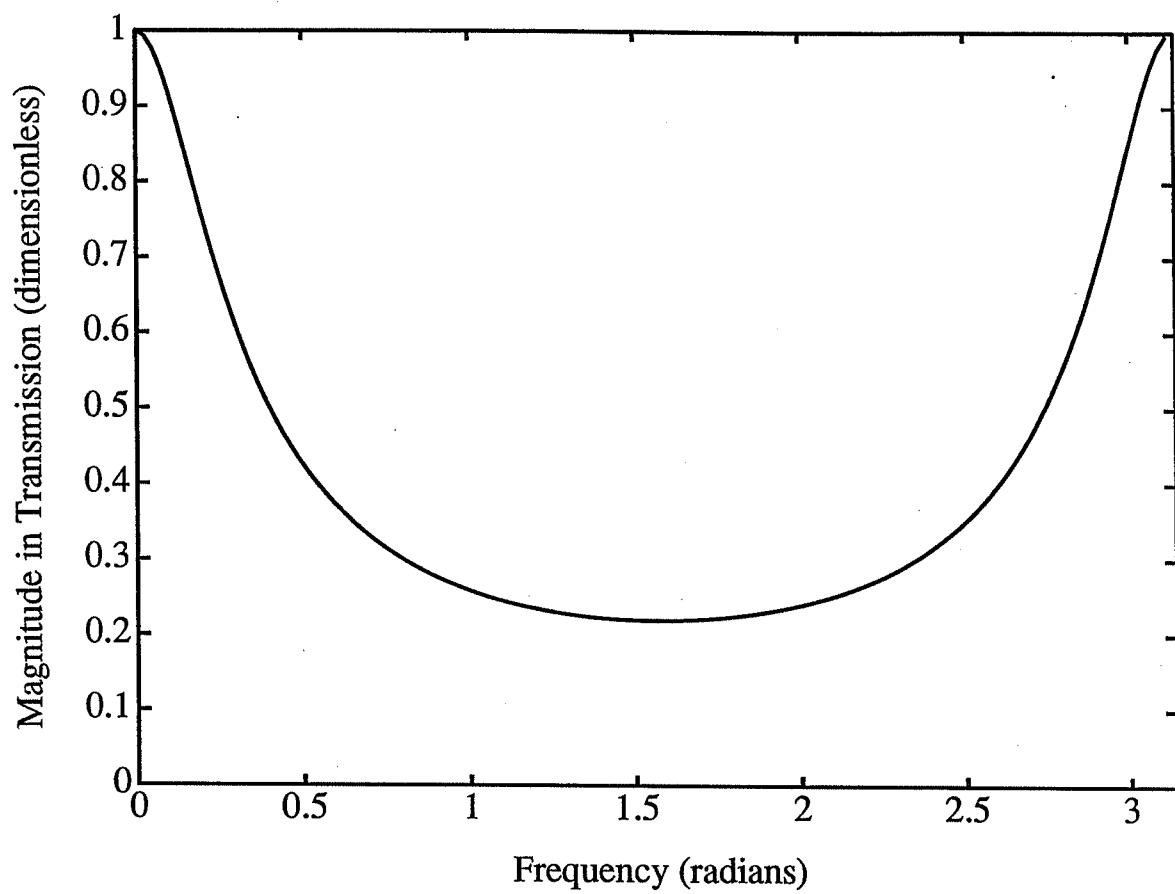


Fig. 2a

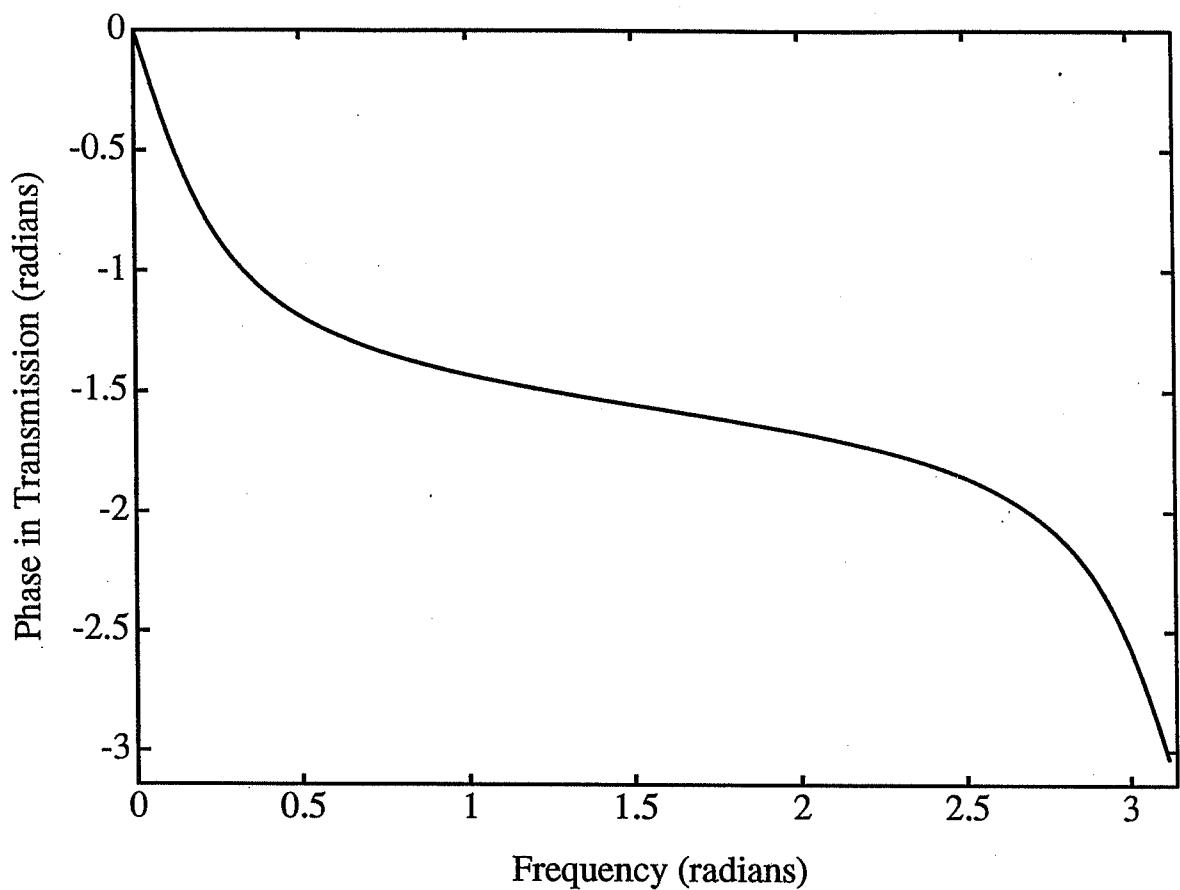
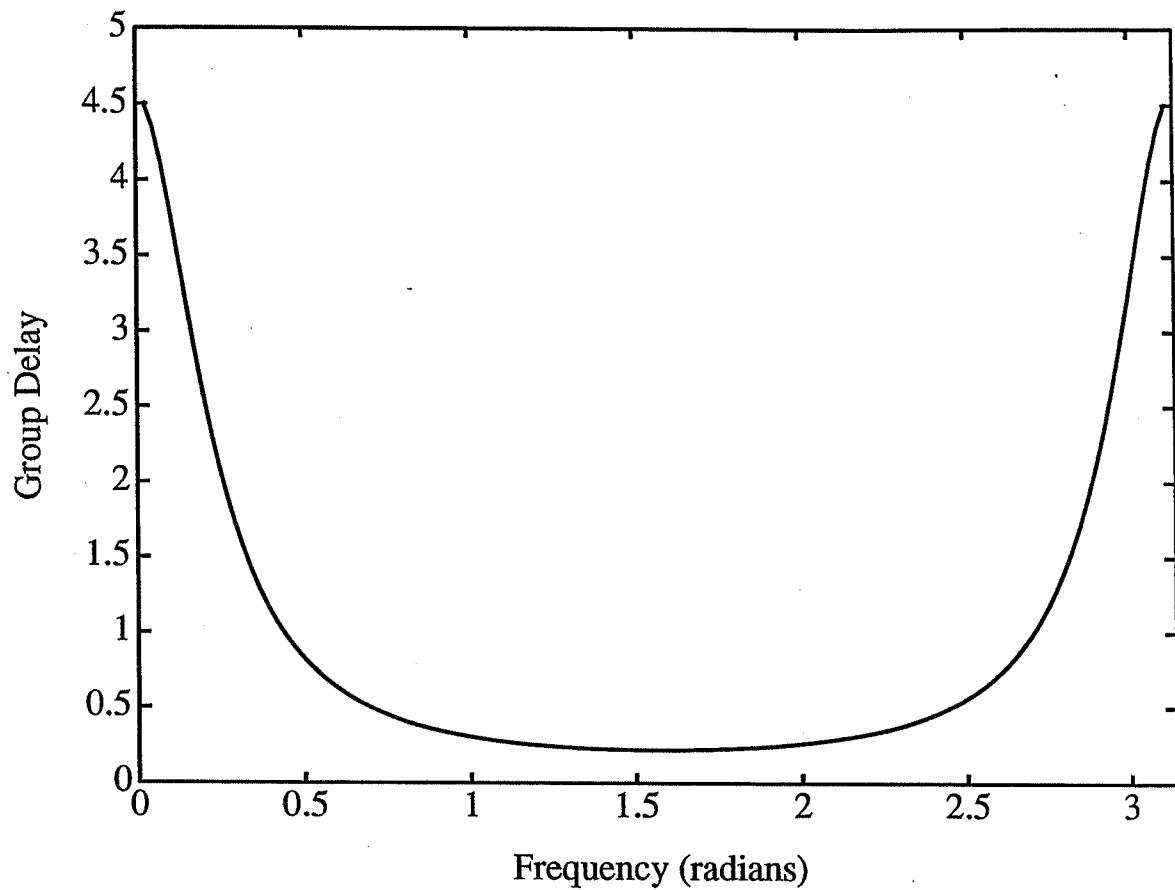


Fig. 2b



$$\tau_g(\omega) = -\frac{d\theta_p}{d\omega}$$

Fig. 2c

Three Mirror Etalons as Optical Bandpass Filters

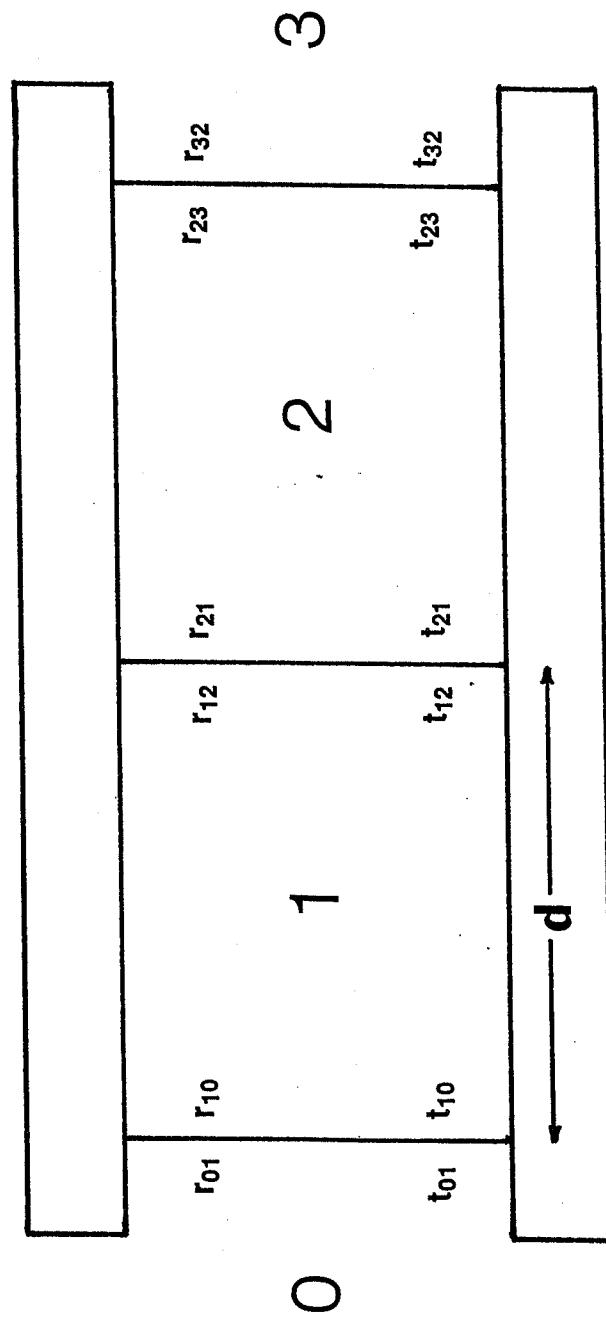


Fig 1a

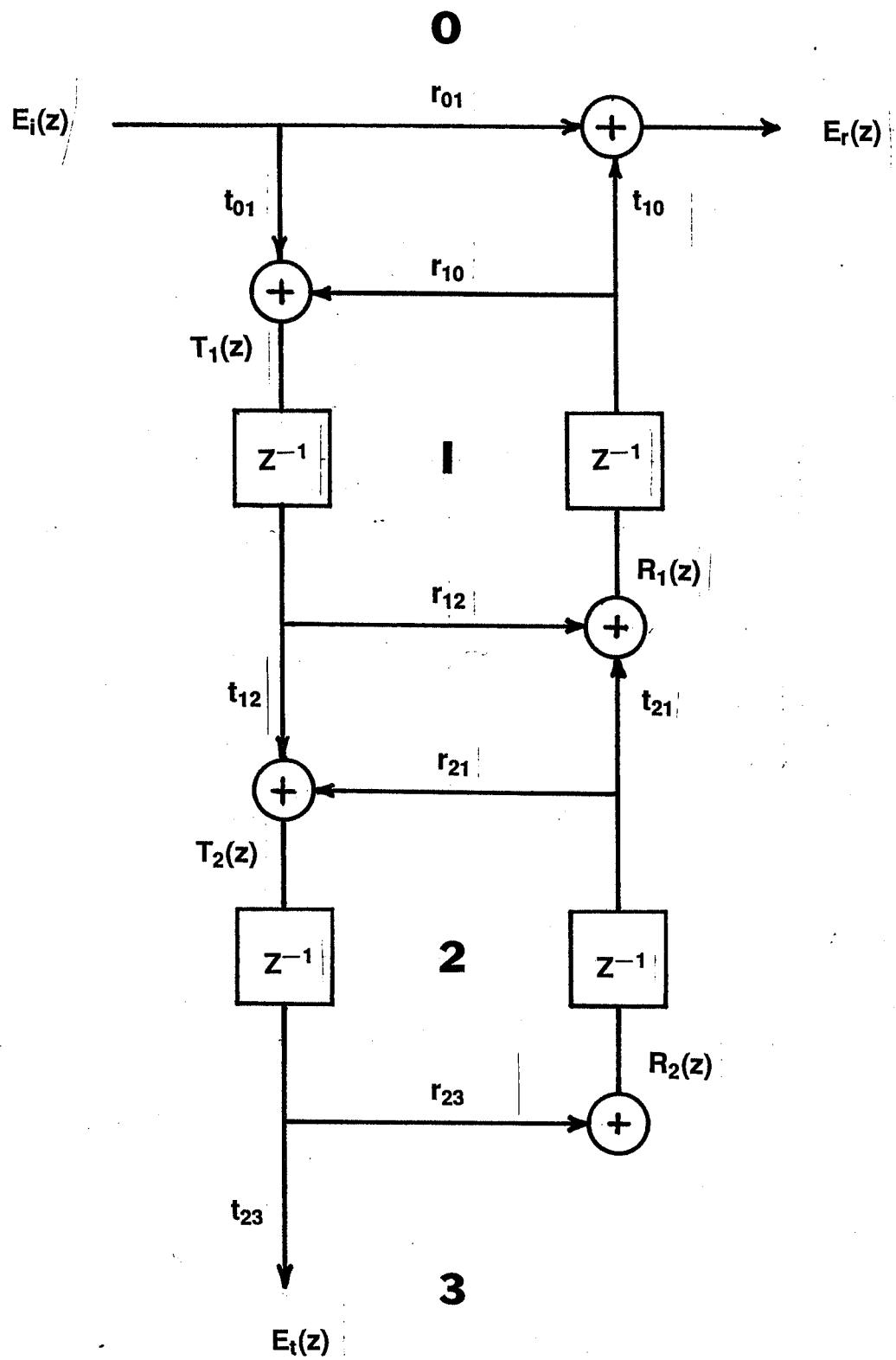


Fig. 3

Three Mirror Etalon

Equations of state:

$$E_r(z) = r_{01}E_i(z) + t_{10}z^{-1}R_1(z)$$

$$T_1(z) = t_{01}E_i(z) + r_{10}z^{-1}R_1(z)$$

$$R_1(z) = r_{12}z^{-1}T_1(z) + t_{21}z^{-1}R_2(z)$$

$$T_2(z) = t_{12}z^{-1}T_1(z) + r_{21}z^{-1}R_2(z)$$

$$R_2(z) = r_{23}z^{-1}T_2(z)$$

$$E_t(z) = t_{23}z^{-1}T_2(z)$$

or in summary,

$$T_k(z) = t_{k-1,k}z^{-1}T_{k-1}(z) + r_{k,k-1}z^{-1}R_k(z)$$

$$R_k(z) = r_{k,k+1}z^{-1}T_k(z) + t_{k+1,k}z^{-1}R_{k+1}(z)$$

Cramer's Rule

$$-r_{01}E_i(z) = -E_r(z) + t_{10}z^{-1}R_1(z)$$

$$-t_{01}E_i(z) = -T_1(z) + r_{10}z^{-1}R_1(z)$$

$$0 = r_{12}z^{-1}T_1(z) - R_1(z) + t_{21}z^{-1}R_2(z)$$

$$0 = t_{12}z^{-1}T_1(z) - T_2(z) + r_{21}z^{-1}R_2(z)$$

$$0 = r_{23}z^{-1}T_2(z) - R_2(z)$$

$$0 = t_{23}z^{-1}T_2(z) - E_t(z)$$

in matrix form

$$\mathbf{Y} = M_0 \mathbf{X}$$

where

$$\mathbf{Y} = \{-r_{01}E_i, -t_{01}E_i, 0, 0, 0, 0\}$$

$$\mathbf{X} = \{E_r, T_1, R_1, T_2, R_2, E_t\}$$

and

$$M_0 = \begin{vmatrix} -1 & 0 & t_{10}z^{-1} & 0 & 0 & 0 \\ 0 & -1 & r_{10}z^{-1} & 0 & 0 & 0 \\ 0 & r_{12}z^{-1} & -1 & 0 & t_{21}z^{-1} & 0 \\ 0 & t_{12}z^{-1} & 0 & -1 & r_{21}z^{-1} & 0 \\ 0 & 0 & 0 & r_{23}z^{-1} & -1 & 0 \\ 0 & 0 & 0 & t_{23}z^{-1} & 0 & -1 \end{vmatrix}$$

the transfer functions:

Continued fraction form:

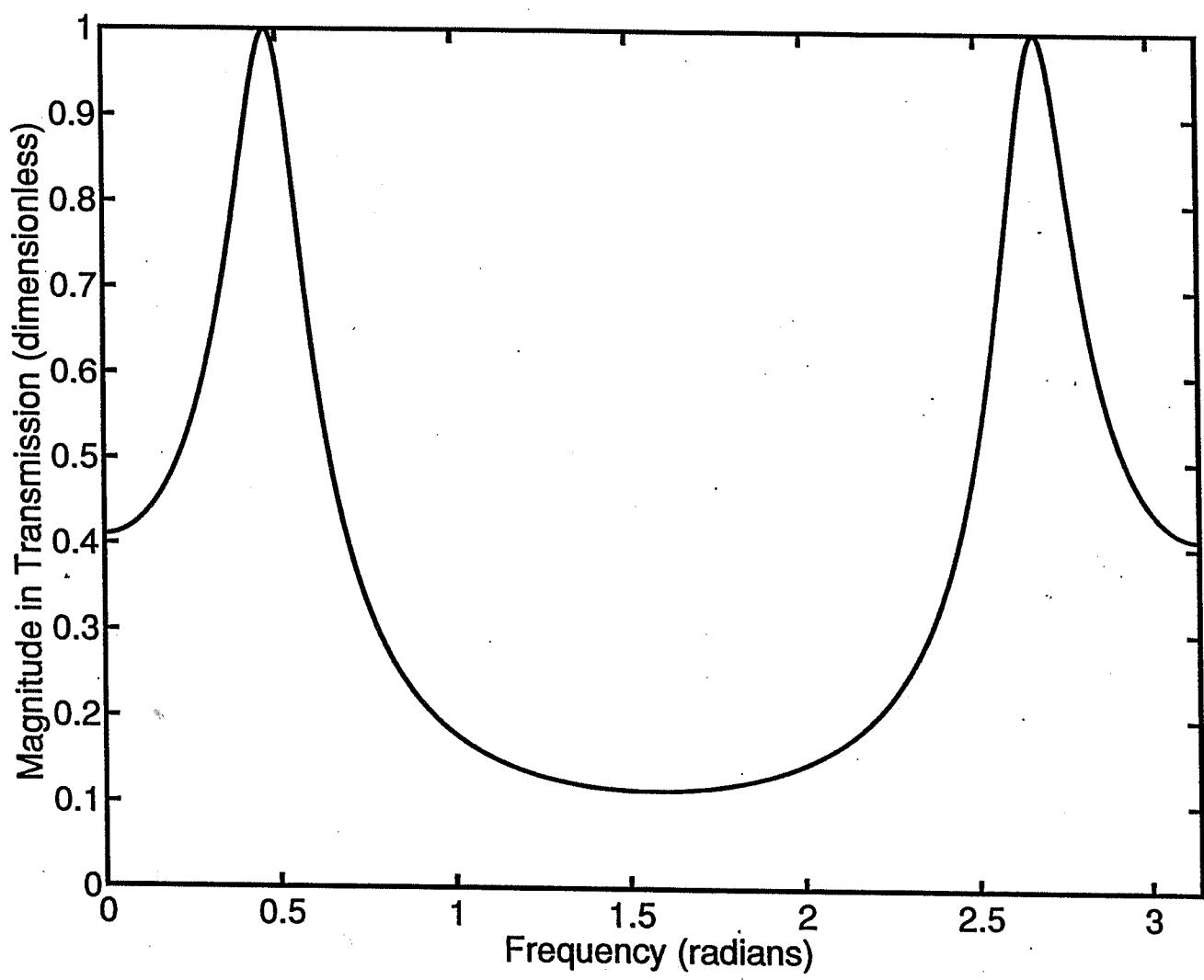
$$\frac{E_t(z)}{E_i(z)} = \frac{t_{01}t_{12}t_{23}z^{-2}}{(1 - r_{21}r_{23}z^{-2}) \left[1 - r_{10}r_{12}z^{-2} - \frac{r_{10}t_{12}t_{21}r_{23}z^{-4}}{1 - r_{21}r_{23}z^{-2}} \right]}$$

$$\frac{E_r(z)}{E_i(z)} = r_{01} + \frac{t_{01}t_{10}r_{12}z^{-2} + \frac{t_{01}t_{12}t_{21}t_{10}r_{23}z^{-4}}{1 - r_{21}r_{23}z^{-2}}}{\left[1 - r_{10}r_{12}z^{-2} - \frac{r_{10}t_{12}t_{21}r_{23}z^{-4}}{1 - r_{21}r_{23}z^{-2}} \right]}$$

Rational form:

$$\frac{E_t(z)}{E_i(z)} = \frac{t_{01}t_{12}t_{23}z^{-2}}{1 - (r_{10}r_{12} + r_{21}r_{23})z^{-2} - r_{10}r_{23}z^{-4}}$$

$$\frac{E_r(z)}{E_i(z)} = \frac{[r_{01} + (r_{12} - r_{01}r_{21}r_{23})z^{-2} + r_{23}z^{-4}]}{1 - (r_{10}r_{12} + r_{21}r_{23})z^{-2} - r_{10}r_{23}z^{-4}}$$



Location of the resonance peaks

$$|T(\omega)| = \frac{t_{01}t_{12}t_{23}}{\sqrt{1 + (r_{10}r_{12} + r_{21}r_{23})^2 + r_{10}^2r_{23}^2 - 2(r_{10}r_{12} + r_{21}r_{23})(1 - r_{10}r_{23})\cos(2\omega) - 2r_{10}r_{23}\cos(4\omega)}}$$

Differentiate

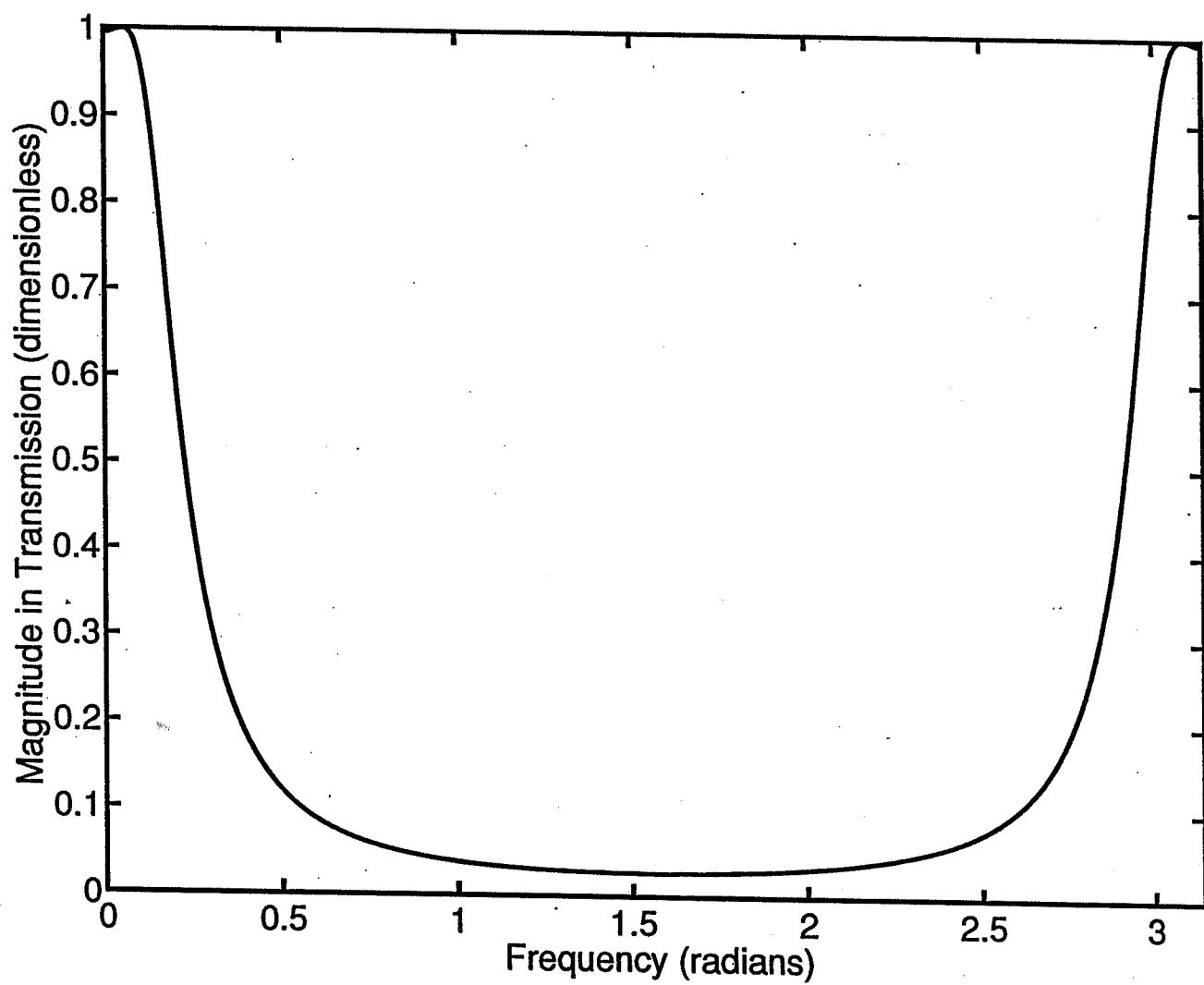
$$\frac{\partial |T(\omega)|}{\partial \omega} = 0 = -4\sin(2\omega)[-(r_{10}r_{12} + r_{21}r_{23})(1 - r_{10}r_{23}) - 4r_{10}r_{23}\cos(2\omega)]$$

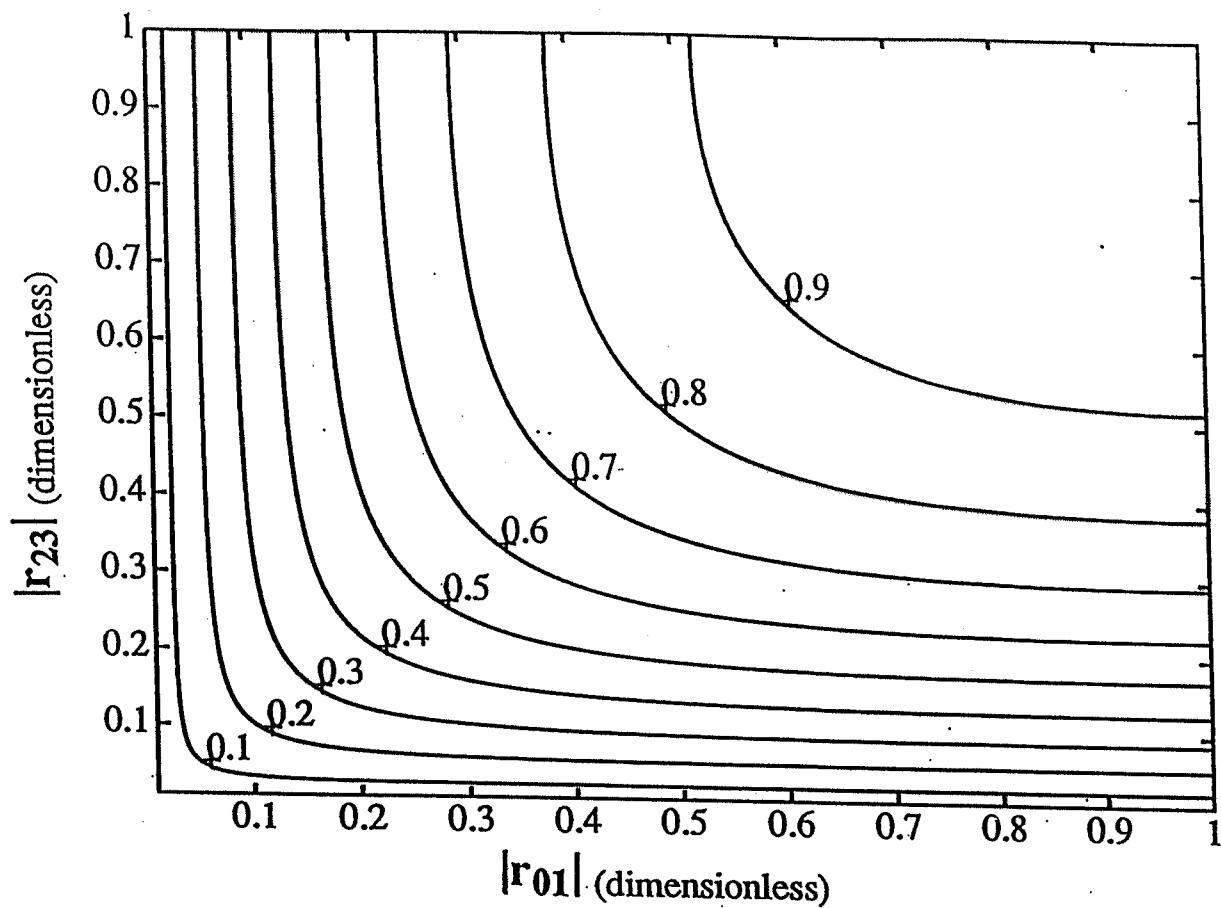
or

$$\omega_m = \frac{1}{2} \arccos \left[\frac{-(r_{10}r_{12} + r_{21}r_{23})(1 - r_{10}r_{23})}{4r_{10}r_{23}} \right]$$

or, for a single resonance peak

$$(r_{10}r_{12} + r_{21}r_{23})(1 - r_{10}r_{23}) = -4r_{10}r_{23}$$





$$\frac{\partial |T(\omega)|}{\partial \omega} = 0 \Rightarrow \omega_m = \frac{1}{2} \cos^{-1} \left[\frac{-(r_{10} r_{12} + r_{21} r_{23})(1 - r_{10} r_{23})}{4 r_{10} r_{23}} \right]$$

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$$(r_{10} r_{12} + r_{21} r_{23})(1 - r_{10} r_{23}) = -4 r_{10} r_{23}$$

Fig. 6

State variable description

$$T_k[n+1] = t_{k-1,k} T_{k-1}[n] + r_{k,k-1} R_k[n]$$

$$R_k[n+1] = r_{k,k+1} T_k[n] + t_{k+1,k} R_{k+1}[n]$$

or in matrix form

$$\mathbf{v}[n+1] = \mathbf{F}\mathbf{v}[n] + \mathbf{q}\mathbf{x}[n]$$

with

$$\mathbf{F} = \begin{vmatrix} 0 & r_{10} & 0 & 0 & 0 & 0 \\ r_{12} & 0 & 0 & t_{21} & 0 & 0 \\ t_{12} & 0 & 0 & r_{21} & 0 & 0 \\ 0 & 0 & r_{23} & 0 & 0 & t_{32} \\ 0 & 0 & t_{23} & 0 & 0 & r_{32} \\ 0 & 0 & 0 & 0 & r_{34} & 0 \end{vmatrix}$$

and

$$\mathbf{q} = \{t_{01}, 0, 0, 0, 0, 0\}$$

$$\mathbf{x} = \{E_i\}$$

$$\mathbf{v}[n] = \{T_1, R_1, T_2, R_2, T_3, R_3\}$$

output equations

$$E_r = r_{01}E_i + t_{10}R_1$$

$$E_t = t_{34}T_3$$

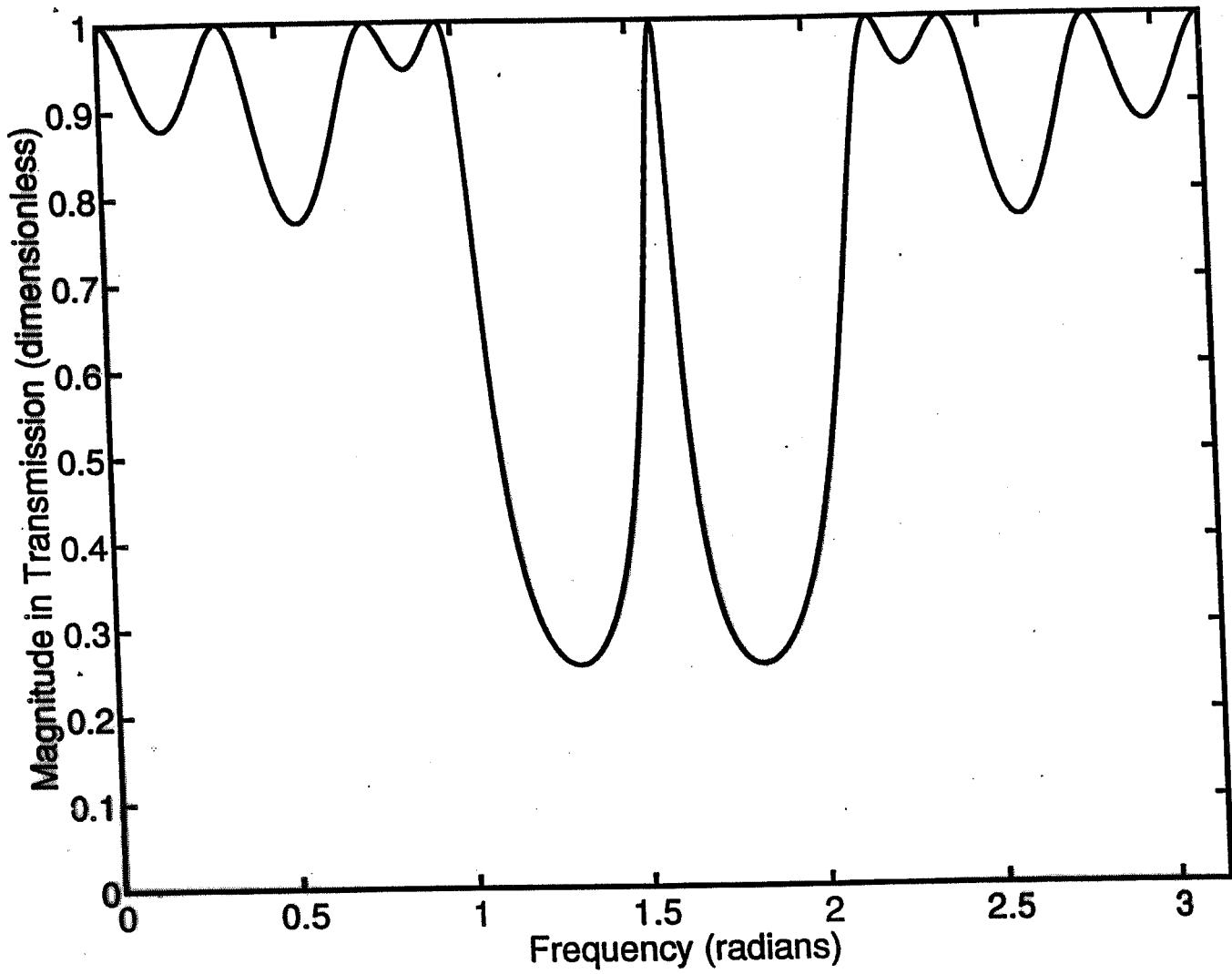
or, in matrix form,

$$\mathbf{y}[n] = \mathbf{g}^t \mathbf{v}[n] + \mathbf{d}x[n]$$

$$\mathbf{y} = \{E_r, E_t\}$$

$$\mathbf{g}^t = \begin{vmatrix} 0 & t_{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & t_{34} & 0 & 0 & 0 \end{vmatrix}$$

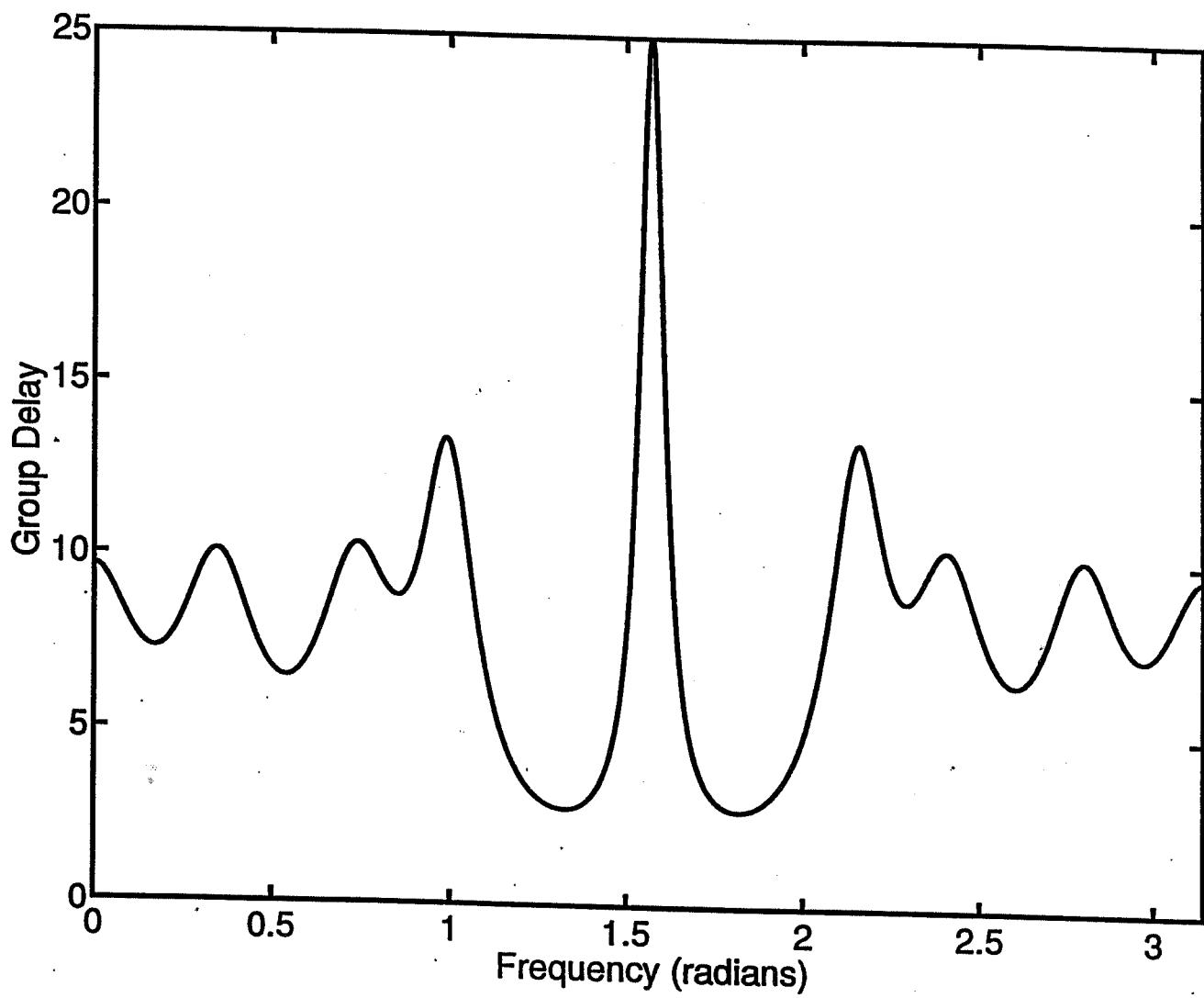
$$\mathbf{d} = \{r_{01}, 0\}$$



THIN FILM STACK HLH-2L-HLH FILTER

Zn Sulfide $n_h = 2.3$

Mg Fluoride $n_L = 1.35$



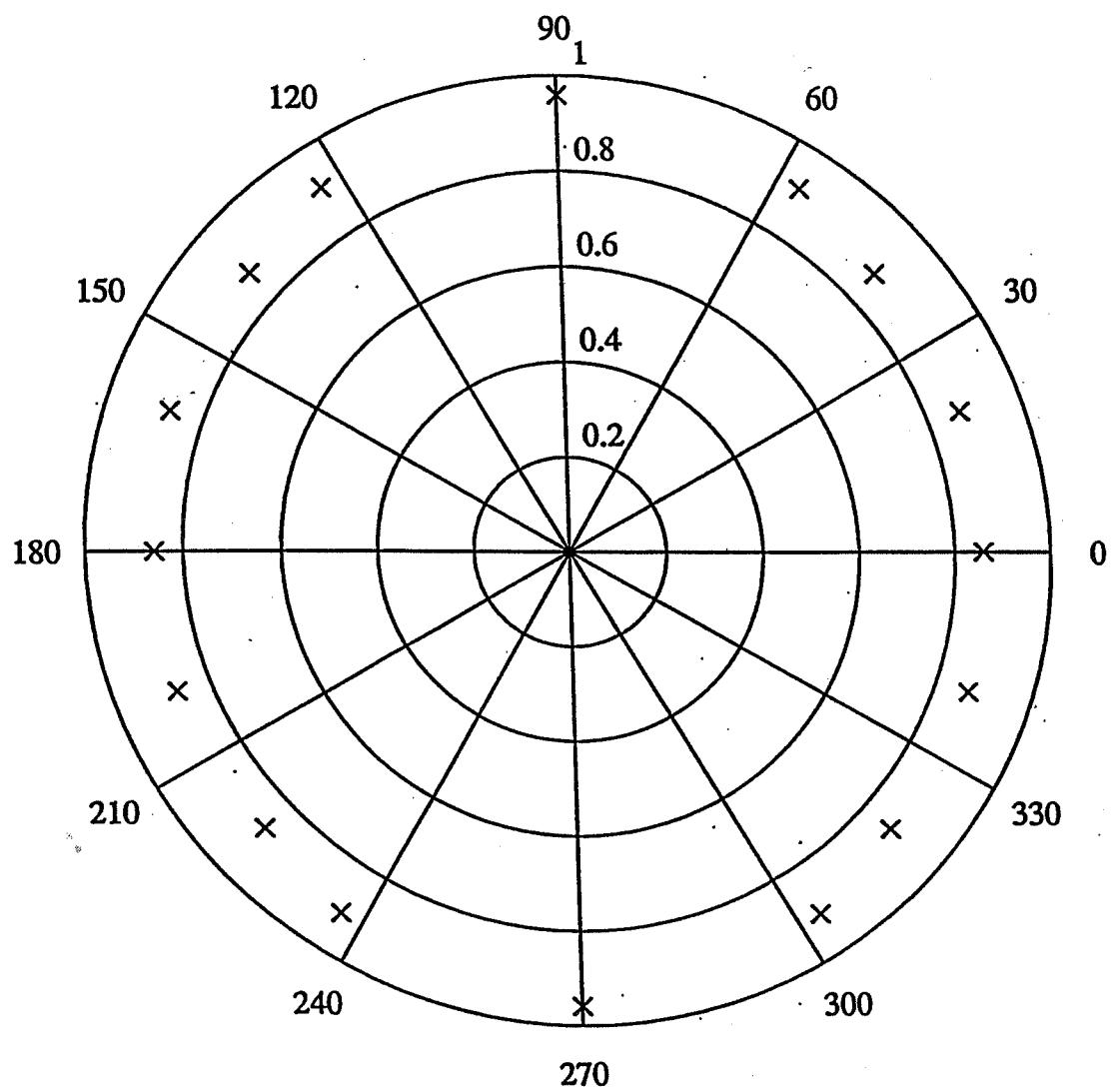


Fig. 8c

Relationship to 2×2 Scattering Matrices

Let us rewrite the recursion relations,

$$T_{k+1}(z) = \frac{z^{-\frac{1}{2}}}{t_{k+1,k}} T_k(z) - \frac{r_{k+1,k} z^{\frac{1}{2}}}{t_{k+1,k}} R'_k(z)$$

and

$$R'_{k+1}(z) = \frac{z^{\frac{1}{2}}}{t_{k+1,k}} R'_k(z) - \frac{r_{k+1,k}}{t_{k+1,k}} z^{-\frac{1}{2}} T_k(z)$$

Or, in matrix form,

$$\begin{bmatrix} T_{k+1}(z) \\ R'_{k+1}(z) \end{bmatrix} = \frac{z^{\frac{1}{2}}}{t_{k+1,k}} \begin{bmatrix} z^{-1} & -r_{k+1,k} \\ -r_{k+1,k} z^{-1} & 1 \end{bmatrix} \begin{bmatrix} T_k(z) \\ R'_k(z) \end{bmatrix}$$

$$\triangleq \frac{z^{\frac{1}{2}}}{t_{k+1,k}} \Phi_{k+1,k} \begin{bmatrix} T_k(z) \\ R'_k(z) \end{bmatrix}$$

This relation holds for all layers k except for the zeroth layer where,

$$\begin{bmatrix} T_1(z) \\ R'_1(z) \end{bmatrix} = \frac{1}{t_{1,0}} \begin{bmatrix} 1 & -r_{1,0} \\ -r_{1,0} & 1 \end{bmatrix} \begin{bmatrix} T_0(z) \\ R'_0(z) \end{bmatrix} \triangleq \frac{1}{t_{1,0}} \Phi_{1,0} \begin{bmatrix} T_0(z) \\ R'_0(z) \end{bmatrix}$$

so that, by induction,

$$\begin{bmatrix} T_{k+1}(z) \\ R'_{k+1}(z) \end{bmatrix} = \frac{z^{\frac{k}{2}}}{t_{k+1,k} \dots t_{1,0}} \Phi_{k,k-1} \Phi_{k-1,k-2} \dots \Phi_0 \begin{bmatrix} T_0(z) \\ R'_0(z) \end{bmatrix}$$

$$\triangleq \tau_{k+1,0} \Phi_{k+1,0} \begin{bmatrix} T_0(z) \\ R'_0(z) \end{bmatrix}$$

The transfer matrix is composed of polynomials of the form,

$$\Phi_{k+1,0} = \begin{bmatrix} A_k^R(z) & B_k^R(z) \\ B_k(z) & A_k(z) \end{bmatrix}$$

where the super R denotes the polynomial coefficient reversal operation,

$$A_k^R(z) = z^{-k} A_k(z^{-1})$$

Can show,

$$A_k(z) = A_{k-1}(z) - r_{k+1,k} z^{-1} B_{k-1}^R(z)$$

$$B_k(z) = -r_{k+1,k} z^{-1} A_{k-1}^R(z) + B_{k-1}(z).$$

To relate these polynomials to the system transfer functions, we must analyze the system model with the appropriate boundary conditions.

We assume the excitation signal, $E_i(z) = T_0(z)$ is applied to the structure, but that no excitation signal is applied from the right, so that $R_{k+1} = 0$.

Thus we must solve the system,

$$\begin{bmatrix} T_{k+1}(z) \\ 0 \end{bmatrix} = \tau_{k+1,0} \begin{bmatrix} A_k^R & B_k^R \\ B_k & A_k \end{bmatrix} \begin{bmatrix} T_0(z) \\ R'_0(z) \end{bmatrix},$$

yielding,

$$T_{k+1}(z) = \tau_{k+1,0} [A_k^R(z)T_0(z) + B_k^R(z)R'_0(z)]$$

and

$$0 = \tau_{k+1,0} [B_k(z)T_0(z) + A_k(z)R'_0(z)]$$

or,

$$R'_0(z) = \frac{-B_k(z)}{A_k(z)} T_0(z).$$

and,

$$T_{k+1}(z) = \frac{z^{\frac{k}{2}} \tau_{k+1,0}}{A_k(z)} T_0(z).$$

Hence, for an arbitrary k -layer system, the two transmission and reflection transfer functions are,

$$H_t(z) = \frac{E_t(z)}{E_i(z)} = \frac{T_{k+1}(z)}{T_0(z)} = \frac{z^{\frac{k}{2}} \tau_{k+1,0}}{A_k(z)}$$

and,

$$H_r(z) = \frac{E_r(z)}{E_i(z)} = \frac{R'_0(z)}{T_0(z)} = \frac{-B_k(z)}{A_k(z)}$$

This basic order-recursive transfer function description provides framework needed to solve the synthesis problem.

Multimirror Etalon Synthesis Algorithm

The synthesis problem is to determine a parameterization (set of reflection coefficients) that will produce a specified transfer function of the correct forms given above.

Write the step-down design recursions in the form

$$\begin{bmatrix} A_k^R(z) & B_k^R(z) \\ B_k(z) & A_k(z) \end{bmatrix} = \begin{bmatrix} z^{-1} & -r_{k+1,k} \\ -r_{k+1,k}z^{-1} & 1 \end{bmatrix} \begin{bmatrix} A_{k-1}^R(z) & B_{k-1}^R(z) \\ B_{k-1}(z) & A_{k-1}(z) \end{bmatrix}.$$

Now apply Cramer's 2×2 matrix inversion rule to obtain,

$$\begin{bmatrix} A_{k-1}^R(z) & B_{k-1}^R(z) \\ B_{k-1}(z) & A_{k-1}(z) \end{bmatrix} = \frac{1}{z^{-1} - r_{k+1,k}^2 z^{-1}} \begin{bmatrix} 1 & r_{k+1,k} \\ r_{k+1,k}z^{-1} & z^{-1} \end{bmatrix} \begin{bmatrix} A_k^R(z) & B_k^R(z) \\ B_k(z) & A_k(z) \end{bmatrix}.$$

Explicitly,

$$A_{k-1}(z) = \frac{r_{k+1,k}}{1 - r_{k+1,k}^2} B_k^R(z) + \frac{1}{1 - r_{k+1,k}^2} A_k(z)$$

$$B_{k-1}(z) = \frac{r_{k+1,k}}{1 - r_{k+1,k}^2} A_k^R(z) + \frac{1}{1 - r_{k+1,k}^2} B_k(z).$$

Note that the $A_{k-1}(z)$ and $B_{k-1}(z)$ are polynomials of order $k - 1$. Thus choose $r_{k+1,k}$ to force the coefficients $a_k(k - 1)$ and $b_k(k - 1)$ to zero:

$$0 = a_k(k - 1)z^{-k} = \frac{r_{k+1,k}}{1 - r_{k+1,k}^2} b_0(k)z^{-k} + \frac{1}{1 - r_{k+1,k}^2} a_k(k)z^{-k}$$

Solving for $r_{k+1,k}$ gives,

$$r_{k+1,k} = -\frac{a_k(k)}{b_0(k)} = -\frac{b_k(k)}{a_0(k)}.$$

Both equalities will be satisfied concurrently by a single value of $r_{k+1,k}$. A final scaling constant, α , stands in the way of a unique solution. Defining,

$$B_k(z) = \alpha \bar{B}_k(z)$$

and using the relation for $r_{k+1,k}$ gives, $\alpha = \sqrt{\frac{a_k(k)}{b_k(k)}}$.

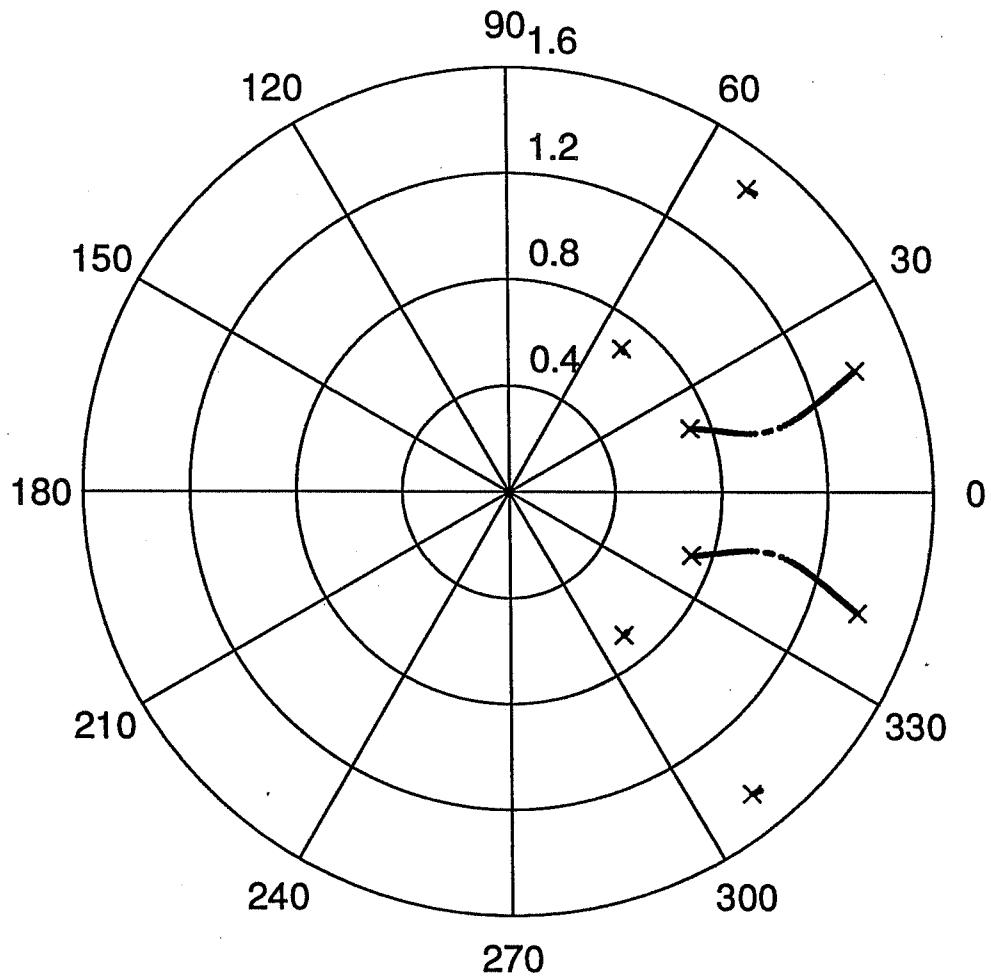
An algorithm can be written which makes use of efficient root-finding routines and autocorrelation routines.

- [1] Set the overall transmission parameter $\tau_{k+1,0}^2$, equal to a step size, ϵ , and input $A_k(z)$.
- [2] Set $\tau_{k+1,0}^2 = \tau_{k+1,0}^2 + \epsilon$ or to the unity intensity gain value.
- [3] Compute and factor $\Psi(z)$ to obtain $\bar{B}_k(z)$.
- [4] Compute the scaling constant α , and from it, the valid $B_k(z)$.
- [5] Compute $r_{k+1,k}$.
- [6] Use the step-down recursions, to move from the order k system to the order $k - 1$ system polynomials, $A_{k-1}(z)$ and $B_{k-1}(z)$.
- [7] Set $k = k - 1$ and return to step [5] until $k = 1$.
- [8] Use $(1 - r_{10}r_{01}) \cdots (1 - r_{k+1,k}r_{k,k+1}) = \tau_{k+1,0}^2$ and solve for r_{10}
- [9] Check to see that $|r_{i+1,i}| < 1$ for $i = 0 \dots k$. If this is the case, report the parameterization.
- [10] Return to step 2 until $\tau_{k+1,0}^2$ or a reflection coefficient approaches unity.

As an example, let's synthesize a multimirror etalon structure to implement the transmission transfer function,

$$H_t(z) = \frac{\tau_{4,0}z^{-2}}{1 - 2.2214z^{-1} + 2.1654z^{-2} - 1.0945z^{-3} + 0.2478z^{-4}}$$

This transfer function is low pass by nature, and has an as yet unspecified gain term in the numerator.



pole migration as the gain coeff & sweep
from zero until one of the $r^k \rightarrow 1$
final values goes to the unit circle.

Fig 9

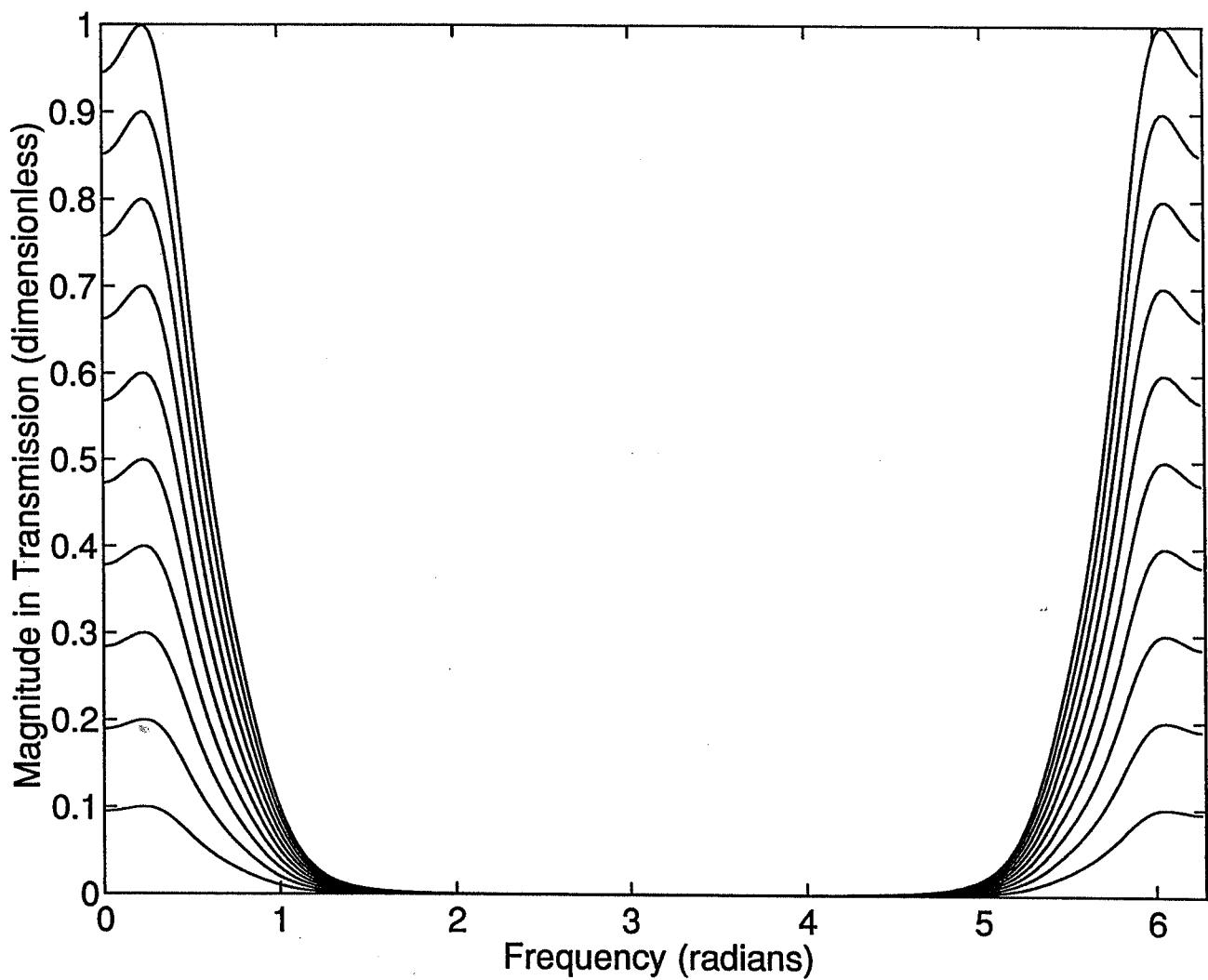


Fig 10

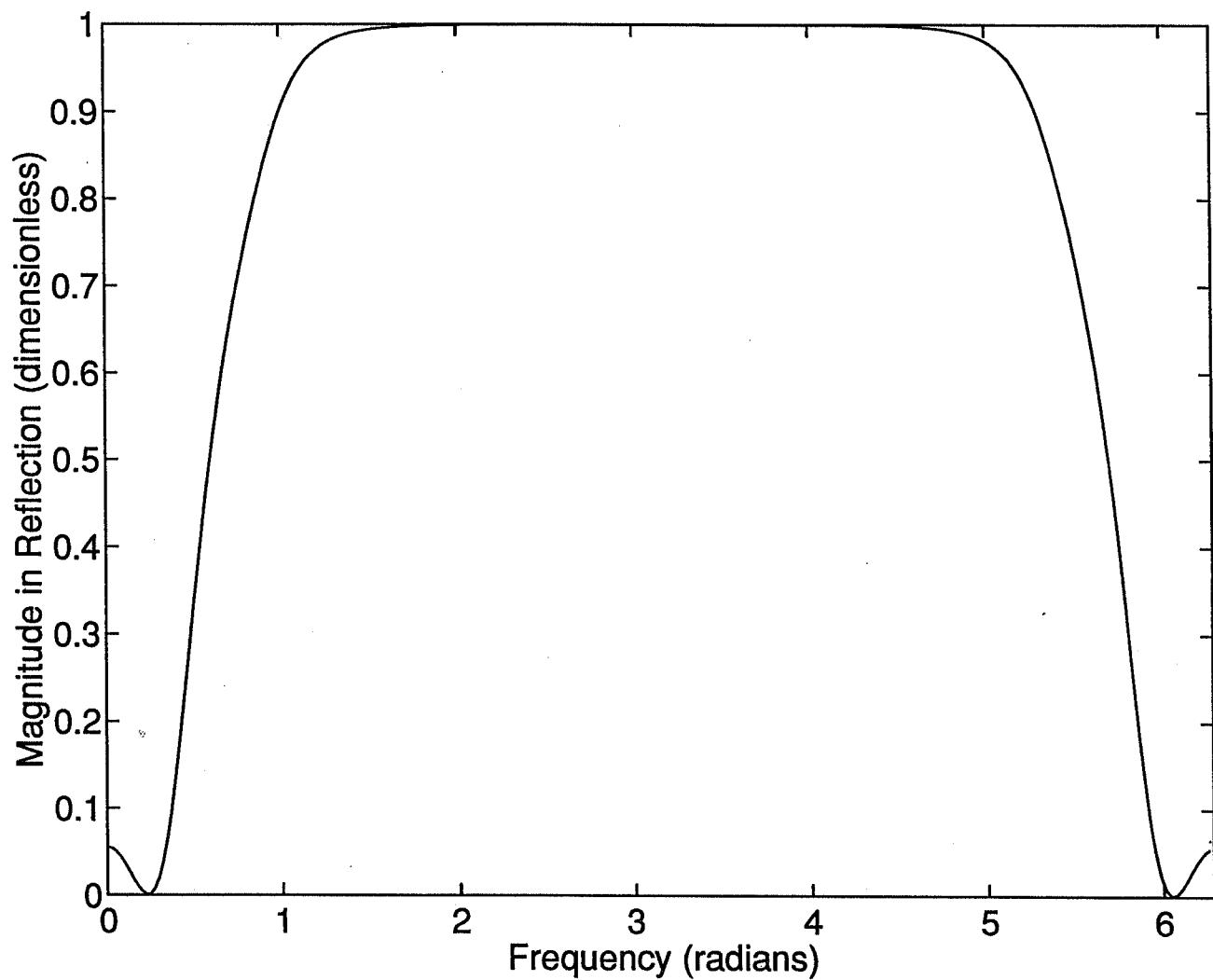


Fig 11

VI. Multimirror Etalon Design

Finally we are in the position of designing N^{th} order devices by first specifying the response function and then synthesizing the associated lattice parameterization.

Digital filter design software may be exploited to generate a response function, and then to convert the response function into an all-pole approximation appropriate for multimirror etalons.

We are especially interested in high finesse structures

We show how to design N^{th} -order interferometer/resonator response functions to achieve unity intensity passband gain, a specified half-power bandwidth (and thus a specified finesse), and a specified out of band rejection level.

Recall the form for the transmission,

$$H_t(z) = \frac{z^{\frac{k}{2}} \tau_{k+1,0}}{A_k(z)}.$$

The numerator represents the scalar loss and the delay of $\frac{k}{2}$ time units through the structure. We'll solve for this using a weighting scheme at the end of our procedure.

Use an autoregressive approximation,

$$H(z) \approx \frac{1}{A_k(z)} \Rightarrow H(z) A_k(z) \approx 1$$

or, in the time domain,

$$h(n) * a_k(n) \approx \delta(n)$$

The time-domain expression, (3), may be written in matrix form as a Toeplitz least squares problem,

$$\begin{bmatrix} h(0) & \dots & 0 \\ \vdots & \ddots & \\ h(k) & & h(0) \\ h(L) & \dots & h(L-k) \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_k \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which we write more compactly as

$$\mathbf{H}\mathbf{a}_k \approx \mathbf{e}_1.$$

L is the block size which is selected to be large enough for the impulse response to be sufficiently close to zero by the L^{th} sample.

The vector of coefficients, \mathbf{a}_k is obtained by solving this equation for a least squares solution.

This gives a reasonable approximation, and autoregressive modified Stiglitz-McBride iterations may be used to further refine the estimate.

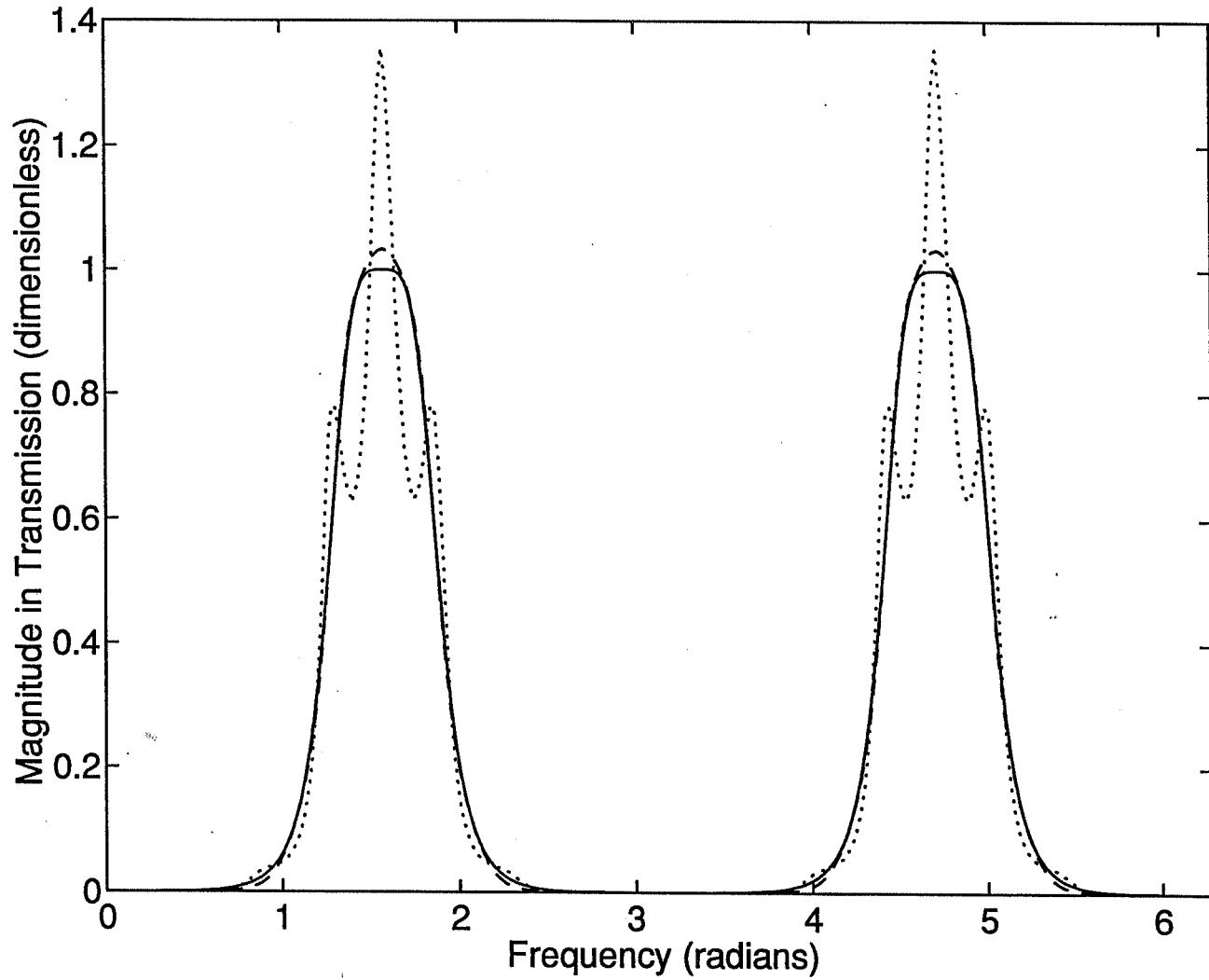


Fig. 12

As another example, consider a resonator response function constructed from a k^{th} -order complex-conjugate pole pair. That is,

$$H_k(z) = \frac{G_k}{(1 - re^{j\phi}z^{-1})^k (1 - re^{-j\phi}z^{-1})^k}.$$

This transfer function has k^{th} order poles at $p = re^{\pm j\phi}$, that is with radius $r < 1$ and angle $\pm\phi$.

Note that there are only four parameters in Eq. (6), $\{k, G_k, r, \phi\}$. For any specified value of k , one can derive a set of parameters $\{G_k, r, \phi\}$ to meet the unity intensity and 1/2 power bandwidth specifications, but systems of higher order, k , will provide better out of band rejection.

Hence the resonator response function design procedure will loop from low to high k producing filters of fixed finesse until the out of band rejection specification is met.

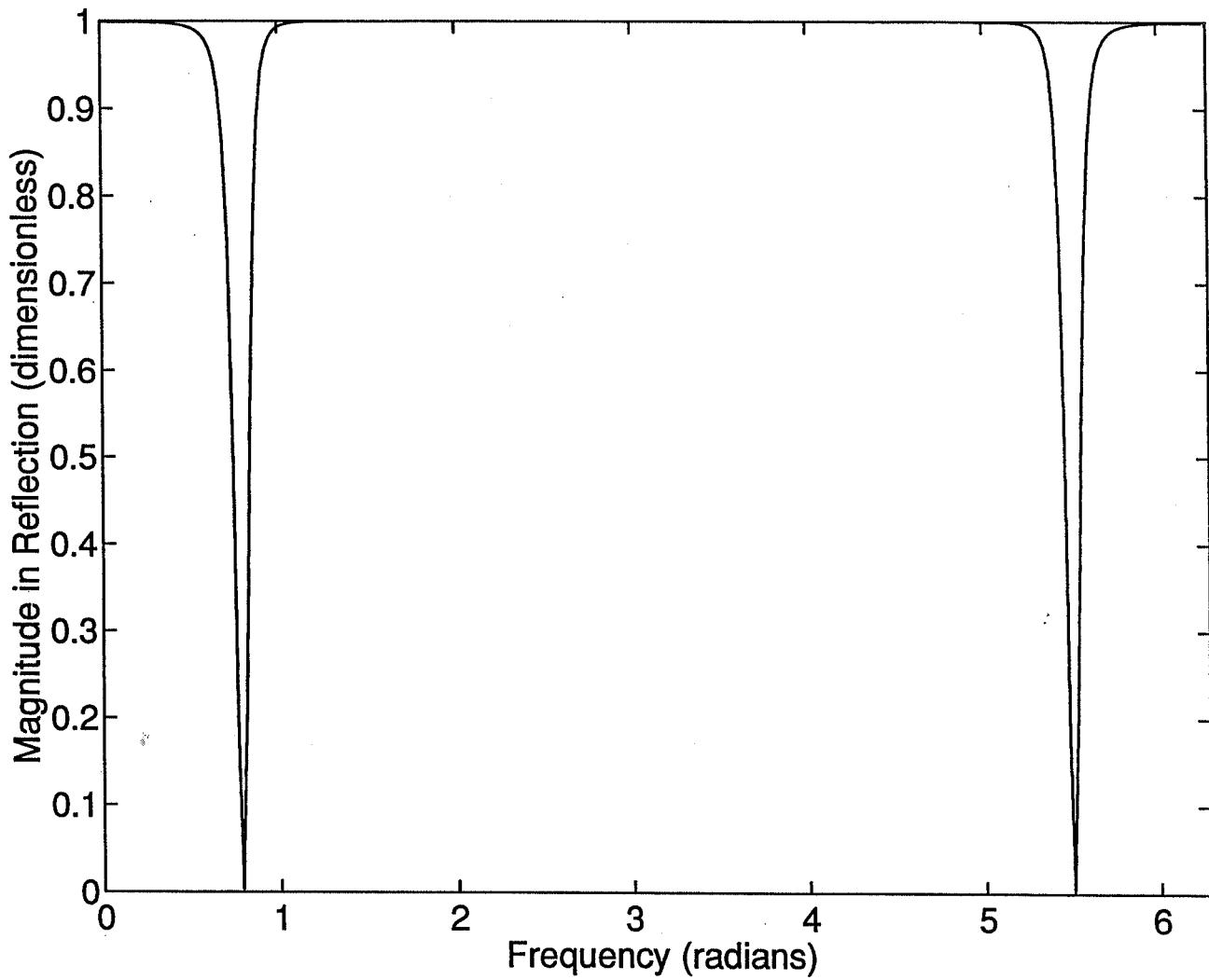


Fig. 14

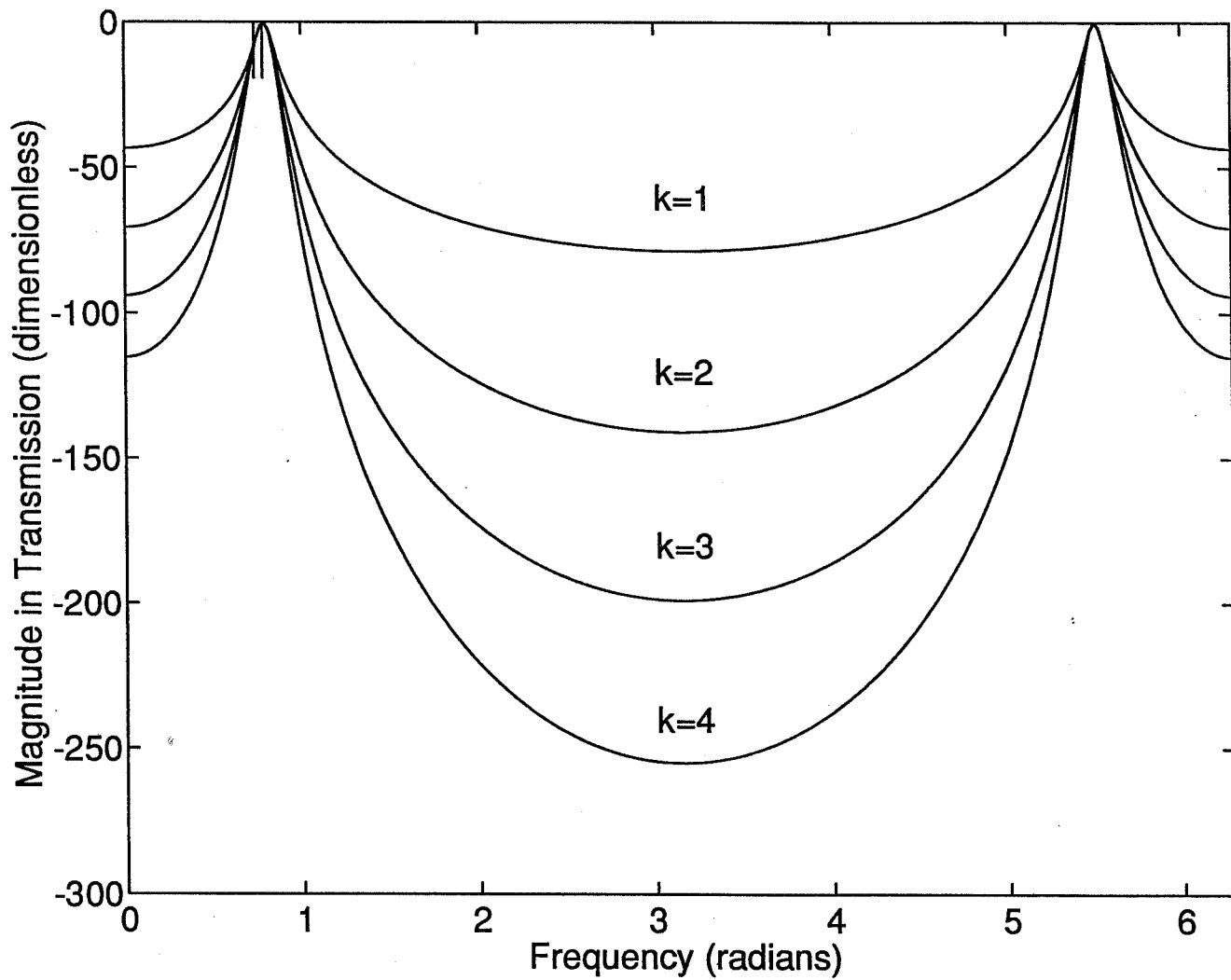


Fig 13

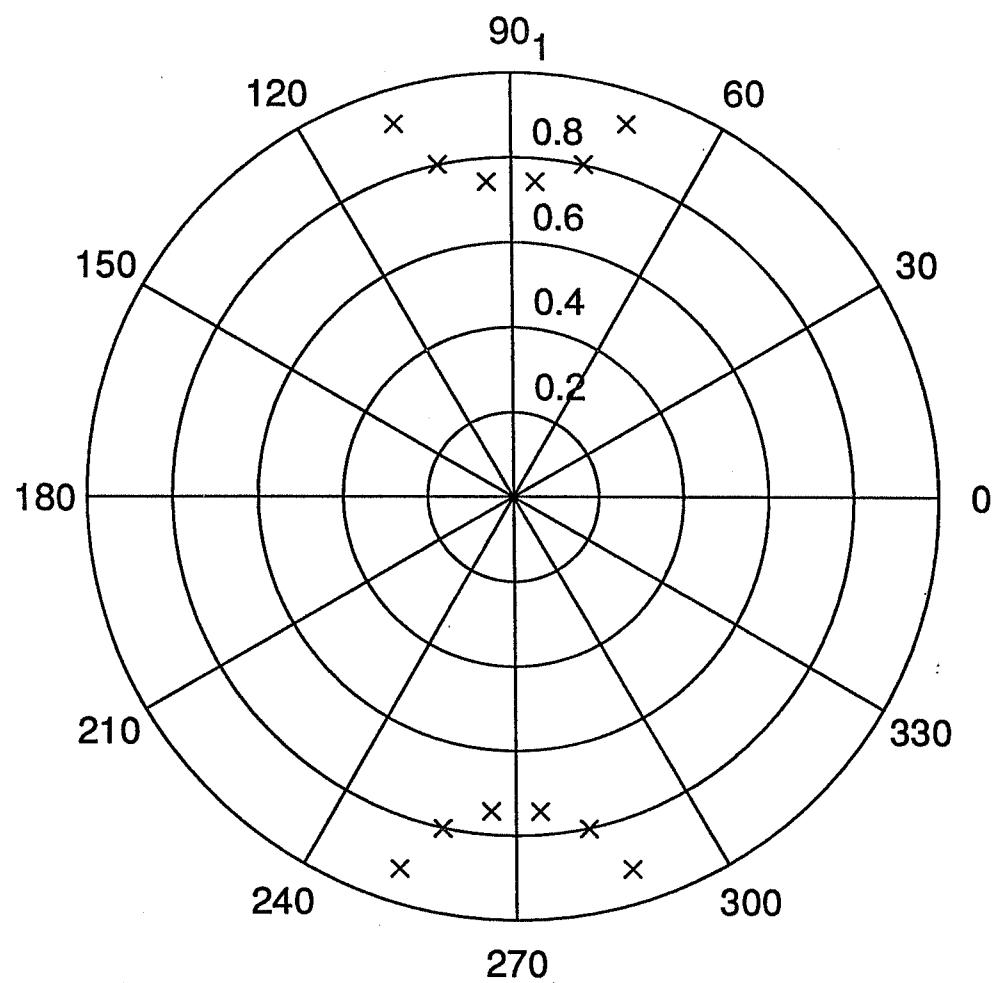


Fig 15

As another example, consider the design of multimirror etalons with flat-top responses.

Instead of starting with the multi-mirror etalon and deriving the condition to get a flat top and no ripples, start with the analog Butterworth filter response function.

The N^{th} -order analog lowpass prototype Butterworth intensity function is given by

$$|H_a(j\omega')|^2 = \frac{1}{1 + \left(\frac{\omega'}{\omega_c}\right)^{2N}}.$$

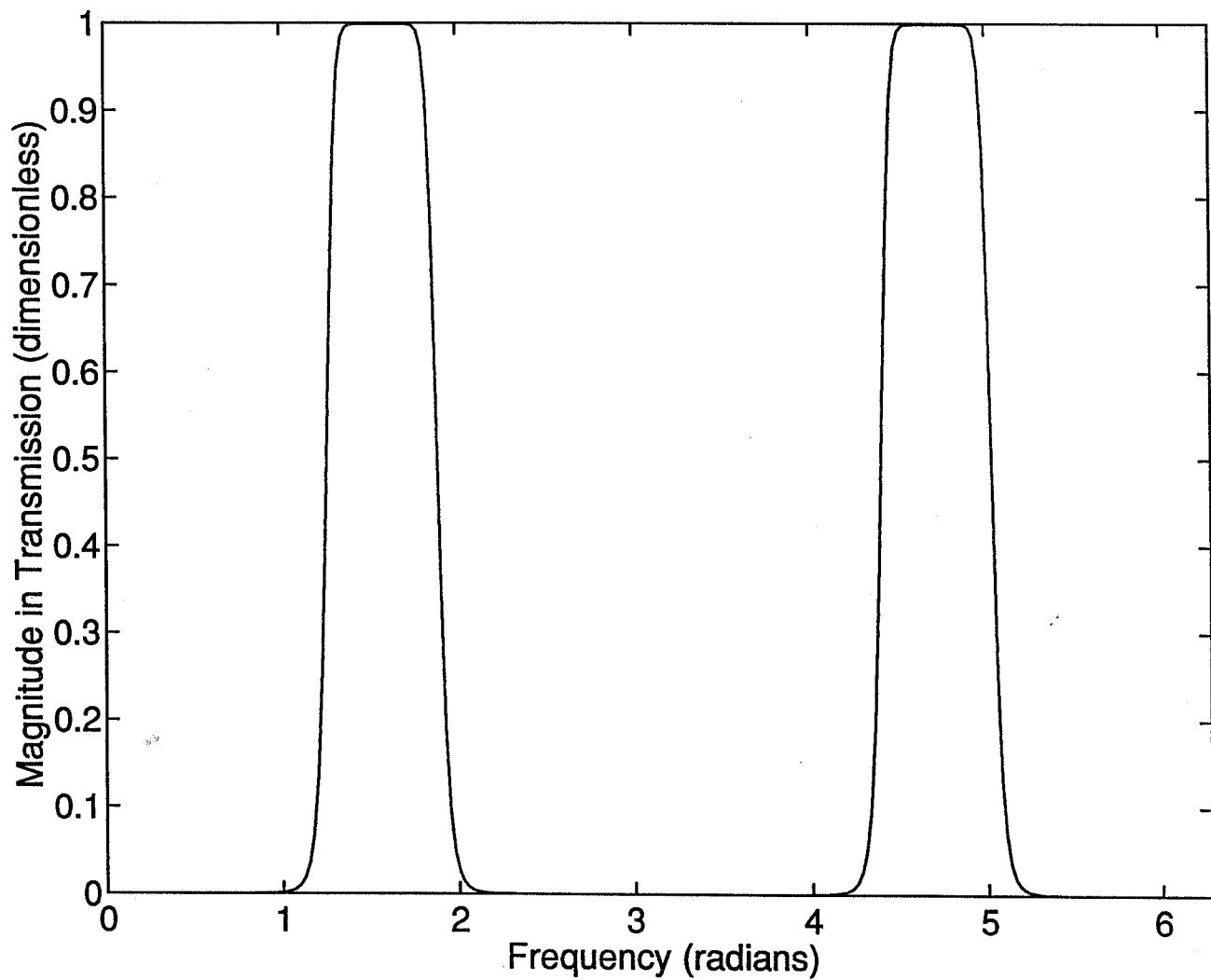


Fig 16

Discussion

- The prime motivation of this work was to introduce z-transform techniques to the analysis of etalon-type optical structures.
- We presented several examples of varying degrees of familiarity in order to illustrate the mechanics of the technique.
- We talked about various matrix techniques in the analysis of these structures.
- We also presented multimirror etalon synthesis recursions that map transfer functions specifications onto sets of reflection coefficients. The synthesis process reverses the direction of the scattering matrix recursions to step down through an order recursive system of polynomials. As the recursion proceeds, one reflection coefficient is revealed each step of the way.
- This is reminiscent of Schur-like “layer peeling” and Levinson-like “layer adjoining” algorithms used in digital lattice filter problems
- Using our results, we can now specify a device in terms of a desired response function and then use the synthesis procedure to compute the corresponding reflection coefficients. The procedure’s order recursive structure by-passes the complicated and non-linear sets of equations normally associated with the design problem.
- The method decouples the design of the response function specification from the design of the device, allowing us to focus separately on the different design issues. We presented a weighted auto-regressive modeling based approach to approximate any arbitrary transfer function with all-pole transfer function suitable for multistage etalon implementation.
- For examples we looked at designing high finesse filters and flat-top response filters.