EE6316 Fields and Waves

Homework Assignment #2 Solutions

Due on: Tue, March 7, 2006
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4. Balanis, Chapter 4, page 174, #4.3

The phasor notation of the H-field is

$$\mathbf{H} = \left(\frac{1}{\eta_0}\right)\left(\hat{\mathbf{a}}_\mathbf{x} - 2\hat{\mathbf{a}}_\mathbf{y}\right)e^{-j\beta z} \tag{1.1}$$

And

$$\mathbf{H} = \frac{1}{\eta_0}\hat{\mathbf{n}} \times \mathbf{E} \tag{1.2}$$

, where in this particular case, the wave goes towards the z direction, so

$$\hat{\mathbf{n}} = \hat{\mathbf{a}}_\mathbf{z} \tag{1.3}$$

(a) Complex E-field

Let (1.2) be written as

$$\mathbf{E} = \eta_0 \mathbf{H} \times \hat{\mathbf{n}} \tag{1.4}$$

, into which we plug (1.1)

$$\begin{aligned} \mathbf{E} &= \eta_0 \begin{vmatrix} \hat{\mathbf{a}}_\mathbf{x} & \hat{\mathbf{a}}_\mathbf{y} & \hat{\mathbf{a}}_\mathbf{z} \\ H_x & H_y & H_z \\ 0 & 0 & 1 \end{vmatrix} = \eta_0\left(H_y\hat{\mathbf{a}}_\mathbf{x} - H_x\hat{\mathbf{a}}_\mathbf{y}\right) \\ &= -\left(2\hat{\mathbf{a}}_\mathbf{x} + \hat{\mathbf{a}}_\mathbf{y}\right)e^{-j\beta z} \end{aligned} \tag{1.5}$$

(b) Instantaneous Poynting vector

$$\mathbf{E}\times\mathbf{H}^{*}=\begin{vmatrix}\hat{\mathbf{a}}_{\mathbf{x}} & \hat{\mathbf{a}}_{\mathbf{y}} & \hat{\mathbf{a}}_{\mathbf{z}}\\ -2e^{-j\beta z} & -e^{-j\beta z} & 0\\ \frac{1}{\eta_0}e^{j\beta z} & \frac{1}{\eta_0}(-2)e^{j\beta z} & 0\end{vmatrix}=\left(\frac{5}{\eta_0}\right)\hat{\mathbf{a}}_{\mathbf{z}} \tag{1.6}$$

$$\mathbf{E}\times\mathbf{H}e^{j2\omega t}=e^{j2\omega t}\begin{vmatrix}\hat{\mathbf{a}}_{\mathbf{x}} & \hat{\mathbf{a}}_{\mathbf{y}} & \hat{\mathbf{a}}_{\mathbf{z}}\\ -2e^{-j\beta z} & -e^{-j\beta z} & 0\\ \frac{1}{\eta_0}e^{-j\beta z} & \frac{1}{\eta_0}(-2)e^{-j\beta z} & 0\end{vmatrix}=\left(\frac{5}{\eta_0}\right)\left(e^{j(2\omega t-2\beta z)}\right)\hat{\mathbf{a}}_{\mathbf{z}} \tag{1.7}$$

So, the instantaneous Poynting vector comes out to be

$$\begin{aligned}\mathbf{p}(x,y,z;t)&=\frac{1}{2}\left[\operatorname{Re}\left(\mathbf{E}\times\mathbf{H}^{*}\right)+\operatorname{Re}\left(\mathbf{E}\times\mathbf{H}e^{j2\omega t}\right)\right]\\&=\left(\frac{5}{2\eta_0}\right)\left[1+\cos\left(2\omega t-2\beta z\right)\right]\hat{\mathbf{a}}_{\mathbf{z}}\end{aligned} \tag{1.8}$$

(c) Time-average Poynting vector

The time average of (1.8) is

$$\mathbf{P}_{\mathbf{avg}}=\left(\frac{5}{2\eta_0}\right)\hat{\mathbf{a}}_{\mathbf{z}} \tag{1.9}$$

5. Balanis, Chapter 4, page 175, #4.8

$$E_0\left(z=0\right)=4\times10^{-3}\ \text{(V/m)} \tag{1.10}$$

$$f_0 = 300\ \text{(MHz)} \tag{1.11}$$

$$\omega = 2\pi f_0 = 1.88\times10^{9}\ \text{(rad/s)} \tag{1.12}$$

$$\beta = 2\pi f_0 / c = 6.28\ \text{(rad/m)} \tag{1.13}$$

(a) Phasor E and H

Let the E-field in the phasor notation be

$$\mathbf{E} = \left(\hat{\mathbf{a}}_{\mathbf{x}}\right) E_0 e^{j\beta z} \tag{1.14}$$

Since

$$\mathbf{H} = \frac{1}{\eta_0}\hat{\mathbf{n}}\times\mathbf{E} \tag{1.15}$$

, where in this particular case, the wave goes into the -z direction, so

$$\hat{\mathbf{n}} = -\hat{\mathbf{a}}_{\mathbf{z}} \tag{1.16}$$

The H-field is therefore

$$\mathbf{H} = \left(-\hat{\mathbf{a}}_{\mathbf{y}}\right)\left(\frac{E_0}{\eta_0}\right) e^{j\beta z} \tag{1.17}$$

, where

$$\frac{E_0}{\eta_0} = 1.06\times10^{-5}\ \text{(A/m)} \tag{1.18}$$

(b) Instantaneous E and H

$$\begin{aligned}\mathbf{e}\left(x,y,z;t\right) &= \mathrm{Re}\left[\mathbf{E}\left(x,y,z\right)e^{j\omega t}\right] \\ &= \mathrm{Re}\left[\left(\hat{\mathbf{a}}_{\mathbf{x}}\right)E_0 e^{j\beta z} e^{j\omega t}\right] \\ &= \left(\hat{\mathbf{a}}_{\mathbf{x}}\right)E_0 \cos\left(\omega t + \beta z\right)\end{aligned} \tag{1.19}$$

$$\mathbf{h}(x,y,z;t) = \mathrm{Re}\left[\mathbf{H}(x,y,z)e^{j\omega t}\right] = \mathrm{Re}\left[\left(-\hat{\mathbf{a}}_{\mathbf{y}}\right)\frac{E_0}{\eta_0}e^{j\beta z}e^{j\omega t}\right] = \left(-\hat{\mathbf{a}}_{\mathbf{y}}\right)\left(\frac{E_0}{\eta_0}\right)\cos(\omega t + \beta z) \tag{1.20}$$

(c) Poynting Vector

The instantaneous Poynting vector

$$\mathbf{p}(x,y,z;t) = \mathbf{e}\times\mathbf{h} = \left(-\hat{\mathbf{a}}_{\mathbf{z}}\right)\left(\frac{E_0^2}{2\eta_0}\right)\left[1+\cos(2\omega t + 2\beta z)\right] \tag{1.21}$$

The time average

$$\mathbf{P}_{\mathbf{avg}} = \left(-\hat{\mathbf{a}}_{\mathbf{z}}\right)\left(\frac{E_0^2}{2\eta_0}\right) \tag{1.22}$$

, where

$$\frac{E_0^2}{2\eta_0} = 2.12\times 10^{-8}\ (\mathrm{W/m^2}) \tag{1.23}$$

(d) Energy Densities

The instantaneous energy densities

$$w_e = \frac{1}{2}\varepsilon_0|\varepsilon|^2 = \frac{1}{4}\varepsilon_0 E_0^2\left[1+\cos(2\omega t + 2\beta z)\right] \tag{1.24}$$

$$w_h = \frac{1}{2}\mu_0|h|^2 = \frac{1}{4}\mu_0\left(\frac{E_0}{\eta_0}\right)^2\left[1+\cos(2\omega t + 2\beta z)\right] \tag{1.25}$$

The time-average energy densities

$$W_e = \frac{1}{4}\varepsilon_0 E_0^2 = 3.54\times 10^{-17}\ (\mathrm{J/m^3}) \tag{1.26}$$

$$W_h = \frac{1}{4}\mu_0\left(\frac{E_0}{\eta_0}\right)^2 = 3.53\times 10^{-17}\ (\mathrm{J/m^3}) \tag{1.27}$$

$W_e = W_h$ as expected.

6. Balanis, Chapter 4, page 176, #4.13

(a) Time-average power density

An antenna transmits a total power of

$$\mathrm{P}_{rad} = 5\times 10^{-2} \text{ (W)} \tag{1.28}$$

The power density at r(m) away from the antenna is

$$S = \frac{\mathrm{P}_{rad}}{4\pi r^2} = 4.42\times 10^{-10} \text{ (W/m}^2\text{)} \tag{1.29}$$

(b) RMS E- and H fields

Let the E-field be written as

$$\mathbf{E} = (\hat{\mathbf{a}}_\mathbf{y}) E_0 e^{-j\beta z} \tag{1.30}$$

and the H-field as

$$\mathbf{H} = (-\hat{\mathbf{a}}_\mathbf{x}) \left(\frac{E_0}{\eta_0}\right) e^{-j\beta z} \tag{1.31}$$

so that the wave travels in the positive direction

$$\mathbf{S} = (\hat{\mathbf{a}}_\mathbf{z}) \frac{1}{2} \mathrm{Re}(\mathbf{E}\times\mathbf{H}^*) = (\hat{\mathbf{a}}_\mathbf{z}) \left(\frac{E_0^2}{2\eta_0}\right) \tag{1.32}$$

Since

$$\frac{E_0^2}{2\eta_0} = S = 4.42\times 10^{-10} \text{ (W/m}^2\text{)} \tag{1.33}$$

The peak value of the E-field is

$$E_0 = 5.77\times 10^{-4} \text{ (V/m)} \tag{1.34}$$

and that of the H-field is

$$E_0/\eta_0 = 1.53\times 10^{-6} \text{ (A/m)} \tag{1.35}$$

The root-mean-square value of the E-field is

$$E_{rms} = E_0/\sqrt{2} = 4.08\times 10^{-4} \text{ (V/m)} \tag{1.36}$$

and that of the H-field is

$$H_{rms} = \left(\frac{E_0}{\eta_0}\right)\frac{1}{\sqrt{2}} = 1.08\times10^{-6} \text{ (A/m)} \tag{1.37}$$

(c) Time average energy density

The time-average electric energy density

$$W_e = \frac{1}{4}\varepsilon_0 E_0^2 = 7.37\times10^{-19} \text{ (J/m}^3\text{)} \tag{1.38}$$

The time-average magnetic energy density

$$W_h = \frac{1}{4}\mu_0\left(\frac{E_0}{\eta_0}\right)^2 = 7.35\times10^{-19} \text{ (J/m}^3\text{)} \tag{1.39}$$

$W_e = W_h$ as expected.