EE6316 Fields and Waves

Homework Assignment #1 Solutions

Due on: Tue, Feb 21, 2006
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4. Balanis, Chapter 1, page 33, #1.5

The instantaneous E-field is

$$\boldsymbol{\varepsilon}(x,y,z;t) = \mathrm{Re}\left[\mathbf{E}(x,y,z)e^{j\omega t}\right] \tag{1.1}$$

, where

$$\mathbf{E}(x,y,z) = \hat{\mathbf{a}}_x A(x+y) + \hat{\mathbf{a}}_y B(x-y) \tag{1.2}$$

is a phasor. Since the medium is source-free,

$$\nabla \bullet \mathbf{D} = \rho = 0 \tag{1.3}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \tag{1.4}$$

$$\begin{aligned} \nabla \bullet \mathbf{D} = \varepsilon_0 \nabla \bullet \mathbf{E} &= \frac{\partial}{\partial x}\left[A(x+y)\right] + \frac{\partial}{\partial y}\left[B(x-y)\right] \\ &= A - B = 0 \end{aligned} \tag{1.5}$$

5. Balanis, Chapter 1, page 35, #1.14

The instantaneous E-field is

$$\boldsymbol{\varepsilon}(x,y,z;t) = \mathrm{Re}\left[\mathbf{E}(x,y,z)e^{j\omega t}\right] \tag{1.6}$$

, where

$$\mathbf{E}(x,y,z) = \hat{\mathbf{a}}_y E_0 \sin\left(\frac{\pi x}{a}\right)e^{-j\beta_z z} \tag{1.7}$$

is a phasor.

(a) Instantaneous H-field

From Faraday's law,

$$\nabla\times\mathbf{E} = -j\omega\mu\mathbf{H} \tag{1.8}$$

$$\begin{aligned}\nabla\times\mathbf{E} &= \begin{vmatrix}\hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & E_y & 0\end{vmatrix} = \left(-\frac{\partial}{\partial z}E_y\right)\hat{\mathbf{a}}_x + \left(\frac{\partial}{\partial x}E_y\right)\hat{\mathbf{a}}_z \\ &= \left(j\beta_z E_0 \sin\left(\frac{\pi x}{a}\right)e^{-j\beta_z z}\right)\hat{\mathbf{a}}_x + \left(E_0\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi x}{a}\right)e^{-j\beta_z z}\right)\hat{\mathbf{a}}_z\end{aligned} \tag{1.9}$$

$$\mathbf{H} = \left(\frac{-\beta_z}{\omega\mu}E_0 \sin\left(\frac{\pi x}{a}\right)e^{-j\beta_z z}\right)\hat{\mathbf{a}}_x + \left(\frac{e^{j\pi/2}}{\omega\mu}E_0\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi x}{a}\right)e^{-j\beta_z z}\right)\hat{\mathbf{a}}_z \tag{1.10}$$

The instantaneous H-field is

$$\begin{aligned}\mathbf{h}(x,y,z;t) &= \mathrm{Re}\left[\mathbf{H}(x,y,z)e^{j\omega t}\right] \\ &= \left(\frac{-\beta_z}{\omega\mu}E_0 \sin\left(\frac{\pi x}{a}\right)\cos(\omega t - \beta_z z)\right)\hat{\mathbf{a}}_x \\ &\quad + \left(\frac{1}{\omega\mu}E_0\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi x}{a}\right)\cos\left(\omega t - \beta_z z + \frac{\pi}{2}\right)\right)\hat{\mathbf{a}}_z\end{aligned} \tag{1.11}$$

The z-component of the H-field is ahead of the x-component by $\pi/2$ in phase.

(b) Phase constant β_z

Referring ahead to Chapter 8 on page 354 of the book,

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 = \omega^2\mu_0\varepsilon_0 \tag{1.12}$$

, where, in this particular case, $\beta_y = 0$ and

$$\beta_x = \frac{\pi}{a} \tag{1.13}$$

So,

$$\beta_z = \sqrt{\omega^2 \mu_0 \varepsilon_0 - \left(\pi/a\right)^2} \tag{1.14}$$

6. Balanis, Chapter 2, page 96, #2.1

$$\mathbf{E} = \hat{\mathbf{a}}_y 2 \text{ (V/m)} \tag{1.15}$$

$$\mathbf{P} = \hat{\mathbf{a}}_y 2.762\times10^{-11} \text{ (C/m}^2\text{)} \tag{1.16}$$

with the cross sectional area of

$$A = 2.4\times10^{-3} \text{ (m}^2\text{)} \tag{1.17}$$

(a)

$$\sigma_{pol} = |\mathbf{P}| = 2.762\times10^{-11} \text{ (C/m}^2\text{)} \tag{1.18}$$

(b)

$$Q_{pol} = \sigma_{pol} A = 6.63\times10^{-14} \text{ (C)} \tag{1.19}$$

(c)

$$q_{vp} = 0 \text{ (C/m}^3\text{)} \tag{1.20}$$

(d)

$$P = \chi\varepsilon_0 E \tag{1.21}$$

Solving this for χ

$$\chi = \frac{P}{\varepsilon_0 E} = \frac{2.762\times10^{-11} \text{ (C/m}^2\text{)}}{2 \text{ (V/m)}\times8.854\times10^{-12} \text{ (F/m)}} = 1.56 \tag{1.22}$$

The dielectric constant comes out to be

$$\varepsilon_r = \chi + 1 = 2.56 \tag{1.23}$$

7. Balanis, Chapter 2, page 97, #2.4

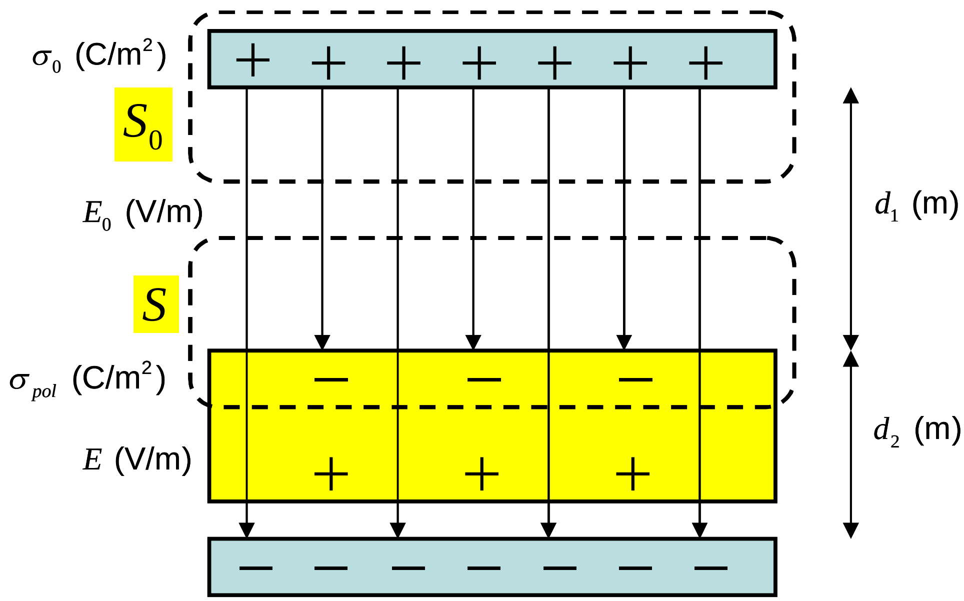

Figure 1: A parallel-plate capacitor partially filled with a dielectric

The parallel plates of an area

$$A = 2\times10^{-2} \text{ [m}^2\text{]} \tag{1.24}$$

are $d_1 + d_2$ in spacing, where

$$d_1 = 2.5\times10^{-4} \text{ [m]} \tag{1.25}$$

$$d_2 = 1\times10^{-3} \text{ [m]} \tag{1.26}$$

Consider a case of free space with no dielectric in there. The capacitance of free space is

$$C_0 = \varepsilon_0 \frac{A}{d} = 8.854\times10^{-12} \times \frac{2\times10^{-2}}{1.25\times10^{-3}} = 1.42\times10^{-10} \text{ [F]} \tag{1.27}$$

and a static charge is

$$Q_0 = C_0 V_0 = \left(1.42\times10^{-10}\right)\left(100\right) = 1.42\times10^{-8} \text{ [C]} \tag{1.28}$$

So, the electrostatic energy is

$$U_0 = \frac{1}{2} C_0 V_0^2 = \frac{1}{2}\left(1.42\times10^{-10}\right)\left(100\right)^2 = 7.1\times10^{-7} \text{ [J]} \tag{1.29}$$

The E-field in free space is

$$E_0 = \frac{V_0}{d} = \frac{100}{1.25\times10^{-3}} = 8\times10^4 \text{ [V/m]} \tag{1.30}$$

So, the energy density comes out to be

$$\frac{1}{2}\varepsilon_0 |E_0|^2 = \frac{1}{2}\left(8.854\times10^{-12}\right)\left(8\times10^4\right)^2 = 2.83\times10^{-2} \text{ [J/m}^3\text{]} \tag{1.31}$$

The total energy is therefore

$$U_0 = \left(\frac{1}{2}\varepsilon_0 |E_0|^2\right)(Ad) = \left(2.83\times10^{-2}\right)\left(2.5\times10^{-5}\right) = 7.1\times10^{-7} \text{ [J]} \tag{1.32}$$

, which agrees with (1.29) as expected.

Now let a dielectric slab slide in as shown in Figure 1. Applying the Gauss's law to the closed surface S0

$$E_0 = \sigma_0 / \varepsilon_0 \tag{1.33}$$

and to the closed surface S

$$E - E_0 = \sigma_{pol} / \varepsilon_0 \tag{1.34}$$

From the KVL

$$E + E_0(0.25) = 10^5 \tag{1.35}$$

The polarization vector

$$\varepsilon_r = \chi + 1 = 5 \tag{1.36}$$

$$\chi = 4 \tag{1.37}$$

$$|\mathbf{P}| = \chi\varepsilon_0 |\mathbf{E}| = |\sigma_{pol}| = -\sigma_{pol} \tag{1.38}$$

We have four equations with four unknowns:

$$\begin{pmatrix} \varepsilon_0 & 0 & -1 & 0 \\ -\varepsilon_0 & \varepsilon_0 & 0 & -1 \\ 0 & 4\varepsilon_0 & 0 & 1 \\ 0.25 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_0 \\ E \\ \sigma_0 \\ \sigma_{pol} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 10^5 \end{pmatrix} \tag{1.39}$$

(a) (b) (c) (d)

Solving this for unknowns, we find

$$E_0 = 2.22\times10^5 \text{ [V/m]} \tag{1.40}$$

$$E = 4.44\times10^4 \text{ [V/m]} \tag{1.41}$$

$$\sigma_0 = 1.97\times10^{-6} \text{ [C/m}^2\text{]} \tag{1.42}$$

$$\sigma_{pol} = -1.57\times10^{-6} \text{ [C/m}^2\text{]} \tag{1.43}$$

The electric flux density in the air gap

$$D_0 = \varepsilon_0 E_0 = 1.96\times10^{-6} \text{ [C/m}^2\text{]} \tag{1.44}$$

and the one in the dielectric

$$D = \varepsilon_0 \varepsilon_r E = 1.96\times10^{-6} \text{ [C/m}^2\text{]} \tag{1.45}$$

The normal component of the E-field flux density is continuous as expected. The charge stored on the plates

$$Q = A\sigma_0 = 3.94\times10^{-8} \text{ [C]} \tag{1.46}$$

(e)

The capacitance

$$C = Q/V_0 = 3.94\times10^{-10} \text{ [F]} \tag{1.47}$$

The voltage across the air gap

$$V_1 = E_0 d_1 = 5.55\times10^1 \text{ [V]} \tag{1.48}$$

and the one running across the dielectric

$$V_2 = E d_2 = 4.44\times10^1 \text{ [V]} \tag{1.49}$$

The capacitance can be thought of as a series combination of C_1 and C_2, where C_1 =capacitance across the air gap, and C_2 =capacitance across the dielectric

$$\begin{gathered} \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \\ C_1 V_1 = C_2 V_2 \end{gathered} \tag{1.50}$$

Solving (1.50) for C_1 and C_2,

$$\begin{gathered} C_1 = 7.10\times10^{-10} \text{ [F]} \\ C_2 = 8.87\times10^{-10} \text{ [F]} \end{gathered} \tag{1.51}$$

(f)

The energy density in free space

$$\frac{1}{2}\varepsilon_0 \left|E_0\right|^2 = 2.18\times10^{-1} \text{ [J/m}^3\text{]} \tag{1.52}$$

and the one in a dielectric

$$\frac{1}{2}\varepsilon_0\varepsilon_r \left|E\right|^2 = 4.36\times10^{-2} \text{ [J/m}^3\text{]} \tag{1.53}$$

The energy in free space

$$\frac{1}{2}\varepsilon_0 \left|E_0\right|^2 \left(Ad_1\right) = 1.09\times10^{-6} \text{ [J]} \tag{1.54}$$

and the one in a dielectric

$$\frac{1}{2}\varepsilon_0\varepsilon_r \left|E\right|^2 \left(Ad_2\right) = 8.72\times10^{-7} \text{ [J]} \tag{1.55}$$

The total energy is therefore

$$U = \left(\frac{1}{2}\varepsilon_0 \left|E_0\right|^2\right)\left(Ad_1\right) + \left(\frac{1}{2}\varepsilon_0\varepsilon_r \left|E\right|^2\right)\left(Ad_2\right) = 1.96\times10^{-6} \text{ [J]} \tag{1.56}$$