Chapter 3 Solutions

Q 3.1.1.

Fourier transform of function $g(t)$ is shown below.

$$F(g(t)) = G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi ft}\, dt$$

From Euler's formula

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\therefore\ G(f) = \int_{-\infty}^{\infty} g(t)\,[\cos 2\pi ft - j\sin 2\pi ft]\, dt$$

$$= \int_{-\infty}^{\infty} g(t)\cos 2\pi ft\, dt - j\int_{-\infty}^{\infty} g(t)\sin 2\pi ft\, dt$$

Even function :- $f(x) = f(-x)$

$$\therefore\ \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

we split limit $-\infty < x < 0$. $0 < x < \infty$

substituting x by $-x$ we can change the limit from $0 < (-x) < \infty$

$$= \int_{0}^{\infty} f(-x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 2\int_{0}^{\infty} f(x)\,dx$$

Odd function $f(-x) = -f(x)$.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx.$$

$$= \int_{0}^{\infty} f(-x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= -\int_{0}^{\infty} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 0.$$

(i) $g(t)$ is even

$\cos(\omega t)$ is also an even function.

$\therefore$ $g(t)\cos(\omega t)$ is an even function. from

$(g(-t)\cos(-\omega t) = g(t)\cos(\omega t)$

definition of even function).

$\sin(\omega t)$ is an odd function.

$\therefore$ $g(t)\sin(\omega t)$ is an odd function.

$(g(-t)\sin(-\omega t) = -g(t)\sin(\omega t))$

$$\therefore G(f) = \underbrace{\int_{-\infty}^{\infty} g(t)\cos 2\pi f t\,dt}_{\text{even}} - \underbrace{j\int_{-\infty}^{\infty} g(t)\sin 2\pi f t\,dt}_{\text{odd}}$$

$$\boxed{G(f) = 2\int_{0}^{\infty} g(t)\cos 2\pi f t\,dt}$$

b) $g(t)$ is an odd function.

$\therefore \quad g(-t) = -g(t)$

$\therefore \quad g(-t)\cos(\omega t) = -g(t)\cos\omega t$ (odd function)

$g(-t)\sin(-\omega t) = g(t)\sin\omega t$ (even function)

$$\therefore G(f) = \underbrace{\int_{-\infty}^{\infty} g(t)\cos 2\pi f t \, dt}_{\text{odd}} - j \underbrace{\int_{-\infty}^{\infty} g(t)\sin 2\pi f t \, dt}_{\text{even}}$$

$$\boxed{G(f) = -2j\int_{0}^{\infty} g(t)\sin 2\pi f t \, dt}$$

c) i) if $g(t)$ is real and even function of t

$$G(f) = 2\int_{0}^{\infty} g(t)\cos 2\pi f t \, dt.$$

$$G(-f) = 2\int_{0}^{\infty} g(t)\cos 2\pi(-f)t \, dt$$

$$= 2\int_{0}^{\infty} g(t)\cos 2\pi f t \, dt \quad [\because \cos\theta \text{ is even } f^n]$$

$\therefore$ $G(f)$ is real & even function.

(ii) $g(t)$ is real & odd function of t

$$G(f) = -2j\int_{0}^{\infty} g(t)\sin 2\pi f t \, dt$$

$$G(-f) = -2j\int_{0}^{\infty} g(t)\sin 2\pi(-f)t \, dt$$

$= -2j \int_0^{\infty} g(t) \sin 2\pi f t \, dt$

$= - G(f)$

$\therefore \quad G(-f) = G(f)$

$\therefore$ $G(f)$ is imaginary and odd function of f

(iii) $g(t)$ is imaginary and even function of t

$\therefore \quad G(f) = 2j \int_0^{\infty} g(t) \cos 2\pi f t \, dt + 0$

$= 2j \int_0^{\infty} g(t) \cos 2\pi f t \, dt$

$\therefore \quad G(-f) = G(f)$

hence $G(f)$ is imaginary & even function of f.

Similarly rest can be proved.

Q 3.1-2

a) $g(t) = \exp(-2|t-3|)$

Fourier transform $G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$

$\therefore \quad G(f) = \int_{-\infty}^{\infty} e^{(-2|t-3|)}\, e^{-j2\pi f t}\, dt$

For $-\infty < t < \infty$ $\quad e^{-2|t-3|}$ can be split as

$e^{2(t-3)} \qquad -\infty < t < 3$

$e^{-2(t-3)} \qquad 3 < t < \infty$

$$\therefore \quad G(f) = \int_{-\infty}^{3} e^{2(t-3)}\, e^{-j2\pi f t}\, dt + \int_{3}^{\infty} e^{-2(t-3)}\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{3} e^{2t} . e^{-6} e^{-j2\pi f t}\, dt + \int_{3}^{\infty} e^{-2t} . e^{6} . e^{-j2\pi f t}\, dt$$

Rearranging terms

$$= e^{-6} \int_{-\infty}^{3} e^{(2-j2\pi f)t}\, dt + e^{6} \int_{3}^{\infty} e^{-(2+2\pi f)t}\, dt$$

$$= e^{-6}\, \frac{(e^{(2-j2\pi f).3} - e^{-\infty})}{2-j2\pi f} - \frac{e^{6}\,(e^{-\infty} - e^{-(2+2\pi f).3})}{2+2\pi f}$$

$$= \frac{(e^{-6} . e^{6} . e^{-j6\pi f} - 0)}{2-j2\pi f} - \frac{(0 - e^{6} . e^{-6} . e^{-j6\pi f})}{2+2\pi f}$$

$$= \frac{e^{-j6\pi f}}{2-j2\pi f} + \frac{e^{-j6\pi f}}{2+j2\pi f}$$

$$= \frac{e^{-j6\pi f}(2+j2\pi f) + e^{-j6\pi f}(2-j2\pi f)}{4(1+\pi^2 f^2)}$$

$$= \frac{4e^{-j6\pi f}}{4(1+\pi^2 f^2)}$$

$$\boxed{G(f) = \frac{e^{-j6\pi f}}{1+\pi^2 f^2}}$$

b) $G(f) = \delta(4\pi f) - \delta(f-2)$

$$g(t) = \mathcal{F}^{-1}(G(f))$$

$$= \int_{-\infty}^{\infty} G(f)\, e^{j2\pi f t}\, df$$

$$= \int_{-\infty}^{\infty} \delta(4\pi f)\, e^{j2\pi f t}\, df - \int_{-\infty}^{\infty} \delta(f-2)\, e^{j2\pi f t}\, df$$

Let $x = 4\pi f$ for first term.

$$\therefore \int_{-\infty}^{\infty} \delta(x)\, e^{j\frac{x}{2}t}\, \frac{dx}{4\pi}$$

This means value of function at $x=0$ as there is impulse function $\delta(x)$.

$$= \frac{e^0}{4\pi} = \frac{1}{4\pi}$$

For the second term.

$$\int_{-\infty}^{\infty} \delta(f-2)\, e^{j2\pi f t}\, df$$

evaluate function for $F=2$ as we have impulse function. $\delta(f-2)$

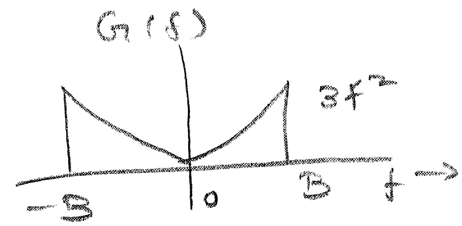
$$\therefore \int_{-\infty}^{\infty} \delta(f-2)\, e^{j2\pi f t}\, df = e^{j2\pi(2)t}$$

$$= e^{j4\pi t}$$

$$\therefore \boxed{g(t) = \frac{1}{4\pi} - e^{j4\pi t}}$$

Q 3.1.5 (a) $G(f) = 3f^2 \quad -B < f < B$

$= 0 \quad$ else



From the graph we can say that function is even.

To prove

$G(-f) = 3(-f)^2 = 3f^2 = G(f)$ $\therefore$ even function

$$g(t) = \int_{-\infty}^{\infty} G(f)\, e^{j2\pi f t}\, df$$

$$= \int_{-B}^{B} 3f^2\, e^{j2\pi f t}\, dt$$

$\left[\because e^{j2\pi f t} = \cos 2\pi f t + j \sin 2\pi f t\right]$

$$= 2\int_{0}^{B} 3f^2 \cos 2\pi f t\, df$$

($\because$ see problem 1 for simplification)

We will use integration by parts.

$$\int_0^B f^2 \cos(2\pi f t)\, df$$

suppose $f^2 = u$ $\quad dv = \cos(2\pi f t)$

$du = 2f\, df$ $\quad v = \dfrac{\sin 2\pi f t}{2\pi t}$

$$\int f^2 \cos(2\pi f t)\, df = \frac{f^2 \sin 2\pi f t}{2\pi t} - \int \frac{\sin 2\pi f t}{2\pi f t} \cdot 2f\, df$$

$\displaystyle\int \frac{\sin 2\pi f t}{2\pi t} \cdot 2f\, df$ solving by integration by parts.

$u = 2f$ $\quad dv = \dfrac{\sin 2\pi f t}{2\pi t}$

$du = 2df$ $\quad v = -\dfrac{\cos 2\pi f t}{4\pi^2 t^2}$

$$\int 2f \frac{\sin 2\pi f t}{2\pi t}\, df = -\frac{2f \cos 2\pi f t}{4\pi^2 t^2} + \int \frac{2\cos 2\pi f t}{4\pi^2 t^2}\, df$$

$$= -\frac{2f \cos 2\pi f t}{4\pi^2 t^2} + \frac{2 \sin 2\pi f t}{8\pi^3 t^3}$$

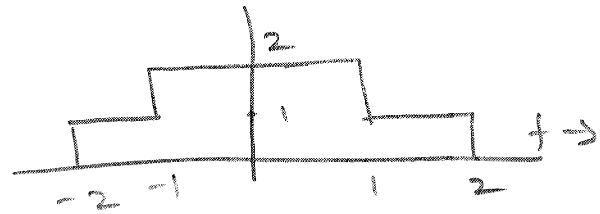
$$\therefore \int f^2 \cos(2\pi f t)\, df = \left[\frac{f^2 \sin 2\pi f t}{2\pi t} + \frac{2f \cos 2\pi f t}{4\pi^2 t^2} - \frac{2\sin 2\pi f t}{8\pi^3 t^3}\right]_0^B$$

Rearranging terms

$$= \left[\frac{(2\pi^2 f^2 t^2 - 1)\sin 2\pi f t + 2\pi f t \cos 2\pi f t}{4\pi^3 t^3}\right]_0^B$$

$\therefore g(t) = 6\left[\dfrac{2\pi B t \cos 2\pi B t + (2\pi^2 B^2 t^2 - 1)\sin 2\pi B t}{4\pi^3 t^3}\right]$

(b) $G(f)$ can be expressed as



$$G(f) = \begin{cases} 0 & -\infty < f < -2 \\ 1 & -2 < f < -1 \\ 2 & -1 < f < 1 \\ 1 & 1 < f < 2 \\ 0 & 2 < f < \infty \end{cases}$$

$$g(t) = \int_{-2}^{-1} G(f) e^{j2\pi f t} df + \int_{-1}^{1} G(f) e^{j2\pi f t} + \int_{1}^{2} G(f) e^{j2\pi f t} df$$

$G(f)$ is even function.

$$\therefore g(t) = 2\int_{1}^{2} \cos 2\pi f t \, df + 2\int_{0}^{1} 2\cos 2\pi f t \, df$$

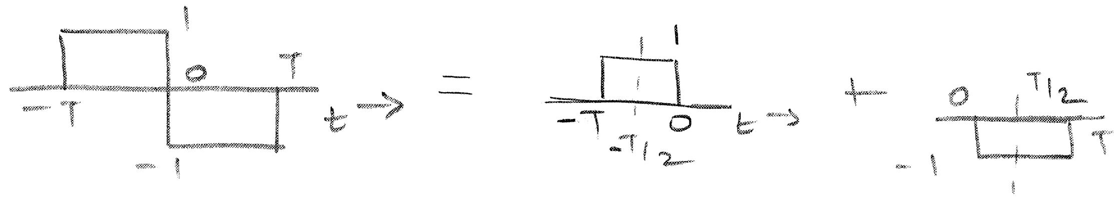
$$= \left[\frac{2\sin 2\pi f t}{2\pi t}\right]_1^2 + \left[4\,\frac{\sin 2\pi f t}{2\pi t}\right]_0^1$$

$$= \frac{2\sin 4\pi t}{2\pi t} - \frac{2\sin 2\pi t}{2\pi t} + \frac{4\sin 2\pi t}{2\pi t}$$

$$= \frac{\sin 4\pi t + \sin 2\pi t}{\pi t}$$

Q. 3.3-3

(a)



$$g(t) = \Pi\left(\frac{t+T/2}{T}\right) - \Pi\left(\frac{t-T/2}{T}\right)$$

From table 3.1

$$\Pi(t/T) = T\,\mathrm{sinc}(\pi f t)$$

$$\therefore \Pi\left(\frac{t+T/2}{T}\right) = T\,\mathrm{sinc}(\pi f T)\,e^{j 2\pi f T/2}$$

$$\Pi\left(\frac{t-T/2}{T}\right) = T\,\mathrm{sinc}(\pi f T)\,e^{-j 2\pi f T/2}$$

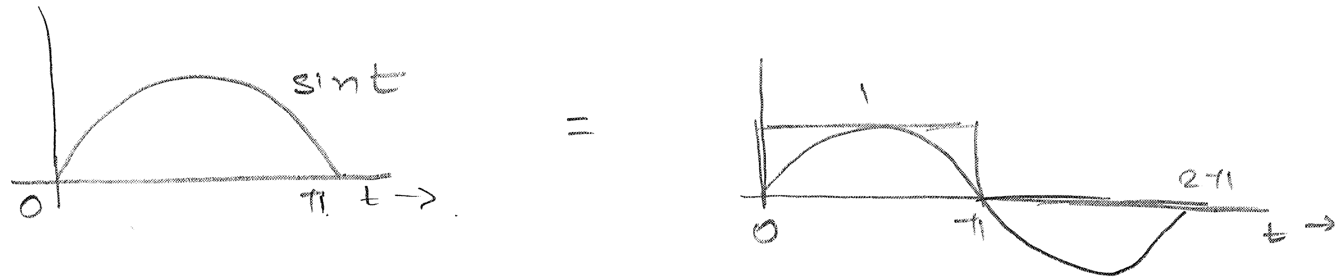
$$\therefore G(f) = T\,\mathrm{sinc}(\pi f T)\left[e^{j\pi f T} - e^{-j\pi f T}\right]$$

$$= T\,\mathrm{sinc}(\pi f T)\left[\cancel{\cos \pi f T} + j\sin \pi f T - \cancel{\cos \pi f T} + j \sin \pi f T\right]$$

$$= T\,\frac{\sin(\pi f T)}{\pi f T}\,(2j \sin \pi f T)$$

$$\boxed{G(f) = \frac{2j\,T\sin^2(\pi f T)}{\pi f T}}$$

b).



From the hint given.

$$g(t) = \sin t\,[u(t) - u(t-\pi)]$$

$$= \sin t\, u(t) - \sin t\, u(t-\pi).$$

Now $\sin t = -\sin(t-\pi)$.

$$\therefore g(t) = \sin t\, u(t) - \sin(t-\pi)\, u(t-\pi)$$

From table 3.1

$$\sin 2\pi f_0 t\, u(t) = \frac{1}{4j}[\delta(f-f_0) - \delta(f+f_0)] + \frac{2\pi f_0}{(2\pi f_0)^2 - (2\pi f)^2}$$

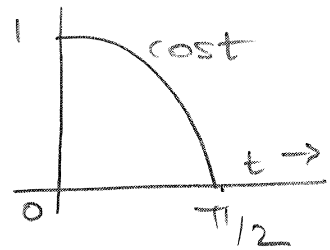
$$\sin t\, u(t) = \frac{1}{4j}[\delta(f - 1/2\pi) - \delta(f + 1/2\pi)] + \frac{1}{1-(2\pi f)^2}$$

$$\sin(t-\pi)\, u(t-\pi) = \left\{\frac{1}{4j}[\delta(f - 1/2\pi) - \delta(f + 1/2\pi)] + \frac{1}{1-(2\pi f)^2}\right\} \times e^{-j2\pi f(\pi)}$$

$$\therefore G(f) = \left[\frac{1}{4j}[\delta(f - 1/2\pi) - \delta(f + 1/2\pi)] + \frac{1}{1-(2\pi f)^2}\right](1 + e^{-j2\pi^2 f})$$

$$= \left[\frac{1}{4j}[1-1] + \frac{1}{1-(2\pi f)^2}\right](1 + e^{-j2\pi^2 f}) = \frac{1 + e^{-j2\pi^2 f}}{1-(2\pi f)^2}$$

(c)



$$g(t) = \cos t\,(u(t) - u(t-\pi/2))$$

$$= \cos t\, u(t) - \cos(t)\,u(t-\pi/2)$$

$$= \cos t\, u(t) + \sin(t-\pi/2)\,u(t-\pi/2)$$

$$(\because \cos t = -\sin(t-\pi/2))$$

$\therefore$ From the table 8.1

$$\cos t\, u(t) \Leftrightarrow \frac{1}{4}\left[\delta(f-1/2\pi) - \delta(f+1/2\pi)\right] + \frac{j2\pi f}{1-(2\pi f)^2}$$

From part (b)

$$\sin(t-\pi/2)\,u(t-\pi/2) \Leftrightarrow \left[\frac{1}{4j}\left[\delta(f-1/2\pi) - \delta(f+1/2\pi)\right] + \frac{1}{1-(2\pi f)^2}\right] e^{-j\pi^2 f}$$

[$e^{-j\pi f t}$ term shows the delay in frequency domain].

Now, $\delta(f-1/2\pi)\,e^{-j\pi^2 f} - \delta(f+1/2\pi)\,e^{-j\pi^2 f}$

$$= e^{-j\pi/2} - e^{-j\pi/2}$$

$$= 0$$

Similarly $\delta(f-1/2\pi) - \delta(f+1/2\pi) = 0$

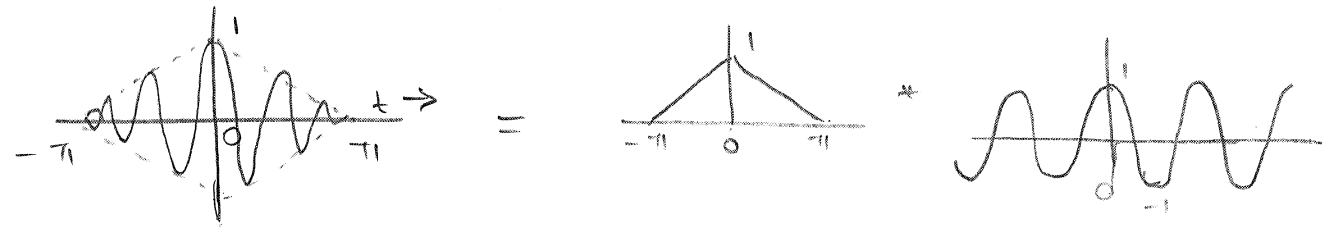
$$\therefore\ G(\omega) = \frac{j2\pi f}{1-(2\pi f)^2} + \frac{e^{-j\pi^2 f}}{1-(2\pi f)^2}$$

$$= \underline{\frac{1}{1-(2\pi f)^2}\left[j2\pi f + e^{-j\pi^2 f}\right]}$$

part (d) can be solved in similar way.

Q 3.3-6

(a)



$$= \Delta(t/2\pi) \cos 10t$$

From property 19

$$\Delta(t/\tau) \Leftrightarrow \frac{\tau}{2} \operatorname{sinc}^2\left(\frac{\pi f \tau}{2}\right)$$

For a modulated signal with

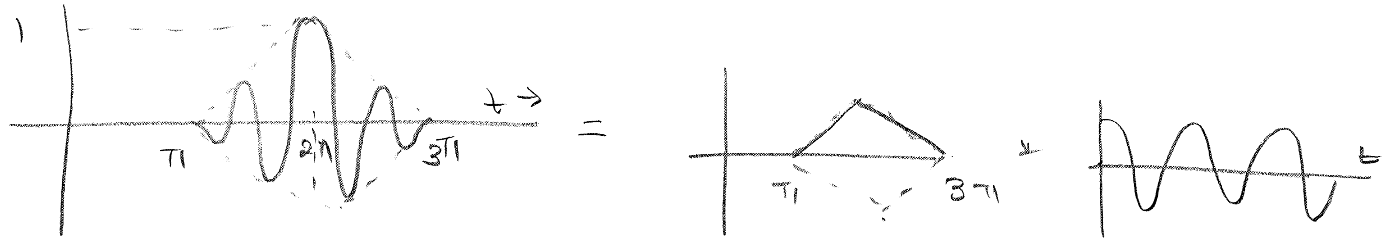
$\Delta(t/2\pi) \cos 10t$ from Frequency shifting property.

$$\mathcal{F}[g(t) \cos 2\pi f_0 t] \Leftrightarrow \frac{1}{2}[G(f-f_0) + G(f+f_0)]$$

$$\Leftrightarrow \frac{1}{2}\left[\frac{2\pi}{2} \operatorname{sinc}^2\left(\frac{2\pi^2 f - 10\pi}{2}\right) + \frac{2\pi}{2} \operatorname{sinc}^2\left(\frac{2\pi^2 f + 10\pi}{2}\right)\right]$$

$$\Leftrightarrow \frac{\pi}{2}\left[\operatorname{sinc}^2(\pi^2 f - 5\pi) + \operatorname{sinc}^2(\pi^2 f + 5\pi)\right]$$

(b)



$\Delta\left(\frac{t-2\pi}{2\pi}\right) \quad * \quad \cos 10t$

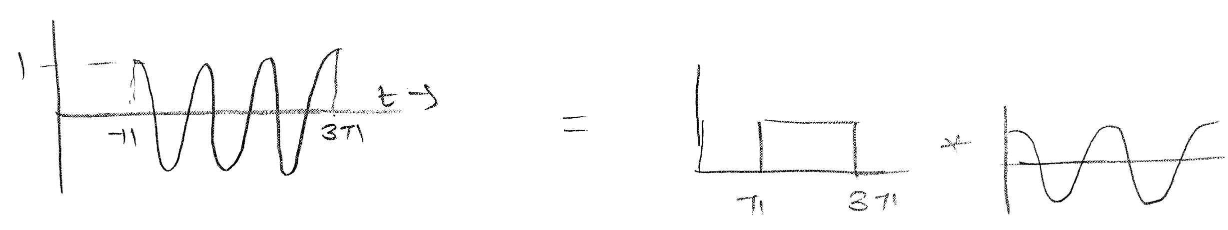
This signal is similar to previous signal in part (a). Only it is delayed by 2π

As seen earlier delay in time domain by 2π can be shown in frequency domain by $e^{-j2\pi f t}$ where $t = 2\pi$

$= e^{-j4\pi^2 f}$

$\therefore G(f) = \frac{\pi}{2}\left[\text{sinc}^2(\pi^2 f - 5\pi) + \text{sinc}^2(\pi^2 f + 5\pi)\right] e^{-j4\pi^2 f}$

(c)



$\Pi(t/2\pi)\ \cos 10t$

From table 3.1 and above problems.

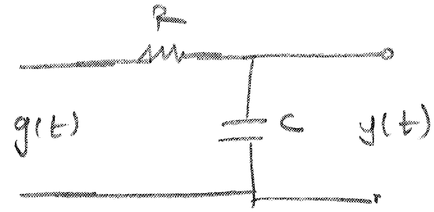
$\Pi(t/\tau) \Leftrightarrow \tau\ \text{sinc}(\pi f \tau)$

$\Pi(t/2\pi)\cos 10t \Leftrightarrow \frac{\pi}{2}\left[\text{sinc}(2\pi^2 f - 10\pi) + \text{sinc}(2\pi^2 f + 10\pi)\right] e^{-j4\pi^2 f}$

Q 3.5.4 Band width $B = 2000 Hz$

$f = 10^5 Hz$

$RC = 10^{-3}$



Now applying voltage division rule

$$H(f) = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{X_c}{R + X_c} = \frac{1/j2\pi f C}{R + 1/j2\pi f C}$$

$$= \frac{1}{1 + j2\pi f RC}$$

$$= \frac{1/RC}{1/RC + j2\pi f}$$

$$= \frac{1000}{1000 + j2\pi f}$$

magnitude of the RC filter $|H(f)| = \dfrac{1000}{\sqrt{1000^2 + (2\pi f)^2}}$

Time delay equation is given by 3.59 in text

$$t_d(f) = -\frac{d\theta_h}{d(2\pi f)} = \frac{1/RC}{(2\pi f)^2 + 1/RC} = \frac{1000}{(2\pi f)^2 + 1000}$$

To check the variation in magnitude & time delay we need check it at low and high frequency band.

Now $f = 100000 Hz$ $B = 2000 Hz$

$\therefore f_L = 99000 Hz$ $f_H = 101000 Hz$

$\therefore B = f_H - f_L$

$\therefore \quad |H(f_L)| = 1.607 \times 10^{-3} \qquad t_d(f_L) = 2.58 \times 10^{-9}$

$|H(f_H)| = 1.576 \times 10^{-3} \qquad t_d(f_H) = 2.48 \times 10^{-9}$

To compute variation following equation is used

$$\Delta |H(f)| = \frac{\big|\,|H(f_h)| - |H(f_l)|\,\big|}{\frac{1}{2}(|H(f_H)| + |H(f_l)|)} \times 100\% = 1.95\% < 2\%$$

$$\Delta t_d(f) = \frac{|t_d(f_h) - t_d(f_l)|}{\frac{1}{2}(|t_d(f_h)| + |t_d(f_l)|)} \times 100\% = 3.9\% > 1\%$$

$\therefore$ Variation is not within tolerance. Hence transmission is not distortionless. For approximate expression for output signal is given by

$$\text{Average} \quad \frac{1}{2}(|H(f_h)| + |H(f_l)|) = 1.59 \times 10^{-3}$$

$$\frac{1}{2}(t_d(f_h) + t_d(f_l)) = 6.14 \times 10^{-4}$$

$$\therefore \quad \boxed{y(t) \approx 1.59 \times 10^{-3}\, g(t - 6.14 \times 10^{-4})}$$

Q 3.7.2

Prove that $\int_{-\infty}^{\infty} g_1(t)\,g_2(t)\,dt = \int_{-\infty}^{\infty} G_1(-f)\,G_2(f)\,df$

Now,

$$g_2(t) = \int_{-\infty}^{\infty} G_2(f)\,e^{j2\pi f t}\,df \quad \text{(Inverse fourier transform)}$$

$$G_1(-f) = \int_{-\infty}^{\infty} g_1(t)\,e^{j2\pi f t}\,dt \quad \left(\because G_1(f) = \int_{-\infty}^{\infty} g_1(t)\,e^{-j2\pi f t}\,dt\right)$$

$$\therefore \int_{-\infty}^{\infty} g_1(t)\,g_2(t)\,dt = \int_{-\infty}^{\infty} g_1(t)\left[\int_{-\infty}^{\infty} G_2(f)\,e^{j2\pi f t}\,df\right]dt$$

Rearranging terms

$$= \int_{-\infty}^{\infty} G_2(f)\left[\int_{-\infty}^{\infty} g_1(t)\,e^{j2\pi f t}\,dt\right]df$$

$$= \int_{-\infty}^{\infty} G_2(f)\,G_1(-f)\,df$$

Similarly,

$$\int_{\infty}^{\infty} g_1(t)\,g_2(t)\,dt = \int_{-\infty}^{\infty} G_1(f)\,G_2(-f)\,df$$ can

be proved.