Details and Rules of the Test

Introduction to Digital Signal Processing
EE 4361
Midterm Exam
Monday-June 21, 2004
Dr. Charles P. Bernardin

Please Print Clearly.

Last Name___________________ First Name _________________ Student ID______________

Instructions:
WRITE CLEARLY AND NEATLY

1. **Exam Duration 1 hour and 15 minutes (Test will end at (9:50)**
2. **One 8.5" x 11" SINGLE-sided Crib sheet**
3. **One calculator is allowed**
4. **Read the test first and note that each problem is worth a different point value**
5. **A Z-Transform Table is provided on the last page of the test**
6. **Answer in the space/sheets provided**
7. **Where possible, show all your work; answers without any justification will not be credited**
8. **ANY copying or cheating will result in appropriate action as per university regulations**

Complex Numbers in Communications Engineering

$$j = i = \sqrt{-1}$$

Euler's Formula:

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

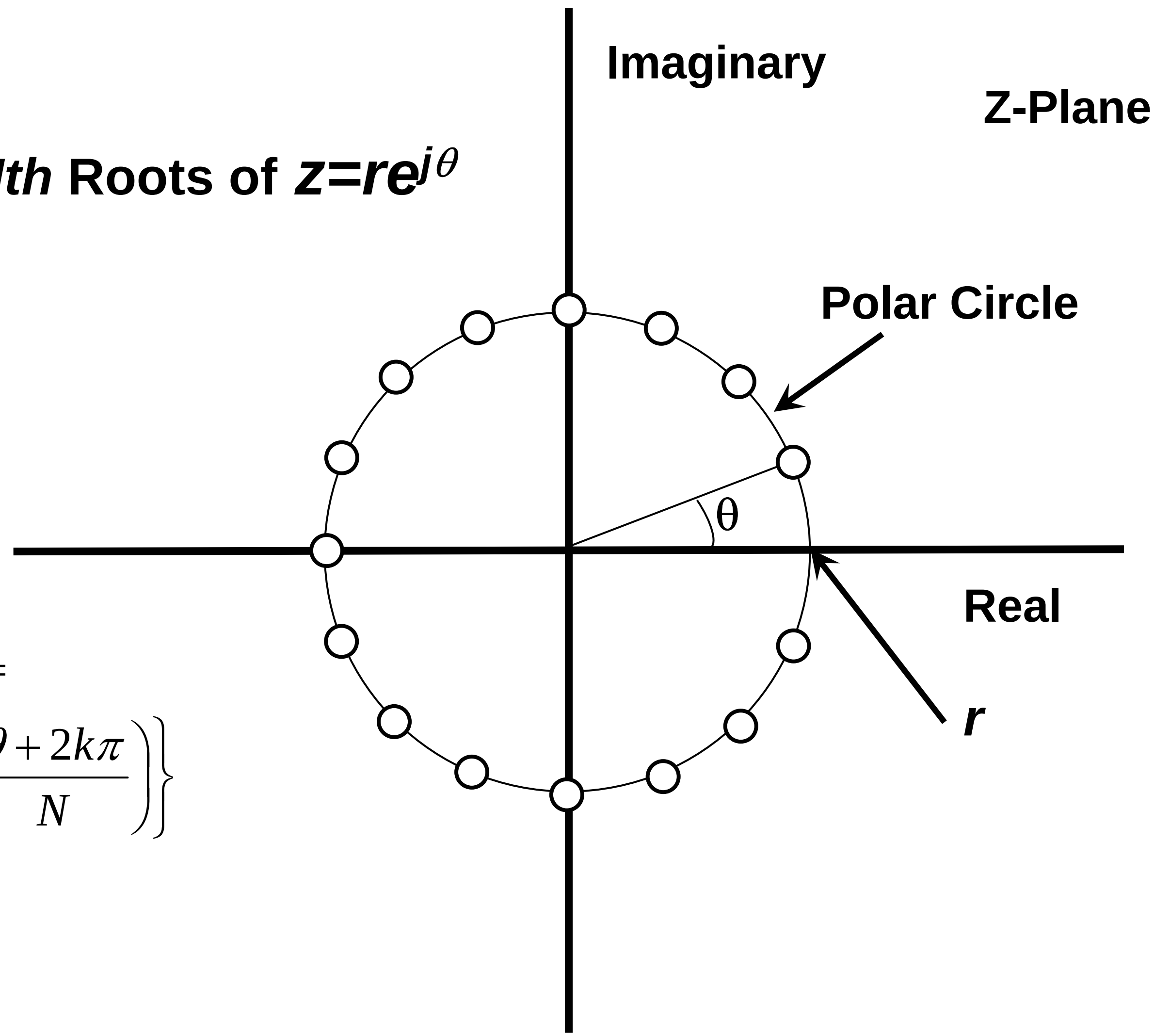
Useful Related Expressions:

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Principal *Nth* root of a complex number
(*N*= 16 for this example)



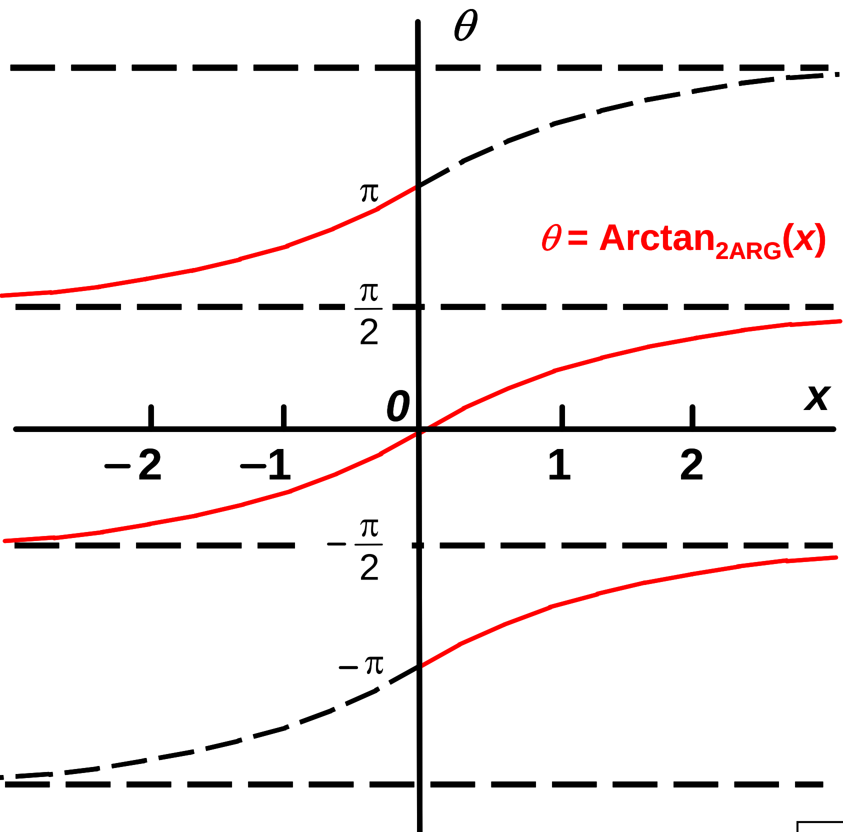
$$z = r(\cos\theta + j\sin\theta)$$

$$z^{1/N} = [r(\cos\theta + j\sin\theta)]^{1/N} =$$

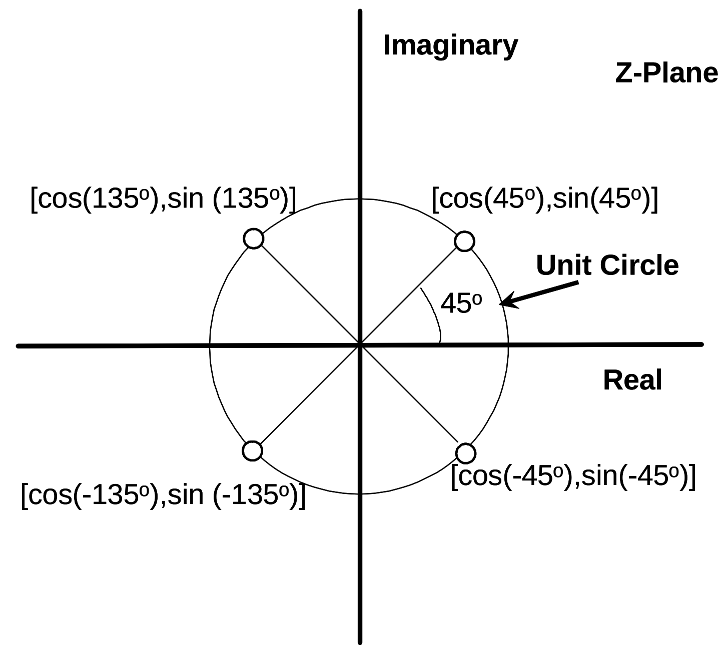
$$r^{1/N}\left\{\cos\left(\frac{\theta + 2k\pi}{N}\right) + j\sin\left(\frac{\theta + 2k\pi}{N}\right)\right\}$$

for $k = 0,1,2,\ldots N\text{-}1$

Phase and Arctangent



Two Argument Arctangent function



***4th* roots of unity**
(*N*= 4 for this example)

$\text{Atan}_{1arg}[\sin(45^o)/\cos(45^o)] = \text{Atan}_{2arg}[\cos(45^o),\sin(45^o)] = 45^o$
$\text{Atan}_{1arg}[\sin(-45^o)/\cos(-45^o)] = \text{Atan}_{2arg}[\cos(-45^o),\sin(-45^o)] = -45^o$
$\text{Atan}_{2arg}[\cos(135^o),\sin(135^o)] = \text{Atan}_{1arg}[\sin(135^o)/\cos(135^o)] + 180^o = 135^o$
$\text{Atan}_{2arg}[\cos(-135^o),\sin(-135^o)] = \text{Atan}_{1arg}[\sin(-135^o)/\cos(-135^o)] - 180^o = -135^o$

$$\theta = \tan^{-1}_{2ARG}\{\text{Real}(z), \text{Imag}(z)\} = \begin{cases} \tan^{-1}_{1ARG}\left(\dfrac{\text{Imag}(z)}{\text{Real}(z)}\right) & \text{for } z \in \text{Quadrant I or IV} \\ \tan^{-1}_{1ARG}\left(\dfrac{\text{Imag}(z)}{\text{Real}(z)}\right) + \pi & \text{for } z \in \text{Quadrant II} \\ \tan^{-1}_{1ARG}\left(\dfrac{\text{Imag}(z)}{\text{Real}(z)}\right) - \pi & \text{for } z \in \text{Quadrant III} \end{cases}$$

More on Phase and Complex Numbers

- **Phase of a real number:** $z = re^{j\theta} = r\,[\cos(\theta) + j\,0]$

  Phase[z] = Phase[$re^{j\theta}$] ; z is real => $\sin(\theta) = 0$. Two cases:

  1) $\theta = 0$ for z positive, since $\cos(0) = 1$
  2) $\theta = \pi$ for z negative, since $\cos(\pi) = -1$

- **Phase of a purely complex number:** $z = re^{j\theta} = r\,[0 + j\sin(\theta)]$

  Phase[z] = Phase[$re^{j\theta}$] ; z is purely complex => $\cos(\theta) = 0$. Two cases:

  1) $\theta = \pi/2$ for z/j positive, since $\sin(\pi/2) = 1$
  2) $\theta = -\pi/2$ for z/j negative, since $\sin(-\pi/2) = -1$

- **Phase of the product of two complex numbers is the sum of the phases of the individual numbers:**

  Phase[$z_1 z_2$] = Phase[$r_1 e^{j\theta 1}\, r_2 e^{j\theta 2}$] = Phase[$r_1\, r_2 e^{j(\theta 1+\theta 2)}$]= $\theta_1 + \theta_2$

- **Phase of the quotient of two complex numbers is the difference between the phase of the numerator and the phase of the denominator:**

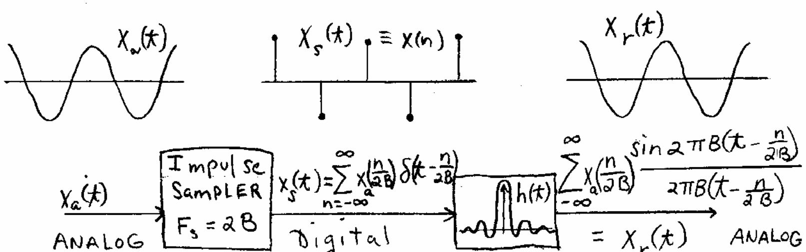
  Phase[z_1/z_2] = Phase[$r_1 e^{j\theta 1}/r_2 e^{j\theta 2}$] = Phase[$r_1/r_2 e^{j(\theta 1-\theta 2)}$]= $\theta_1 - \theta_2$

Chapter 1

Sampling Theorem

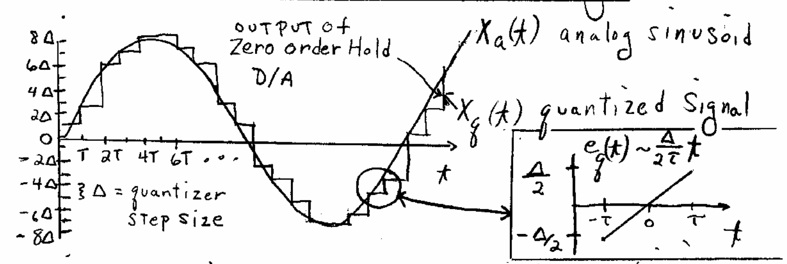
If the highest frequency contained in an analog signal is $F_{MAX} = B$ and the signal is sampled at a rate $F_s > 2F_{MAX} = 2B$, then the analog signal can be exactly recovered from its sample values using the interpolation function:

$$h(t) = \frac{\sin 2\pi B t}{2\pi B t}$$



- This is like putting a Tomato in a vegematic and then putting it back together again
- Note that $h(t)$ can be considered a continuous time lowpass antialias filter
- We can use the sampling theorem to resample the signal at any arbitrary resampling frequency without having to go back to analog domain (noisy)

A/D Quantization and Dynamic Range



For a bandlimited signal $x_a(t)$ that is sufficiently oversampled, the quantization error, $e_q(t)$, is approximately piecewise linear over the time τ that $x_a(t)$ remains within a quantization level. The power of the quantization error is approximately:

- $P_q = \frac{1}{\tau}\int_0^{\tau}\left(\frac{\Delta}{2\tau}\right)^2 t^2\,dt = \frac{\Delta^2}{4\tau^3}\left[\frac{t^3}{3}\right]_0^{\tau} = \frac{\Delta^2}{12}$

The power of the sinusoid is

- $P_s = \frac{1}{T}\int_0^{T}(A\cos 2\pi f_0 t)^2\,dt = \frac{A^2}{2}$
- Quantizer has $\underline{b}$ bits $\Rightarrow \Delta = 2A/2^b \Rightarrow P_q = \frac{A^2/3}{2^{2b}}$
- SQNR (i.e. dynamic range) $= P_s/P_q = \frac{3}{2}\cdot 2^{2b}$

Signal to Quantization Noise Ratio in dB = $\underline{1.76 + 6.02b}$

Properties of Linear Shift Invariant (LSI) Systems

- Any system that is described by the following constant-coefficient difference equation is **Linear Shift Invariant (LSI)**

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

- **Causality:** A system is causal if *y(n)*,for $n=n_1$, depends on *x(n)* **only** for $n < n_1$, i.e., the impulse response $h(n)=0$ for $n<n_1$

- **Memory** defined through examples:
  - Memoryless (Instantaneous) e.g., $y(n) = 2x(n)$ (memory of zero length)
  - Finite Memory e.g., $y(n) = 2x(n)+3x(n-1)+4x(n-2)$ (i.e., FIR with memory of length 3)
  - Infinite memory e.g. $y(n) = ay(n-1) + x(n)$ (first order recursive lowpass filter)

- **BIBO** (bounded input bounded output) Stability
- A system is **BIBO** stable **if and only if for every** input *x(n)* that is bounded on the ordinate, **t**here is a resulting output *y(n)* that is also bounded on the ordinate i.e., $|x(n)|<\infty$ and $|y(n)|<\infty$
- Equivalently, a system is **BIBO** stable **if and only if** it's impulse response series is absolutely convergent

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$$

Examples BIBO Stability

•Consider the first order causal recursive lowpass filter with difference equation

$y(n) = ay(n-1) + x(n)$

• The impulse response is $h(n) = a^n u(n), n=0,1,2,\ldots$ (why?)

$$\sum_{n=-\infty}^{N}|h(n)| = \sum_{n=-\infty}^{N}\left|a^n\right| = \frac{1-|a|^{N+1}}{1-|a|}; \quad \text{for } N \geq 0$$

•This is a geometric series which converges to

•The system is stable if $\sum_{n=-\infty}^{+\infty}|h(n)| < \infty$

•This means that for the system to be stable, $|a|<1$

e.g., 1) $h(n) = 2^n u(n)$ is **not** stable, since $\sum_{n=-\infty}^{\infty}|h(n)| = \lim_{N\to\infty}\sum_{n=-\infty}^{+N}\left|a^n\right| = \lim_{N\to\infty}\frac{1-|2|^{N+1}}{1-|2|} = 2^{N+1}-1 = \infty$

2) $h(n) = (1/2)^n u(n)$ **is** stable, $\sum_{n=-\infty}^{\infty}|h(n)| = \lim_{N\to\infty}\sum_{n=-\infty}^{+N}\left|a^n\right| = \lim_{N\to\infty}\frac{1-|1/2|^{N+1}}{1-|1/2|} = 2 < \infty$

3) $h(n) = (1)^n u(n)$ is **not** stable. However, this system is sometimes referred to as marginally stable because it is stable for many input waveforms. However, note that if $x(n) = u(n)$ (bounded input), then $y(n) = r(n)$ (unbounded output) => unstable

System Stability iff Unit Circle is in ROC of Z-Transform

- A LSI System is stable iff the ROC of the Z-Transform of its impulse response includes the unit circle

- Convergence iff $H(z) = \sum_{n} |h(n)\ z^{-n}| < \infty$

- Stability iff ***H(z)*** converges for |***z***| = 1,
  i.e., the ROC includes the unit circle
  hence, $\sum_{n} |h(n)| < \infty$

Chapter 2 & Chapter 3

- A discrete-time (DT) signal is defined only at discrete instants of time.
- A DT signal is usually represented as a sequence of values $x(n)$ for integer values of n.
- A DT signal $x(n)$ is periodic with period N if $x(n+N) = x(n)$ for some integer N.
- The DT unit-step and impulse functions are related as

$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

$$\delta(n) = u(n) - u(n-1)$$

- Any DT signal $x(n)$ can be expressed in terms of shifted impulse functions as

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

- The complex exponential $x(n) = \exp[j\Omega_0 n]$ is periodic only if $\Omega_0/2\pi$ is a rational number.
- The set of harmonic signals $x_k(n) = \exp[jk\Omega_0 n]$ consists of only N distinct waveforms.
- Time scaling of DT signals may yield a signal that is completely different from the original signal.
- Concepts such as linearity, memory, time invariance, and causality in DT systems are similar to those in continuous-time (CT) systems.
- A DT LTI system is completely characterized by its impulse response.
- The output $y(n)$ of an LTI DT system is obtained as the convolution of the input $x(n)$ and the system impulse response $h(n)$:

$$y(n) = h(n) * x(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m)$$

- The convolution sum gives only the forced response of the system.
- An alternative representation of a DT system is in terms of the difference equation (DE)

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k), \qquad n \geq 0$$

- The DE can be solved either analytically or by iterating from known initial conditions. The analytical solution consists of two parts: the homogeneous (zero-input) solution and the particular (zero-state) solution. The homogeneous solution is determined by the roots of the characteristic equation. The particular solution is of the same form as the input $x(n)$ and its delayed versions.
- The impulse response is obtained by solving the system DE with input $x(n) = \delta(n)$ and all initial conditions zero.
- The simulation diagram for a DT system can be obtained from the DE using summers, coefficient multipliers, and delays as building blocks.
- The following conditions for the BIBO stability of a DT LTI system are equivalent:
  (a) $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$
  (b) The roots of the characteristic equation are inside the unit circle. **These are also the poles of the system transfer function H(z)=Y(z)/X(z).**

- The Z-transform is the discrete-time counterpart of the Laplace transform.
- The bilateral Z-transform of the discrete-time sequence $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

- The unilateral Z-transform of a causal signal $x(n)$ is defined as

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \quad \text{i.e., x(n) is a right-sided sequence}$$

- The region of convergence (ROC) of the Z-transform consists of those values of z for which the sum converges.
- For causal sequences, the ROC in the z plane lies outside a circle containing all the poles of $X(z)$. For anticausal signals, the ROC is inside the circle such that all poles of $X(z)$ are external to this circle. If $x(n)$ consists of both a causal and an anticausal part, then the ROC is an annular region, such that the poles outside this region correspond to the anticausal part of $x(n)$, and the poles inside the annulus correspond to the causal part. **For finite sequences, the ROC is $0 < |z| < \infty$**
- The Z-transform of an anticausal sequence $x\_(n)$ can be determined from a table of unilateral transforms as

$$X_-(z) = Z\{x_-(-n)\}$$

- Expanding $X(z)$ in partial fractions and identifying the inverse of each term from a table of Z-transforms is the most convenient method for determining $x(n)$. If only the first few terms of the sequence are of interest, $x(n)$ can be obtained by expanding $X(z)$ in a power series in z^{-1} by a process of long division.
- The properties of the Z-transform are similar to those of the Laplace transform. Among the applications of the Z-transform are the solution of difference equations and the evaluation of the convolution of two discrete sequences.
- The time-shift property of the Z-transform can be used to solve difference equations.
- If $y(n)$ represents the convolution of two discrete sequences $x(n)$ and $h(n)$, then

$$Y(z) = H(z)X(z)$$

- The transfer function $H(z)$ of a system with input $x(n)$, impulse response $h(n)$, and output $y(n)$ is

$$H(z) = Z\{h(n)\} = \frac{Y(z)}{X(z)}$$

- The relation between the Laplace transform and the Z-transform of the sampled analog signal $x_a(t)$ is

$$X(z)\big|_{z=\exp[Ts]} = X_s(s)$$

- The transformation $z = \exp[Ts]$ represents a mapping from the s plane to the z plane in which the left half of the s plane is mapped inside the unit circle in the z plane, the $j\omega$-axis is mapped into the unit circle, and the right half of the s plane is mapped outside the unit circle. The mapping efffectively divides the s plane into horizontal strips of width ω_s, each of which is mapped into the entire z plane.