

5

Statistics

Statistics evolved from the need to analyze social and political trends. Despite its origin, statistics has developed into a highly sophisticated numerical science. Modern statistics goes beyond data reduction, forming the basis of all applied probability. Statistics is the foundation of many new and diverse scientific fields such as reliability theory, decision theory, and epidemiology.

This chapter attempts to cover those areas of statistics that are more frequently used by engineers and scientists, rather than those that are found in advanced statistics books.

5.1 EXPECTATION AND RANDOM VARIABLES

In the chapter on probability, the concept of the **mean** and **variance** of a random variable was introduced. Recall that a **random variable** is any function that maps all the events of a probabilistic experiment (the sample space) to the x -axis. As discussed previously, each random variable has its own characteristic probability density function and probability distribution function.

This section defines the **expectation** of a random variable. The mean and variance are discussed in more detail and are shown to be special cases of the more general concept of expectation.

5.1.1 Expectation of a Function

If X is a random variable and if $Y = H(X)$ is a function of X , then Y is also a random variable with its own probability density function. The **expectation** of the random variable Y will be denoted as $E(Y)$ and is defined below.

If X is a discrete random variable, the expectation of a function of X , $H(X)$, is defined by:

$$E(Y) = E(H(X)) = \sum_{j=1}^{\infty} H(x_j)p(x_j)$$

where $p(x_j)$ is the discrete probability density of X .

Similarly, if X is a continuous variable, the expectation of a function of X , $H(X)$, is defined by

$$E(Y) = E(H(X)) = \int_{-\infty}^{+\infty} H(x)f(x) dx$$

where $f(x)$ is the continuous probability density of X .

Properties of Expectation. There are a number of important properties of the expected value that hold for both discrete and continuous random variables. They are listed here without proof:

1. If C is a constant, and $X = C$, then $E(X) = C$.
2. If C is a constant, then $E(CX) = CE(X)$.
3. Given two random variables, X and Y , $E(X + Y) = E(X) + E(Y)$.
4. If X and Y are independent, then $E(XY) = E(X)E(Y)$.

5.1.2 Mean and Variance of a Random Variable

In the previous section, the expectation of a function of a random variable was defined. This definition may be used to formally define the mean and variance of a random variable.

The Mean of a Random Variable. The **mean** μ of a random variable X is simply the expected value of X , $E(X)$. Thus, if X is a discrete random variable, its mean is defined by

$$E(X) = \sum_{i=1}^{\infty} x_i p(x_i)$$

where $p(x_i)$ is the discrete probability density of X .

Similarly, if X is a continuous variable, its mean is defined by

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx$$

where $f(x)$ is the continuous probability density of X .

The mean represents the average value of the probability density function.

The Variance of a Random Variable. The variance s^2 of a random variable X is the expected value of $(X - u)^2$, and is denoted $\text{VAR}(X)$.

Properties of Variance. There are several important properties of the variance of discrete and continuous random variables:

1. $\text{VAR}(X) = E(X - E(X))^2 = E(X^2) - E^2(X)$, where $E(X) = u$.
2. If C is a constant, then $\text{VAR}(X + C) = \text{VAR}(X)$.
3. If C is a constant, then $\text{VAR}(CX) = C^2\text{VAR}(X)$.
4. If X and Y are independent, then $\text{VAR}(X + Y) = \text{VAR}(X) + \text{VAR}(Y)$.

The variance quantifies the amount of spread that a random variable has about its mean value.

Sample Variance. In statistical sampling, the above definition of variance is often referred to as the **population variance**. In practice, the entire population can rarely be observed and if the variance is desired, it must be estimated from a small sample of the population. To ensure that the mean of estimate equals the true variance (i. e., an unbiased estimate), the above variance definition must be modified slightly. The **sample variance** of a sample of size m is given by

$$s_s^2 = [(X_1 - u_x)^2 + (X_2 - u_x)^2 + \dots + (X_m - u_x)^2]/(m - 1)$$

Similarly, the **sample standard deviation** is s_s . Note that this variance expression is similar to the true variance definition except for the factor of $m - 1$ in the denominator.

See Function: `stat()`.

5.1.3 Chebyshev's Theorem

The concepts of mean and variance are very useful in predicting the outcomes of random events. Chebyshev's theorem relates the mean and variance of any arbitrary random variable. This famous inequality is an explicit mathematical bound which shows how the variance directly measures statistical variability. **Chebyshev's theorem** is as follows:

If a random variable has a probability density with a mean u and a standard deviation s , the probability that the variable deviates from the mean by at least k standard deviations is at most $1/k^2$.
Mathematically,

$$P(|X - u| \geq ks) \leq 1/k^2$$

Chebyshev's inequality is a very powerful statistical statement. For example, this theorem proves that the probability that a random variable will achieve a value that is two standard deviations away from the mean is at most 1/4. Likewise, the probability

that a random variable will achieve a value that is 10 standard deviations away from the mean is at most $1/100$.

Example:

Consider 1600 flips of a fair coin. The number of heads is a binomial random variable with mean $\mu = 1600/2 = 800$ and standard deviation $s = (1600/4)^{0.5} = 20$. For $k = 5$, Chebyshev's theorem states that there is a probability of at least 0.96 that there will be between 700 and 900 heads. Equivalently, the *proportion* of heads will fall between 0.4375 and 0.5625.

5.2 REGRESSION AND LEAST-SQUARES

Errors are inherent in the measurement of any physical quantity. In making measurements, usually the goal is to determine the true values from the observations as accurately as possible. **Regression** is a statistical technique that reduces random errors by effectively smoothing multiple measurements.

Few numerical techniques have had a more pervasive effect on applied mathematics than the field of least-squares analysis. The contributions of least-squares methods are most obvious in the areas of:

1. Data smoothing
2. Statistical trend analysis (regression)
3. Fourier analysis
4. Functional approximation and minimization.

It is interesting to note that the mathematician who first discovered least-squares (Gauss) considered his findings trivial and did not report them until a decade after his discovery.

5.2.1 Generalized Least-Squares

One of the most powerful curve-fitting techniques is the method of **generalized least-squares**. For any set of one-dimensional functions, this technique determines the linear combination that best fits the data.

The hypothesis is that a set of M observations (x_i, y_i) can be represented by the equation:

$$y(x) = a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) + \dots + a_N f_N(x)$$

The goal is to find the coefficients a_j of the best-fitting linear combination of the N one-dimensional basis functions $f_j(x)$. For a solution to exist, the number of observations must be greater than the number of basis functions, (i. e., $M \geq N$). Solving for the coefficients a_j is equivalent to determining the composite function $y(x)$ that minimizes the **mean-square error** across the entire data interval:

$$\text{MSE} = (y(x_1) - y_1)^2 + (y(x_2) - y_2)^2 + \dots + (y(x_M) - y_M)^2$$

Note that if y_i exactly equals each $y(x_i)$, then the error of the approximation is zero. This is generally more than one can expect with real data. In practice, there is usually some minimal nonzero error.

Before proceeding it will be helpful to introduce a shorthand notation. Let Σ denote the finite sum over the M data points. For example, Σf_j means the sum of the j th basis function across all the data points:

$$\Sigma f_j = f_j(x_1) + f_j(x_2) + \dots + f_j(x_M)$$

Similarly, $\Sigma y f_j$ is defined as:

$$\Sigma y f_j = y_1 f_j(x_1) + y_2 f_j(x_2) + \dots + y_M f_j(x_M)$$

and the expression $\Sigma f_k f_j$ denotes:

$$\Sigma f_k f_j = f_k(x_1) f_j(x_1) + f_k(x_2) f_j(x_2) + \dots + f_k(x_M) f_j(x_M)$$

where $1 \leq k \leq N$ and $1 \leq j \leq N$.

The minimum MSE solution to the generalized least-squares problem can be found by solving:

$$\begin{bmatrix} \Sigma f_1 f_1 & \Sigma f_1 f_2 & \Sigma f_1 f_3 & \dots & \Sigma f_1 f_N \\ \Sigma f_2 f_1 & \Sigma f_2 f_2 & \Sigma f_2 f_3 & \dots & \Sigma f_2 f_N \\ \Sigma f_3 f_1 & \Sigma f_3 f_2 & \Sigma f_3 f_3 & \dots & \Sigma f_3 f_N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Sigma f_N f_1 & \Sigma f_N f_2 & \Sigma f_N f_3 & \dots & \Sigma f_N f_N \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \Sigma y f_1 \\ \Sigma y f_2 \\ \Sigma y f_3 \\ \vdots \\ \Sigma y f_N \end{bmatrix}$$

These are called the **normal equations**. Note that the normal equations are symmetric. This property can be exploited in determining least-squares solutions.

See Function: `least_sq()`.

5.2.2 Polynomial Regression

One of the more effective fitting functions is the N th order polynomial:

$$y = a_1 + a_2 x + a_3 x^2 + \dots + a_N x^N$$

The normal equations for a polynomial fit are

$$\begin{bmatrix} M & \Sigma x & \Sigma x^2 & \dots & \Sigma x^N \\ \Sigma x & \Sigma x^2 & \Sigma x^3 & \dots & \Sigma x^{N+1} \\ \Sigma x^2 & \Sigma x^3 & \Sigma x^4 & \dots & \Sigma x^{N+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Sigma x^N & \Sigma x^{N+1} & \Sigma x^{N+2} & \dots & \Sigma x^{2N} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \Sigma y \\ \Sigma yx \\ \Sigma yx^2 \\ \vdots \\ \Sigma yx^N \end{bmatrix}$$

These equations can be solved with a variety of computer approaches such as LU decomposition or Gaussian elimination. It should be noted that computer precision usually limits the order of the polynomial to less than ten.

5.2.3 Linear Regression

The simplest type of approximating curve is a line. The constants a and b must be found which give the closest agreement between the empirical data and the equation:

$$y = bx + a$$

This is just a special case of the polynomial regression described above. The solution can be found by using Cramer's Rule and the following normal equations:

$$\begin{bmatrix} M & \Sigma x \\ \Sigma x & \Sigma x^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \Sigma y \\ \Sigma yx \end{bmatrix}$$

Using Σ to denote summation over all the observations, the solution is given by:

$$b = \frac{\Sigma yx - \Sigma x \Sigma y / M}{\Sigma x^2 - (\Sigma x)^2 / M}$$

$$a = [\Sigma y - b \Sigma x] / M.$$

The **coefficient of determination** r^2 indicates how closely a function approximates a data set. Sometimes called the **correlation coefficient**, the value of r^2 is always between zero and one. The closer r^2 is to one, the better the approximation. For a linear approximation, the coefficient is given by:

$$r^2 = \frac{[\Sigma yx - \Sigma x \Sigma y / M]^2}{[\Sigma x^2 - (\Sigma x)^2 / M][\Sigma y^2 - (\Sigma y)^2 / M]}$$

See Function: linreg().

5.2.4 Logarithmic Regression

Another useful approximating function is the natural logarithm:

$$y = a + b \ln(x)$$

where $x_i > 0$.

The normal equations for the logarithmic fit are

$$\begin{bmatrix} M & \Sigma \ln(x) \\ \Sigma \ln(x) & \Sigma \ln^2(x) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \Sigma y \\ \Sigma y \ln(x) \end{bmatrix}.$$

Solving for the logarithmic coefficients yields

$$b = \frac{\Sigma y \ln(x) - \Sigma y \Sigma \ln(x) / M}{\Sigma \ln^2(x) - (\Sigma \ln(x))^2 / M}$$

$$a = [\Sigma y - b \Sigma \ln(x)] / M$$

The correlation coefficient for the logarithmic fit is given by

$$r^2 = \frac{[\Sigma \ln(x)y - \Sigma y \Sigma \ln(x) / M]^2}{[\Sigma y^2 - (\Sigma y)^2 / M][\Sigma \ln^2(x) - (\Sigma \ln(x))^2 / M]}$$

5.2.5 Fourier Regression

Any data set can be represented by a combination of a finite sine and cosine series:

$$y = a_0 + a_1 \cos(wx) + a_2 \cos(2wx) + \dots + a_N \cos(Nwx) \\ + b_1 \sin(wx) + b_2 \sin(2wx) + \dots + b_N \sin(Nwx)$$

The normal equations for a finite sinusoidal series fit are

$$\begin{bmatrix} M & \Sigma \cos(wx) & \Sigma \sin(wx) & \dots \\ \Sigma \cos(wx) & \Sigma \cos^2(wx) & \Sigma \sin(wx) \cos(wx) & \dots \\ \Sigma \sin(wx) & \Sigma \sin(wx) \cos(wx) & \Sigma \sin^2(wx) & \dots \\ \vdots & \vdots & \vdots & \ddots \\ \Sigma \cos(Nwx) & \Sigma \cos(Nwx) \cos(wx) & \Sigma \cos(Nwx) \sin(wx) & \dots \\ \Sigma \sin(Nwx) & \Sigma \sin(Nwx) \cos(wx) & \Sigma \sin(Nwx) \sin(wx) & \dots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_N \\ b_N \end{bmatrix} = \begin{bmatrix} \Sigma y \\ \Sigma y \cos(wx) \\ \Sigma y \sin(wx) \\ \vdots \\ \Sigma y \cos(Nwx) \\ \Sigma y \sin(Nwx) \end{bmatrix}$$

The Fourier coefficients can be determined by any number of approaches for solving linear systems of equations. However, it should be mentioned that this is neither the most efficient nor the most accurate approach to sinusoidal analysis. This is especially true as the number of harmonics N increases. For large N ($N > 10$), the FFT algorithm described in the Digital Signal Processing chapter is a far more efficient and accurate method. The FFT method is also a least-squares fit and has almost no limit on the order (N) of the sinusoidal series.

5.2.6 Nonlinear Regression

This section describes a few nonlinear forms of regression. For generalized least-squares, the fitting function can be expressed as a linear combination of arbitrary basis functions. In this context the term nonlinear does **not** mean “not a line,” but refers to functions that cannot be represented by a linear combination of basis functions. Although these regression types are not currently implemented in this package, you could easily implement them yourself.

Exponential Regression. The exponential function is one of the most commonly used curve-fitting functions. A linear combination of weighted exponentials can be a very effective regression formula:

$$y = c_1 + c_2 e^x + \dots$$

This is the exponential form that is implemented in the GRAFIX program.

One of the simplest nonlinear approximating functions is the exponential curve:

$$y = a e^{bx}$$

where $a > 0$, and $y_i > 0$.

This problem can be linearized by taking the logarithm of both sides of the equation:

$$\ln(y) = \ln(a) + bx$$

The normal equations for the exponential fit are

$$\begin{bmatrix} M & \Sigma x \\ \Sigma x & \Sigma x^2 \end{bmatrix} \begin{bmatrix} \ln(a) \\ b \end{bmatrix} = \begin{bmatrix} \Sigma \ln(y) \\ \Sigma x \ln(y) \end{bmatrix}$$

Solving for the coefficients of the exponential function yields

$$b = \frac{\Sigma x \ln(y) - \Sigma x \Sigma \ln(y)/M}{\Sigma x^2 - (\Sigma x)^2/M}$$

$$a = \exp[\Sigma \ln(y)/M - b \Sigma x/M]$$

The correlation coefficient of the exponential fit is given by

$$r^2 = \frac{[\Sigma \ln(y)x - \Sigma x \Sigma \ln(y)/M]^2}{[\Sigma x^2 - (\Sigma x)^2/M][\Sigma \ln^2(y) - (\Sigma \ln(y))^2/M]}$$

Geometric Power Curve Regression. Like the exponential curve, the geometric power curve can be useful in approximating data:

$$y = ax^b$$

where $a > 0$, $x_i > 0$, and $y_i > 0$.

This problem can be linearized by taking the logarithm of both sides of the equation:

$$\ln(y) = \ln(a) + b \ln(x)$$

The normal equations for the geometric power fit are

$$\begin{bmatrix} M & \Sigma \ln(x) \\ \Sigma \ln(x) & \Sigma \ln^2(x) \end{bmatrix} \begin{bmatrix} \ln(a) \\ b \end{bmatrix} = \begin{bmatrix} \Sigma \ln(y) \\ \Sigma \ln(y) \ln(x) \end{bmatrix}$$

Solving for the coefficients of the geometric power function yields

$$b = \frac{\Sigma \ln(x) \ln(y) - \Sigma \ln(x) \Sigma \ln(y)/M}{\Sigma \ln^2(x) - (\Sigma \ln(x))^2/M}$$

$$a = \exp[\Sigma \ln(y) - b \Sigma \ln(x)]/M$$

For the geometric power fit the correlation coefficient is given by

$$r^2 = \frac{[\Sigma \ln(y) \ln(x) - \Sigma \ln(x) \Sigma \ln(y)/M]^2}{[\Sigma \ln^2(x) - (\Sigma \ln(x))^2/M][\Sigma \ln^2(y) - (\Sigma \ln(y))^2/M]}$$

Hyperbolic Regression. In speech analysis and synthesis, the generalized hyperbolic curve is a useful approximating function:

$$y = \frac{1}{a_1 + a_2x + a_3x^2 + \dots + a_Nx^N}$$

where $y_i > 0$.

The normal equations for the hyperbolic fit are

$$\begin{bmatrix} M & \Sigma x & \Sigma x^2 & \dots & \Sigma x^N \\ \Sigma x & \Sigma x^2 & \Sigma x^3 & \dots & \Sigma x^{N+1} \\ \Sigma x^2 & \Sigma x^3 & \Sigma x^4 & \dots & \Sigma x^{N+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \Sigma x^N & \Sigma x^{N+1} & \Sigma x^{N+2} & \dots & \Sigma x^{2N} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \Sigma 1/y \\ \Sigma x/y \\ \Sigma x^2/y \\ \vdots \\ \Sigma x^N/y \end{bmatrix}$$

Again, a variety of computer approaches can be used to determine the best-fitting coefficients. To avoid computer precision problems, the order of the denominator polynomial should be limited to 10.

TABLE 1. Values of the Standard Normal Distribution Function*

$$\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = P(Z \leq z)$$

z	0	1	2	3	4	5	6	7	8	9
-3.0	0.0013	0.0010	0.0007	0.0005	0.0003	0.0002	0.0002	0.0001	0.0001	0.0000
-2.9	0.0019	0.0018	0.0017	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0126	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0238	0.0233
-1.8	0.0359	0.0352	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0300	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0570	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0722	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2297	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

*B. W. Lindgren, *Statistical Theory*, The Macmillan Company, 1960.

TABLE 1 (Continued)

$$\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = P(Z \leq z)$$

z	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6701	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9430	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9648	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9700	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9762	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000